Stat Papers (2013) 54:911–930
DOI 10.1007/s00362-012-0497-3

REGULAR ARTICLE

When bubbles burst: econometric tests based on structural breaks

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Received: 17 July 2012 / Revised: 12 December 2012 / Published online: 9 January 2013

Abstract Speculative bubbles have played an important role ever since in financial economics. During an ongoing bubble it is relevant for investors and policy-makers to know whether the bubble continues to grow or whether it is already collapsing. Prices are typically well approximated by a random walk in absence of bubbles, while periods of bubbles are characterised by explosive price paths. In this paper we first propose a conventional Chow-type testing procedure for a structural break from an explosive to a random walk regime. It is shown that under the null hypothesis of a mildly explosive process a suitably modified Chow-type statistic possesses a standard normal limiting distribution. Second, a monitoring procedure based on the CUSUM statistic is suggested. It timely indicates such a structural change. Asymptotic results are derived and small-sample properties are studied via Monte Carlo simulations. Finally, two empirical applications illustrate the merits and limitations of our suggested procedures.

Keywords Speculative bubbles · Structural breaks · Mildly explosive processes · Monitoring

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JEL Classification C12 (Hypothesis Testing) · C22 (Time-Series Models) · G10 (General Financial Markets)

1 Introduction

Speculative bubbles have a long history in stock, commodity and real estate markets. They are often considered to be responsible for severe economic crises such as the speculative excess on stock prices prior to the great depression from 1930–1933 or the recent financial crisis of 2007–2008 that was preceded by the US housing bubble. All bubbles are characterised by an explosive path of the underlying market prices, whereas in "normal times", prices are well approximated by a random walk process. A standard model to rationalise the occurrence and persistence of speculative bubbles is the framework of rational bubbles [see, e.g., Blanchard and Watson (1982)]. In such models it is economically rational to invest in an obviously overpriced asset as long as the investor asserts that the price continues to rise at an exponential rate. An alternative approach is adapted by Froot and Obstfeld (1991). It is based on the concept of intrinsic bubbles. This framework directly links the evolution of the bubble to fundamentals like dividend payments.

A natural empirical approach to identify the start of speculative bubbles is therefore the application of tests for a structural break from a random walk to an explosive regime. Such tests were originally proposed by Phillips et al. (2011c) and further developed by Phillips et al. (2011a,b), Phillips and Yu (2011) and Homm and Breitung (2012). Kruse and Frömmel (2012) consider a CUSUM of squares-based test for a structural break in the fractional differencing parameter for the dividend-price ratio instead. A collapse of a bubble can be detected by testing for a structural break from an explosive regime to a random walk. Phillips et al. (2011c) identify the dates of the emergence and the collapse of the bubble by the time period where the sequence of right-tailed recursive Dickey–Fuller (DF) tests crosses the critical level from below and from above, respectively. Their analysis suggests that the so-called dot-com bubble starts at July 1995 and ends March 2001. As shown by Homm and Breitung (2010) the identification of break dates suggested by the recursive testing approach suffers from a built-in time delay that may be sizable in small and moderate sample sizes.[1]

The inherent time delay is particularly relevant when date-stamping the burst of the bubble. Assume that some price series exhibits a very substantial and sustainable bubble process. In this case, the recursive DF statistic will be driven far above the critical value. When the bubble process comes to an end, the DF statistic starts to decrease gradually towards the critical value. Some periods later, it falls below the threshold. This is nicely illustrated by Fig. 3 of Phillips et al. (2011c). The sequence of DF statistics reaches its maximum already October 2000 but it requires another five months until the statistic falls below the critical value in March 2001.

[1] Phillips et al. (2011c) suggest critical values that slowly tend to infinity in order to ensure that the break dates are estimated consistently. In finite samples, however, the distribution of estimated break dates is severely skewed leading to a considerable delay of 5–15 periods [cf. Homm and Breitung (2010)].

In this paper we consider tests for a collapsing bubble that attempt to avoid the delay in detecting the burst of the bubble. Following Homm and Breitung (2012) we consider two different approaches. First, a conventional Chow type testing procedure for a structural break from an explosive to a random walk regime is proposed. Note that in this framework we assume an explosive process under the null hypothesis, whereas Phillips et al. (2011c) and Homm and Breitung (2012) consider the null hypothesis that the series is a random walk. Accordingly, both approaches are not directly comparable as the approach suggested by Phillips et al. (2011c) does not control the size of the bubble hypothesis. Second, we propose a monitoring procedure based on the CUSUM statistic that timely indicates a change from an explosive to a random walk process.

The paper is organised as follows. The testing framework is developed in Sect. 2. It is argued that if the series is composed of (i) a fundamental component that exhibit the martingale property and (ii) a bubble process that is governed by a mildly explosive autoregressive process [in the sense of Phillips and Magdalinos (2007)], then the resulting process has an explosive ARIMA(1,1,1) representation. Moreover, it is argued that this process can be reasonably well approximated by a simple autoregressive process of finite order. In Sect. 3 the LM version of the Chow test for a structural break is analysed. It is shown that the Chow test has a degenerate limiting null distribution. We therefore propose a modified test statistic that is shown to have a standard normal limiting distribution. A monitoring procedure based on the CUSUM statistic is considered in Sect. 4 and in Sect. 5 we propose to estimate the unknown break date using a maximum likelihood (ML) approach. The empirical size and power properties of alternative test statistics are compared in Sect. 6 and two applications to the NASDAQ 100 and Hang Seng stock price indexes are presented. Section 7 concludes.

2 Testing framework

The stock price is decomposed as $P_t = P_t^f + B_t$, where the fundamental value of the stock is represented by the present value of the expected stream of dividends (D_{t+i}):

$$P_t^f = \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} E_t(D_{t+i})$$

and the (rational) bubble component which is characterised by

$$E_t(B_{t+1}) = (1+r)B_t, \tag{1}$$

where r is the risk free rate which is assumed to be constant. Following earlier work starting with Bachelier (1900) we assume that the fundamental price is a martingale process with

$$E_t(P_{t+1}^f) = P_t^f.$$

There are two possible rationales to motivate this assumption. First, assuming that D_t is a random walk with drift

$$D_{t+1} = \mu + D_t + \varepsilon_{t+1}$$

it follows that

$$P_t^f = \frac{(1+r)\mu}{r^2} + \frac{1}{r} D_t$$

and, thus, the fundamental value of the stock is also characterised by a random walk with drift.

In many application the sampling frequency of the stock price is higher than the dividend periods. Assume for example that dividends are payed out annually, whereas stock prices form a monthly series. Accordingly, the dividend series D_t is zero in eleven out of 12 months. Accordingly, the change in stock price during periods without dividend payment is solely due to the update of expectations about the future stream of dividends. It follows that without dividend payment the updates of the expectations from period t to $t+1$ form a martingale difference sequence with

$$E_t(P_{t+1}^f - P_t^f) = 0$$

and, thus, P_t^f can be represented by a random walk.[2]

It is important to note that the expected bubble premium $E_t(B_{t+1} - B_t)$ includes a premium for a possible collapse of the bubble. Denote the probability that the bubble continues by $0 < \pi < 1$, whereas the probability of a bubble crash is $1 - \pi$. It follows that condition (1) becomes $\pi E_t(B_{t+1}) = (1+r)B_t$ implying

$$E_t(B_{t+1}) = \frac{1+r}{\pi} B_t \equiv \varrho B_t$$

with $\varrho > r$ [cf. Blanchard and Watson (1982)]. Based on this framework the stock prices can be represented as

$$P_t = \delta_t B_t + P_t^f, \tag{2}$$

where δ_t is an indicator function that is equal to one if a bubble is present and zero otherwise. Our empirical testing strategy for identifying a bubble crash is equivalent to testing for $\delta_{[\tau T]+\kappa} = 0$ $(\kappa = 1, 2, \ldots)$ conditional on $\delta_{[\tau T]-s} = 1$ for $s = 0, 1, \ldots$. Let $[a]$ indicate the integer part of a. In our notation $\tau \in (0, 1)$ represents the break period relative to the sample span, i.e., $\tau = 0.5$ means that the break occurs in the first period of the second half of the sample. The actual break date measured by the absolute time scale is $[\tau T]$.

The bubble regime is characterised by an explosive ARIMA(1,1,1) series with

$$(1 - \varrho L)(1 - L)P_t = (1 - L)u_t + (1 - \varrho L)\varepsilon_t \tag{3}$$

[2] Note that on exchange markets the stock price immediately adjusts for dividends so that the price represents the *ex dividend* price of the share. It follows that in periods with dividend payment the ex dividend price also have the martingale difference property.

Table 1 Implied MA coefficient of η_t

ϱ	$\lambda^2 = 0.5$	$\lambda^2 = 1$	$\lambda^2 = 2$	$\lambda^2 = 5$	$\lambda^2 = 10$
1.005	0.9955	0.9965	0.9978	0.9990	0.9995
1.010	0.9911	0.9930	0.9955	0.9980	0.9990
1.015	0.9868	0.9895	0.9933	0.9971	0.9985
1.020	0.9824	0.9860	0.9911	0.9961	0.9980
1.025	0.9781	0.9826	0.9889	0.9951	0.9975

Entries report the implied MA coefficient β for $\eta_t = (1 - \varrho L)(1 - L)P_t = \nu_t - \beta\nu_{t-1}$ in (3)

where u_t and ε_t denote the innovation of the bubble process and the fundamental price, respectively. After a bubble crash (indicated by switch from $\delta_t = 1$ to $\delta_{t+1} = 0$) the price series follows a random walk process with $(1-L)P_t = \varepsilon_t$. Accordingly, our strategy is to test for a structural break from an explosive regime in periods $1, 2, \ldots, [\tau T]$ to a random walk process in the subsequent periods $[\tau T] + 1, \ldots, T$.

It should be noted that the root of the MA polynomial of the bubble regime is typically close to unity. The first order autocorrelation of $\eta_t = (1 - \varrho L)(1 - L)P_t$ can be represented as

$$\text{corr}(\eta_t, \eta_{t-1}) = -\frac{1}{2 + \theta} \quad \text{for } t < [\tau T]$$

with $\theta = (1 - \varrho)^2/(\lambda^2 + \varrho)$ and $\lambda^2 = \sigma_u^2/\sigma_\varepsilon^2$. A unit MA coefficient results if $\theta = 0$ (resp. $\varrho = 1$). For a monthly series the range of plausible values is $\varrho \in [1.005, 1.025]$ yielding a bubble premium between 6 and 30 percent a year. Table 1 presents the implied MA coefficient of the MA(1) representation for various reasonable values of ϱ and λ. It turns out that the implied MA coefficients is typically very close to unity and therefore the usual estimation procedures (nonlinear least-squares or ML estimators) may perform poorly [eg. Davis and Dunsmuir (1996)]. Therefore, we refrain from estimating the ARIMA(1,1,1) representation and consider a convenient but nevertheless reliable approximation.

To derive an approximation of the stochastic process within the bubble regime we follow Phillips et al. (2011c) and represent the bubble process as a *mildly explosive* process with

$$B_t = \underbrace{\left(1 + \frac{c}{T^\gamma}\right)}_{= \varrho} B_{t-1} + u_t \quad \text{with } 0 < \gamma < 1.$$

In empirical applications it is not very likely that the bubble runs more than 5 years (or 60 months), say. In such small samples it makes sense to express the deviation of the autoregressive parameter from unity relative to the sample size T. For example, in a sample with $T = 60$ the bubble parameter of $\varrho = 1.02$ results for $c = 0.43$ and $\gamma = 0.75$. Using this representation, it is possible to assess the relative magnitude of the bubble and fundamental component. Note that

$$(1-\varrho L)P_t = u_t + \varepsilon_t - \frac{c}{T^\gamma}P_{t-1}^f.$$

Since P_{t-1}^f is $O_p(\sqrt{T})$ it follows that for $0.5 < \gamma < 1$ the stock price can be approximated by an mildly explosive AR(1) process as $T \to \infty$. Since it is not very likely that the bubble runs for a long time span, the test is typically performed using fairly small sample sizes. Therefore, the asymptotic properties may be unreliable in relevant sample sizes (eg. $T = 60$ months). It is thus reasonable to allow for some serial correlation in the residuals e_t of the first order autoregressive representation $y_t = \varrho y_{t-1} + e_t$ for a time series of prices y_t.

3 Test statistics

In the previous section we argue that the collapse of a speculative bubble can be represented by a change from an explosive regime to a random walk. Accordingly, we consider alternative tests for a structural break in the autoregressive parameter.

The first statistic is the LM version of Chow (1960) test statistic [cf. Krämer and Sonnberger (1986)], which is asymptotically equivalent to the t-statistic for $\phi = 0$ in the regression[3]

$$y_t = \varrho y_{t-1} + \phi y_{t-1}^\tau + e_t \quad t = 2, 3, \ldots, T, \tag{4}$$

where

$$y_t^\tau = \begin{cases} 0 & \text{for } t = 1, \ldots, [\tau T] \\ y_t & \text{for } t = [\tau T] + 1, \ldots, T \end{cases}$$

and $\tau \in (0, 1)$ denotes the break date measured relative to the sample size. Let $\tilde{e}_t^0$ denote the residual computed under the null hypothesis $\phi = 0$, i.e., $\tilde{e}_t^0 = y_t - \widehat{\varrho}^0 y_{t-1}$, where

$$\widehat{\varrho}^0 = \frac{\sum_{t=2}^T y_t y_{t-1}}{\sum_{t=2}^T y_{t-1}^2}.$$

The LM test statistic for $\phi = 0$ results as

$$\psi_\tau = \frac{\sum_{t=[\tau T]+1}^T \tilde{e}_t^0 \, y_{t-1}}{\sqrt{\frac{1}{T-1}\sum_{t=2}^T (\tilde{e}_t^0)^2}\sqrt{\sum_{t=[\tau T]+1}^T y_{t-1}^2}}. \tag{5}$$

It is well known that if y_t is stationary (i.e. $|\varrho| < 1$) this test statistic has a standard normal limiting distribution under the null hypothesis $\phi = 0$. If $\varrho = 1$ the asymptotic

[3] To simplify the exposition we neglect a possible constant or time trend in the autoregressive representation. In our empirical examples a constant is included.

null distribution of the test statistic is nonstandard and can be represented as a functional of Brownian motions [cf. Homm and Breitung (2012)]. For the case of a mildly explosive process Theorem 1 states that the test statistic has a degenerate limiting null distribution. It is, however, possible to modify the test statistic such that its limiting distribution is standard normal.

Theorem 1 *Let y_t be generated by a mildly explosive process with $y_t = \varrho y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} (0, \sigma^2)$, $\varrho = 1 + c/T^\gamma$, $c > 0$ and $0 < \gamma < 1$. As $T \to \infty$,*

$$\psi_\tau \overset{p}{\to} 0.$$

The modified test statistic

$$\psi_\tau^* = \frac{\sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 \, y_{t-1}}{\sqrt{\frac{1}{[\tau T]-1} \sum_{t=2}^{[\tau T]} (\tilde{e}_t^0)^2} \sqrt{\sum_{t=2}^{[\tau T]} y_{t-1}^2}} \tag{6}$$

possesses a standard normal limiting distribution for all $\tau \in (0, 1)$.

Proof See Appendix.

Three remarks are in order. □

Remark 1 The finding that the LM Chow statistic converges to zero in probability may appear counterintuitive. We provide an illustration for this surprising result. As shown in the proof of this theorem the numerator can be written as

$$\sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 \, y_{t-1} = \sum_{t=[\tau T]+1}^{T} \varepsilon_t \, y_{t-1} - \kappa_T(\tau) \sum_{t=2}^{T} \varepsilon_t \, y_{t-1},$$

where

$$\kappa_T(\tau) = \frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}{\sum_{t=2}^{T} y_{t-1}^2}.$$

As $T \to \infty$ the factor $\kappa_T(\tau)$ converges in probability to one for all $\tau \in (0, 1)$ and, therefore,

$$\sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 \, y_{t-1} \approx - \sum_{t=2}^{[\tau T]} \varepsilon_t \, y_{t-1}.$$

This shows that the sum of squares in the denominator of the test statistic should run from 2 to $[\tau T]$ instead of $[\tau T] + 1$ to T in order to obtain a non-degenerate limiting distribution. The limiting normal distribution is then obtained by using results of Phillips and Magdalinos (2007).

Remark 2 Note that the residual variance is estimated by using the observations of the first regime. This is due to the fact that under the alternative the residual variance of the second regime tends to increase exponentially which would imply a substantial loss of power.

Remark 3 The behavior of the LM Chow statistic in the explosive AR model carries over to the usual Wald (or LM) version of the Chow test. It is well known that the alternative test statistics differ only in the estimator for the error variance. The LM statistic is computationally convenient as the same estimator of the residual variance is used to compute the test statistic for a range of possible break dates. Using the results of the proof it is straightforward to show that under the null hypothesis the usual residual variance estimator computed from the residuals of regression (4) is asymptotically equivalent to the variance estimator employed for the LM statistic.

As argued in Sect. 2, a series that is decomposed into an explosive and a random walk component possesses an ARIMA(1,1,1) representation. It follows that approximating the series by an (explosive) AR(1) process implies that the errors are autocorrelated. To adjust for this serial correlation, HAC standard errors are employed [cf. Newey and West (1987)]. Let

$$\widehat{V}_\ell(\tau) = \widehat{\gamma}_0(\tau) + 2\sum_{j=1}^{\ell} w_j \widehat{\gamma}_j(\tau)$$

where $w_j = (\ell + 1 - j)/(\ell + 1)$ and

$$\widehat{\gamma}_j = \sum_{t=j+1}^{[\tau T]} \tilde{e}_t^0 \tilde{e}_{t-j}^0 y_{t-1} y_{t-j-1}.$$

The HAC version of the test statistic results as

$$\widetilde{\psi}_\tau^* = \frac{\sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 \, y_{t-1}}{\sqrt{\widehat{V}_\ell(\tau)}}. \tag{7}$$

To investigate the small sample properties of the alternative test statistics for the composed series $y_t = B_t + P_t^f$, where $B_t = \varrho B_{t+1} + u_t$ and $P_t^f = P_{t-1}^f + \varepsilon_t$, we obtain the components by generating the error process as $\varepsilon_t \overset{iid}{\sim} \mathcal{N}(0, 1)$ and $u_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_u^2)$. Note that the variance ratio $\lambda^2 = \sigma_u^2/\sigma_\varepsilon^2$ measures the relative importance of the bubble. It is assumed that the bubble crashes at period $T_0 = [\tau_0 T]$. The (mildly explosive) autoregressive parameter is determined as $\varrho = 1 + c/T^{3/4}$. Table 2 presents the rejection frequencies of alternative tests for a structural break in the autoregressive coefficient at period $[\tau_0 T]$. The original LM Chow statistic ψ_{τ_0} is computed as in (5). The modified version $\psi_{\tau_0}^*$ is presented in Eq. (6) of Theorem 1. Finally, the HAC version ψ_τ^* is computed using Schwert's (1989) rule $\ell = [4(T/100)^{1/4}]$. This version

Table 2 Actual size for various bubble processes

c	$\tau_0 = 0.5$			$\tau_0 = 0.75$		
	ψ_{τ_0}	$\psi^*_{\tau_0}$	$\psi^*_{\tau_0}(4)$	ψ_{τ_0}	$\psi^*_{\tau_0}$	$\psi^*_{\tau_0}(4)$
Large bubble: $\lambda = 10$						
0.5	0.0062	0.0308	0.0198	0.0129	0.0167	0.0137
1	0.0016	0.0284	0.0144	0.0036	0.0247	0.0157
2	0.0000	0.0331	0.0156	0.0001	0.0400	0.0197
3	0.0000	0.0410	0.0139	0.0000	0.0535	0.0202
4	0.0000	0.0531	0.0094	0.0000	0.0631	0.0168
5	0.0000	0.0596	0.0076	0.0000	0.0672	0.0092
Moderate bubble: $\lambda = 1$						
0.5	0.0099	0.0227	0.0139	0.0189	0.0115	0.0082
1	0.0018	0.0197	0.0078	0.0065	0.0216	0.0124
2	0.0000	0.0455	0.0134	0.0005	0.0972	0.0591
3	0.0000	0.1210	0.0408	0.0000	0.1978	0.1370
4	0.0000	0.1834	0.0696	0.0000	0.2613	0.1731
5	0.0000	0.2338	0.0775	0.0000	0.2816	0.1616
Small bubble: $\lambda = 0.1$						
0.5	0.0224	0.0168	0.0114	0.0384	0.0063	0.0052
1	0.0146	0.0182	0.0128	0.0352	0.0067	0.0049
2	0.0023	0.0040	0.0019	0.0064	0.0138	0.0020
3	0.0001	0.0139	0.0003	0.0004	0.1691	0.0735
4	0.0000	0.1104	0.0054	0.0000	0.2798	0.1725
5	0.0000	0.2211	0.0272	0.0000	0.3225	0.1882

Entries are the rejection frequencies for 10.000 replications of the process $y_t = B_t + F_t$ with $\lambda = \sigma_u/\sigma_\varepsilon$. The sample size is $T = 100$ and the nominal size is 0.05. ψ_{τ_0} denotes the LM Chow statistic and $\psi^*_{\tau_0}(4)$ employs HAC standard errors using Schwert (1989) rule with factor 4. The autoregressive coefficient is $\varrho = 1 + c/T^{0.75}$

is indicated as $\widetilde{\psi}^*_{\tau_0}(4)$. The nominal size of the test is 0.05 and the sample size is $T = 100$.

The results reported in Table 2 generally confirm the asymptotic results of Theorem 1. For larger values of c, the actual sizes of the original LM Chow statistic tend to zero, whereas the modified statistic $\psi^*_{\tau_0}$ performs reasonably well, at least if the series exhibits a sizable explosive component. On the other hand, if the explosive bubble process is not dominating, the approximation by an AR(1) process gives rise to severe size distortions. In these cases, the HAC version $\widetilde{\psi}^*_{\tau_0}(4)$ performs better but it still suffers from size distortions in particular if c is large and break date is located towards the end of the sample. It should be noted, however, that empirically realistic values of the explosive parameter are around $c = 1$ (yielding $\varrho = 1.032$), where the statistics $\psi^*_{\tau_0}$ and $\widetilde{\psi}^*_{\tau_0}(4)$ perform acceptable (albeit conservative).

Theorem 1 assumes that the break date $[\tau T]$ is known. In practice, the date of the bubble crash is usually unknown. The conventional approach is to consider all

Table 3 Critical value for the minimum Chow statistic

$\overline{\tau}$	$c = 0.5$	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$	indep.
$T = 60$							
0.05	–2.755	–2.823	–2.898	–2.926	–2.964	–2.998	–3.106
0.10	–2.699	–2.770	–2.823	–2.899	–2.971	–2.948	–3.071
0.15	–2.729	–2.745	–2.821	–2.881	–2.935	–2.999	–3.031
$T = 200$							
0.05	–2.934	–2.965	–3.055	–3.126	–3.176	–3.188	–3.254
0.10	–2.877	–2.925	–3.303	–3.108	–3.141	–3.161	–3.220
0.15	–2.851	–2.885	–3.014	–3.045	–3.103	–3.129	–3.182

Critical values of the min(LM-Chow) statistic, where the test is computed for $\tau \in [\overline{\tau}, 1 - \overline{\tau}]$. The column labeled as "indep." indicates the critical values obtained by assuming that all statistics are independent standard normal random variables

possible break dates within some pre-specified interval and compare the maximal (here: minimal as we consider a one-sided test version) statistic to an appropriate critical value. An important problem with this approach is that in our case the limiting distribution of the minimum statistic depends on the parameter c in the autoregressive coefficient $\varrho = 1 + c/T^{\gamma}$. For small values of c the sequence of test statistics is highly autocorrelated. If c increases, the correlation of the test statistics tends to zero as the weight on the last observation increases exponentially with c. Since this parameter cannot be estimated consistently (which is typical), the suitable critical value is unknown. For illustration, Table 3 presents the 0.05 critical values for $T \in \{60, 100\}$, $\gamma = 3/4$, a search in the interval $[\overline{\tau}, 1 - \overline{\tau}]$, and various values of c. As expected, the critical values depend on the parameter c. As a benchmark, the last column of the table presents the 5 % critical value of the minimum of $(1 - 2\overline{\tau})T$ independent $\mathcal{N}(0, 1)$ distributed random variables. It turns out that for large values of c the sequence of test statistics behave like a sequence of independent and identically distributed standard normal random variables. Therefore, we suggest to use the α-quantile of the minimum of $n_{\overline{\tau}} = [(\overline{\tau}_u - \overline{\tau}_\ell)T]$ independent standard normally distributed random variables as an approximation for the critical value, where the search interval is $\tau \in [\overline{\tau}_\ell, \overline{\tau}_u]$. For a significance level α, the critical value can be computed as

$$c_\alpha = \Phi_z^{-1}\left(1 - (1 - \alpha)^{1/n_{\overline{\tau}}}\right), \tag{8}$$

where $\Phi_z^{-1}(a)$ denotes the inverse of the c.d.f. of the standard normal distribution. The resulting test is somewhat conservative for small values of c.

4 Monitoring based on the CUSUM statistic

Provided that the bubble collapses at the end of the sample, the forecast version of the Chow test may be preferable. Assume that we want to assess the evidence against the bubble process within a monitoring exercise [cf. Chu et al. (1996) and Homm and Breitung (2012)]. To this end, we split the sample into a training (sub)sample $t = 1, \dots, [\tau T]$ and a monitoring (sub)sample with $t = [\tau T] + 1, \dots, T$. The training sample is merely employed for estimation but not for the monitoring exercise. Recursive forecast errors are computed for the monitoring sample. If the explosive process switches to a random walk, the forecast errors tend to be negative. Accordingly the crash of a bubble is indicated by high negative cumulated forecast errors.

Let $[\tau T]$ be the number of observations in the training sample and denote by $n = [(1 - \tau)T]$ the number of monitoring periods. The CUSUM statistic is computed as

$$\phi_\tau(k) = \frac{1}{\sqrt{n\,\widehat{\upsilon}_e^2}} \sum_{t=[\tau T]+1}^{[\tau T]+k} \tilde{e}_t^\tau, \quad \text{for } k = 1, 2, \dots, n,$$

where $\tilde{e}_t^\tau = y_t - \widehat{\varrho}_{t-1} y_{t-1}$ denotes the forecast residual[4] and $\widehat{\varrho}_{t-1}$ denotes the least-squares estimator of the autoregressive coefficient based on the observations $y_1, y_2, \dots, y_{t-1}$. The residual variance is estimated from the training sample as

$$\widehat{\upsilon}_e^2 = \frac{1}{[\tau T]} \sum_{t=2}^{[\tau T]} (y_t - \widehat{\varrho}_{[\tau T]} y_{t-1})^2.$$

If the AR(1) process is correctly specified (e.g. if y_t is identical to the bubble process), then $\tilde{e}_t^\tau$ forms a martingale difference sequence with respect to the increasing sigma-field generated by $\{y_t, y_{t-1}, \dots, y_1\}$ and, therefore, as $n \to \infty$

$$\phi_\tau(k) \Rightarrow W(r), \text{ where } r = \frac{k}{n}.$$

Following Chu et al. (1996) and Homm and Breitung (2012) the monitoring procedure is based on the decision rule

$$\text{reject } H_0 \text{ if } \phi_\tau(k) < -\zeta_\tau \sqrt{k} \text{ for } k = 1, 2, \dots, n \tag{9}$$

where $\zeta_\tau = \sqrt{b_\alpha + \log(1 + k/[\tau T])}$. As in Homm and Breitung (2012) we use $b_{0.05} = 4.6$ for a significance level of 0.05.

An important advantage of this approach compared to testing approach based on the Chow statistic is that the decision rule (9) does not depend on the unknown parameter c in the sequence of mildly explosive autoregressive coefficients. A serious drawback of this approach is, however, that the ARIMA(1,1,1) representation for the composed price process $y_t = B_t + P_t^f$ is approximated by a AR(1) process and, thus, its forecast residuals are correlated. This may lead to severe size distortions. To cope with this

[4] As pointed out by Ploberger and Krämer (1992), the forecast residuals can be replaced by the ordinary (in-sample) residuals. In the explosive model, however, the distribution of the cumulated residuals depends on the unknown parameter c. We therefore do not consider the OLS CUSUM approach in what follows.

Table 4 Actual sizes of the monitoring approach

	$\lambda = 10$		$\lambda = 1$		$\lambda = 0.1$	
c	$\mathcal{M}(1)$	$\mathcal{M}(6)$	$\mathcal{M}(1)$	$\mathcal{M}(6)$	$\mathcal{M}(1)$	$\mathcal{M}(6)$
$T = 60$						
0.5	0.0360	0.0452	0.0498	0.0479	0.0405	0.0495
1	0.0430	0.0531	0.0496	0.0603	0.0521	0.0564
2	0.0481	0.0620	0.1030	0.1004	0.1701	0.1682
3	0.0535	0.0695	0.1912	0.1233	0.2361	0.1712
4	0.0617	0.0839	0.2635	0.1308	0.2960	0.1420
5	0.0836	0.0760	0.3018	0.1264	0.3437	0.1189
c	$\lambda = 10$		$\lambda = 1$		$\lambda = 0.1$	
	$\mathcal{M}(1)$	$\mathcal{M}(8)$	$\mathcal{M}(1)$	$\mathcal{M}(8)$	$\mathcal{M}(1)$	$\mathcal{M}(8)$
$T = 100$						
0.5	0.0398	0.0503	0.0449	0.0560	0.0375	0.0460
1	0.0416	0.0511	0.0516	0.0554	0.0616	0.0605
2	0.0488	0.0539	0.1220	0.1004	0.1806	0.1694
3	0.0502	0.0619	0.2246	0.1233	0.2686	0.1440
4	0.0607	0.0731	0.3014	0.1175	0.3263	0.1125
5	0.0736	0.0760	0.3471	0.1093	0.3599	0.0947

Rejection rates for the monotoring approach with a nominal significance level of 0.05 based on 10.000 replications

problem we approximate the ARIMA(1,1,1) process by an AR(p) process, where the lag order increases with the sample size. Specifically we choose the lag order according to the rule $p = [\sqrt{[\tau T]}]$.[5] The monitoring statistic based on AR(p) forecast residuals is denoted by $\mathcal{M}(p)$. Table 4 reports the actual sizes of the monitoring approach referring to a nominal size of 0.05. The results suggest that the monitoring approach performs well if the bubble component is large ($\lambda = 10$). For moderate ($\lambda = 1$) or small ($\lambda = 0.1$) bubble components the approximation of the ARIMA(1,1,1) process by an AR(p) process reduces the size distortions substantially, but still the test tends to reject too often. Fortunately, the monitoring procedure works well in the empirically relevant range of $c \in [0.5, 1.5]$ so that in empirical applications the monitoring procedure is likely to yield reliable results.

5 Dating the crash

As argued in Homm and Breitung (2012) the break date can be estimated consistently by maximising the likelihood function. Conditional on the first observation,

[5] We also tried out various other rules such as the ones suggested by Schwert (1989). Overall the square-root rule performs best in our simulations.

maximising the log-likelihood (assuming normally distributed innovations) is equivalent to minimising the sum of squared residuals. Let

$$\hat{e}_t^\tau = y_t - \left(\frac{\sum_{t=2}^{[\tau T]} y_t y_{t-1}}{\sum_{t=2}^{[\tau T]} y_{t-1}^2} \right) y_{t-1} \quad \text{for } t = 2, 3 \ldots, [\tau T]$$
$$\hat{e}_t^\tau = y_t - \left(\frac{\sum_{t=[\tau T]+1}^{T} y_t y_{t-1}}{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2} \right) y_{t-1} \quad \text{for } t = [\tau T] + 1, \ldots, T.$$

The ML estimator for the (relative) break date results as

$$\widehat{\tau} = \arg\min_{\tau} \left\{ \sum_{t=2}^{T} (\hat{e}_t^\tau)^2 \right\} \tag{10}$$

[eg. Bai (1994)]. In stationary regressions, minimising the residual sum of squares is equivalent to maximising the F-statistic for a structural break. In Remark 2 we have argued, however, that the usual Chow test for a structural break is invalid if the process is (mildly) explosive. For the modified test suggested in Theorem 1 there is no monotonic relationship between the statistic ψ_τ^* and the sum of squared residuals. Therefore, the ML estimator for the structural break date cannot be obtained from minimising ψ_τ^* with respect to τ. Accordingly, the estimator for the break date must be computed as in (10).

6 Empirical applications

Before considering two empirical applications based on stock price indices we first report the results of a "realistic" Monte Carlo experiment, where the parameters and sample sizes are chosen to match the empirical examples. Specifically, we set the autoregressive parameter to $\varrho = 1.02$ (implying an annual return of the bubble of 24 percent) and the sample sizes are $T = 69$ (to match the Hang Seng data) and $T = 121$ (which corresponds to the NASDAQ data). Furthermore, a constant term is included in the autoregressive representations. The truncation lag of the HAC kernel is set according to Schwert (1989) rule with the factor 4. The lag length of the autoregressive representation in the monitoring procedure is $p = [\sqrt{\tau T}]$. The date when the bubble collapses is assumed to be unknown and, therefore, the test procedure searches in the interval $\tau \in [0.4, 0.8]$. We follow Blanchard and Watson (1982) and impose that the bubble continues with probability π in each period. Only bubbles that run at least for $[0.4T]$ periods are included in the experiment. The critical value is computed as in (8). For the monitoring procedure, test decisions are based on (9).

From Table 5 it be seen that the actual sizes of all tests are reasonably close to the nominal size of 0.05. As expected, the power of the tests depends on the size of the bubble (measured by the variance ratio λ^2). Furthermore, if π is large, the probability of a continuing bubble is large and, therefore, the tests will identify only a small number of crashes. Hence, the rejection rates are expected to increase with a

Table 5 Size and power in a realistic scenario

π		Large bubble: $\lambda = 10$		Moderate bubble: $\lambda = 1$		Small bubble $\lambda = 0.1$	
		min $\psi^*(3)$	$\mathcal{M}(5)$	min $\psi^*(3)$	$\mathcal{M}(5)$	min $\psi^*(3)$	$\mathcal{M}(5)$
$T = 69$	1 (size)	0.0490	0.0625	0.0670	0.0780	0.0990	0.0735
0.98		0.3930	0.2420	0.3145	0.2175	0.1165	0.0925
0.96		0.6550	0.4420	0.5430	0.3855	0.1675	0.1175
0.94		0.7625	0.5640	0.6810	0.5070	0.2495	0.1775
0.92		0.8240	0.6630	0.7410	0.6085	0.3445	0.2815
0.90		0.8490	0.7285	0.8025	0.6895	0.4645	0.3575
π		Large bubble: $\lambda = 10$		Moderate bubble: $\lambda = 1$		Small bubble $\lambda = 0.1$	
		min $\psi^*(4)$	$\mathcal{M}(6)$	min $\psi^*(4)$	$\mathcal{M}(6)$	min $\psi^*(4)$	$\mathcal{M}(6)$
$T = 121$	1 (size)	0.0100	0.0630	0.0100	0.0850	0.0860	0.0620
0.98		0.6280	0.4600	0.5430	0.4290	0.1560	0.1470
0.96		0.8150	0.6920	0.7570	0.6420	0.3350	0.3080
0.94		0.8730	0.7900	0.8410	0.7680	0.5140	0.4790
0.92		0.8950	0.8310	0.8760	0.7890	0.7050	0.6270
0.90		0.8960	0.8500	0.8880	0.8530	0.7970	0.7270

Entries report rejection frequencies for the minimum LM Chow statistics (with HAC standard errors) and the monotoring procedure. The parameter π is the probability that the bubble continues in each period. The nominal size is 0.05 and the tests search for a break in $\tau \in [0.4, 0.8]$. The explosive parameter for the bubble component is $\rho = 1.02$

decreasing probability π. Our results presented in Table 5 indicate that the tests are able to indicate a crash of the bubble reasonably well. The test approach based on the minimum LM Chow statistic tends to be more powerful than the monitoring approach although the differences are not dramatic. Overall, the performance of our test appears to be encouraging in realistic empirical applications that we will consider next.

The two applications we are studying are based on the real NASDAQ 100 index and the real Hang Seng index. The data is analysed at a monthly frequency and the sample runs from June 1993 to June 2003 (121 observations) for the NASDAQ index and from November 2004 to July 2010 (69 observations) for the Hang Seng index, respectively. For both series (see the Figs. 1a and 2a), the relative breakpoint varies between $\tau_l = 0.4$ and $\tau_u = 0.8$ which corresponds to the following intervals of potential breakpoints: May 1997–June 2001 (NASDAQ) and January 2007–June 2009 (Hang Seng). It can be seen from the figures that these intervals include periods were the bubbles are likely to collapse.

In what follows, we include a constant in the test regressions and account for serial correlation of the errors (i.e. HAC standard errors are used for the minimum Chow statistic and an autoregressive approximation is employed for the monitoring approach). First, we summarise and discuss the results for the NASDAQ series (see Fig. 1a). Figure 1b presents the sequence of $\widetilde{\psi}_\tau^*$ statistics for $[0.4T] + 1, \ldots, [0.8T]$. The critical

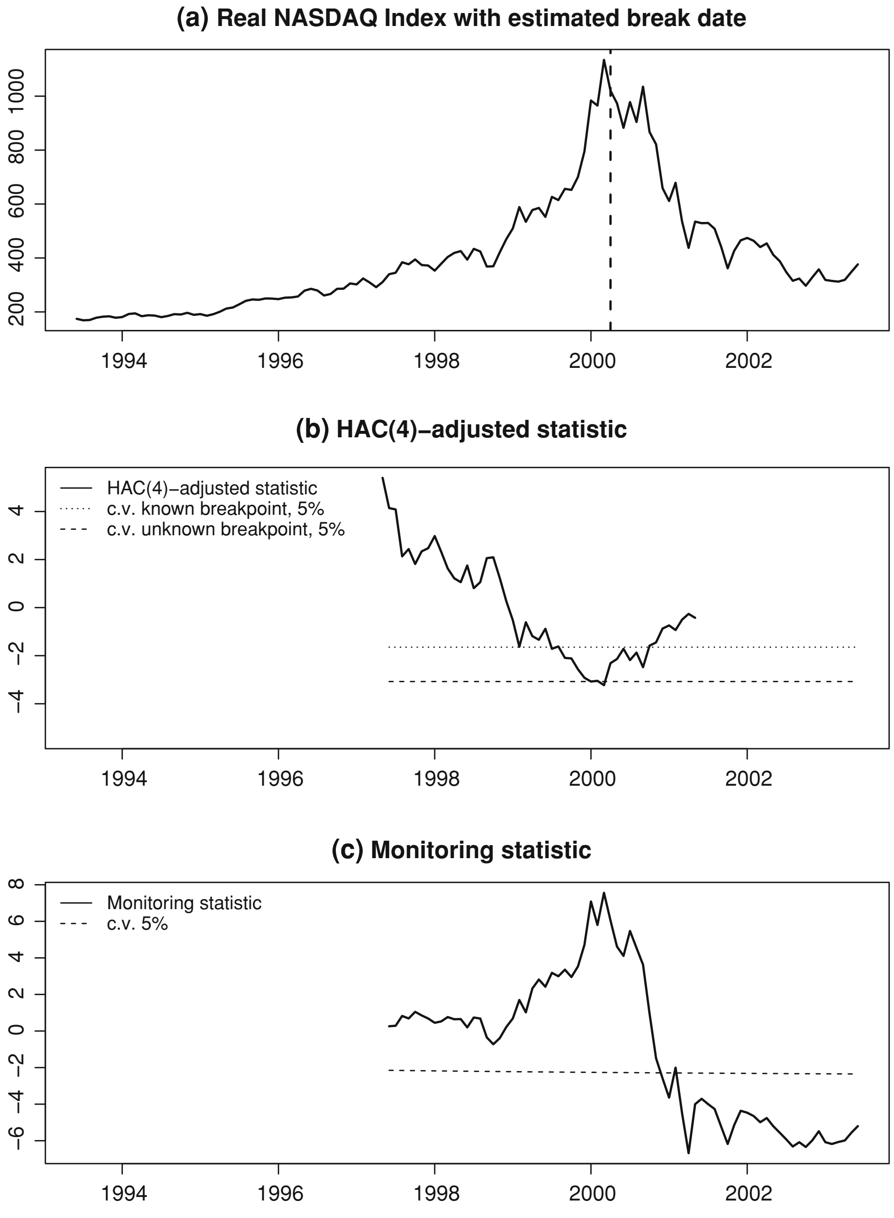

Fig. 1 Time series plot of real NASDAQ index (**a**), HAC(4)-adjusted statistic (**b**), Monitoring statistic (**c**)

value for the minimum Chow statistic corresponding to the 0.05 significance level equals –3.077 and is shown as a dotted line. The dashed-dotted line shows the critical value –1.65 for the Chow test with known break date (see Theorem 1). It turns out that the null hypothesis of no structural break in a mildly explosive autoregressive process has to be rejected even if the break date is assumed to be unknown. The evidence

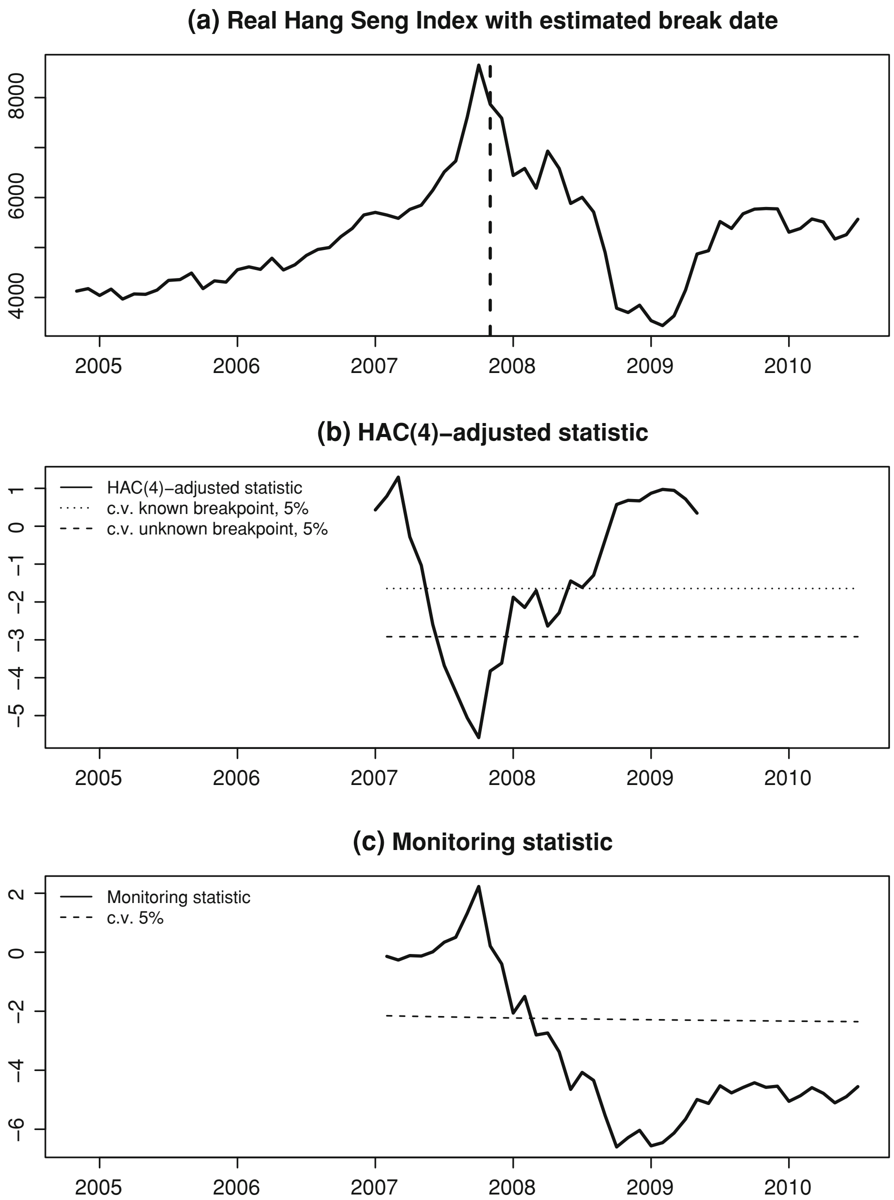

Fig. 2 Time series plot of real Hang Seng index (**a**), HAC(4)-adjusted statistic (**b**), monitoring statistic (**c**)

for a structural change from an explosive process to a unit root is, however, not overwhelming: the minimal statistic equals –3.228. The ML break estimate for the break date yields March 2000, implying that the process became non-explosive from April 2000 onwards. When applying the date-stamping method proposed by Phillips et al. (2011c), we find a bubble collapse in October 2000 which makes a difference of seven months.[6] During these seven months, the real NASDAQ declined by 21.64 %. Figure 1c depicts the sequence of monitoring statistics $\phi_\tau(k)$ for $k = [\tau T] + 1, \ldots, T$. This test statistic becomes significant after December 2000. Note that the evidence for a crash of the bubble is much stronger compared to the Chow test procedure. There is however a substantial delay in identifying the crash which is comparable to the Phillips et al. (2011c) approach.

Second, we briefly discuss the results for the real Hang Seng index. Figure 2 provides similar information on the time series and test statistics. The ML estimate of the break date is indicated by a vertical line. Overall we find similar, but more clear-cut results. The minimum Chow statistic is –5.581, well below the critical value of –2.917. Similar to the results for the NASDAQ, we find the break date (October 2007) is located right after the peak in 2007. The sequence of monitoring statistics is presented in Fig. 2(c). The statistic becomes significant in March 2008 which corresponds well with the date where the recursive ADF statistic of Phillips et al. (2011c) fall below the critical value.

7 Conclusions

Explosive price paths are a typical feature of speculative bubbles. In a recent contribution, Phillips et al. (2011c) suggest a recursive right-tailed Dickey–Fuller test against the explosive alternative. Their testing framework additionally allows date-stamping of the start and collapse of a bubble. Unfortunately, their procedure exhibits an inherent delay which can be sizable in small and moderate samples. In this study we propose tests for collapsing bubbles which attempt to avoid the delay in detecting the burst of the bubble. Under the null hypothesis, prices are mildly explosive while prices exhibit a structural change from explosiveness to a random walk under the alternative. Our theoretical results show that a modified Chow-type statistic exhibits a standard normal limiting distribution under the null which eases its application. Even though this result is derived for a fixed breakpoint, it turns out that in case of an unknown breakpoint and for a large autoregressive coefficient, the distribution of the sequence of test statistics behaves like a sequence of independent and identically distributed standard normal random variables. Thus, relevant critical values can easily be found in the case of unknown breakpoints as well. Another contribution is the proposal of a monitoring procedure based on a CUSUM statistic. Simulation results confirm that the monitoring procedure works well for parameter settings which are relevant in empirical situations. Our applications to two stock price indices show that the newly suggested techniques allow a timely identification of bursting bubbles in practice.

Acknowledgements We would like to thank an anonymous referee and a Guest Editor for their helpful comments. Robinson Kruse gratefully acknowledges financial support from CREATES funded by the Danish National Research Foundation.

[6] Please note that this estimate differs from the one reported in Phillips et al. (2011c) as we consider a sub-sample running from June 1993 to June 2003.

Appendix: Proof of Theorem 1

(i) Under the null hypothesis we have

$$\begin{aligned}\sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 y_{t-1} &= \sum_{t=[\tau T]+1}^{T} \varepsilon_t y_{t-1} + (\varrho - \widehat{\varrho}^0) \sum_{t=[\tau T]+1}^{T} y_{t-1}^2 \\ &= \sum_{t=[\tau T]+1}^{T} \varepsilon_t y_{t-1} - \left(\frac{\sum_{t=2}^{T} \varepsilon_t y_{t-1}}{\sum_{t=2}^{T} y_{t-1}^2} \right) \sum_{t=[\tau T]+1}^{T} y_{t-1}^2 \\ &= \sum_{t=[\tau T]+1}^{T} \varepsilon_t y_{t-1} - \left(\frac{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}{\sum_{t=2}^{T} y_{t-1}^2} \right) \sum_{t=2}^{T} y_{t-1} \varepsilon_t \\ &= \sum_{t=[\tau T]+1}^{T} \varepsilon_t y_{t-1} - \kappa_T \sum_{t=2}^{T} y_{t-1} \varepsilon_t,\end{aligned}$$

where

$$\kappa_T = \frac{\sum_{[\tau T]+1}^{T} y_{t-1}^2}{\sum_{t=2}^{T} y_{t-1}^2}.$$

We now show that $\kappa_T = 1 - o_p(1)$. From Phillips and Magdalinos (2007) we have as $T \to \infty$

$$\sum_{t=2}^{T} y_{t-1}^2 = O_p \left(\varrho^{2T} T^{2\gamma} \right)$$

and, thus,

$$\kappa_T = 1 - \frac{\sum_{t=2}^{[\tau T]} y_{t-1}^2}{\sum_{t=2}^{T} y_{t-1}^2} = 1 - \frac{O_p(\varrho^{2\tau T} (\tau T)^{2\gamma})}{O_p(\varrho^{2T} T^{2\gamma})} = 1 - O_p \left[\tau^{2\gamma} \left(\varrho^{-2(1-\tau)T} \right) \right].$$

Using

$$\begin{aligned}\lim_{T\to\infty} \varrho^{-2(1-\tau)T} &= \lim_{T\to\infty} \left[\left(1 + \frac{c}{T^\gamma} \right)^{T^\gamma / c} \right]^{-2(1-\tau)cT/T^\gamma} \\ &= \lim_{T\to\infty} e^{-2c(1-\tau)T^{1-\gamma}} \to 0\end{aligned}$$

for any $\gamma < 1$, it follows that $\kappa_T \xrightarrow{p} 1$.

In a similar manner it can be shown that

$$\frac{\sum_{[\tau T]+1}^{T} \varepsilon_t y_{t-1}}{\sum_{t=2}^{T} \varepsilon_t y_{t-1}} = 1 + o_p(1).$$

Therefore, $\sum_{t=[\tau T]+1}^{T} \varepsilon_t y_{t-1}$ is $O_p\left(\varrho^T T^\gamma\right)$ (cf. Phillips and Magdalinos (2007), Theorem 4.3a). It follows that

$$\frac{\varrho^{-\tau T}}{(\tau T)^\gamma} \sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 y_{t-1} = -\frac{\varrho^{-\tau T}}{(\tau T)^\gamma} \sum_{t=2}^{[\tau T]} \varepsilon_t y_{t-1} + o_p(1).$$

Using $(\tau T)^{-1} \sum_{t=2}^{[\tau T]} (\tilde{e}_t^0)^2 \xrightarrow{p} \sigma_u^2$ we obtain

$$\begin{aligned}
\psi_\tau &= \frac{\sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 \, y_{t-1}}{\sigma_u \sqrt{\sum_{t=[\tau T]+1}^{T} y_{t-1}^2}} + o_p(1) \\
&= \tau^\gamma \left(\varrho^{-(1-\tau)T}\right) \frac{\frac{\varrho^{-\tau T}}{(\tau T)^\gamma} \sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 \, y_{t-1}}{\sigma_u \sqrt{\frac{\varrho^{-2T}}{T^{2\gamma}} \sum_{t=[\tau T]+1}^{T} y_{t-1}^2}} + o_p(1) \\
&= o_p(1),
\end{aligned}$$

and, therefore, ψ_τ converges to zero in probability as $T \to \infty$ and $c > 0$.

(ii) From Theorem 4.3 of Phillips and Magdalinos (2007) it follows that[7]

$$\frac{\varrho^{-\tau T}}{(\tau T)^\gamma} \sum_{t=2}^{[\tau T]} \varepsilon_t y_{t-1} \Rightarrow XY$$

$$\frac{\varrho^{-2\tau T}}{(\tau T)^{2\gamma}} \sum_{t=2}^{[\tau T]} y_{t-1}^2 \Rightarrow \frac{1}{2c} Y^2$$

for all $0 < \tau < 1$, where X and Y are independent $\mathcal{N}(0, \sigma_\varepsilon^2/2c)$ random variables. Thus,

$$\begin{aligned}
\psi_\tau^* &= \frac{\frac{\varrho^{-\tau T}}{(\tau T)^\gamma} \sum_{t=[\tau T]+1}^{T} \tilde{e}_t^0 \, y_{t-1}}{\sigma_\varepsilon \sqrt{\frac{\varrho^{-2\tau T}}{(\tau T)^{2\gamma}} \sum_{t=[\tau T]+1}^{T} y_{t-1}^2}} + o_p(1) \\
&\Rightarrow \frac{XY}{\frac{\sigma_\varepsilon}{\sqrt{2c}} |Y|} = \frac{\sqrt{2c}}{\sigma_\varepsilon} \operatorname{sign}(Y) X \sim \mathcal{N}(0, 1). \qquad \diamond
\end{aligned}$$

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[7] Note that there is an obvious typo in Theorem 4.3(a) of Phillips and Magdalinos (2007), cf. their Eq. (9).

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