# Constructing indices of multivariate polarization *

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January 30, 2008

*We are indebted to Satya R. Chakravarty, Joan Esteban, Christoph Scheicher, and Esfandiar Maasoumi for critical and helpful remarks on previous versions of this paper. We are also grateful to two anonymous referees who provided insightful suggestions. The usual caveat applies.

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**Abstract**

Multivariate indices of polarization are constructed to measure effects of non-income attributes like wealth and education. Polarization is considered as the presence of groups which are internally homogeneous, externally heterogeneous, and of similar size. We propose a class of polarization indices which is built from measures of relative groups size and from decomposable indices of socio-economic inequality. For the latter, we employ the special inequality indices of Maasoumi (1986), Tsui (1995, 1999) and Koshevoy and Mosler (1997). Then, postulates for multidimensional polarization measurement are stated and discussed. The approach is illustrated by an empirical application to the population of the East and West Germany with polarization defined on income and education.

Keywords: Decomposable inequality indices, multidimensional inequality, multivariate social evaluation, polarization index.

JEL= C43, D63.

# 1 Introduction

Polarization is commonly connected with the division of a society into groups as possible cause of social conflicts. It is measured by quantifying and comparing socio-economic disparity, not only in terms of differences among individuals (as inequality measurement does) but also in terms of differences among population groups.

The first systematic investigations into indices and postulates of polarization measurement are due to Wolfson (1994, 1997) and Esteban and Ray (1994). These pioneering papers have been followed by many others, among them Chakravarty and Majumder (2001), Esteban et al. (2007), Wang and Tsui (2000), D´Ambrosio (2001), Gradín (2000), Duclos et al. (2004). All these papers study polarization in terms of the distribution of incomes and measure how much this distribution spreads out from its center, dividing the population into at least two groups that are homogeneous and well separated from each other.

In case of two groups, the phenomenon can be also seen as a decline of the central part of the distribution. Correspondingly, two strands are distinguished in the literature on univariate polarization: the first one, going back to Wolfson (1994), describes the decline of the middle class, measuring how the center of the income distribution is emptied. The second strand, originating from Esteban and Ray (1994), focuses on the rise of separated income groups; polarization is the larger the more homogeneous the groups are, the more separate, and the more equal in size.

But, societal status of a person and distance between persons (and groups) is not determined by income alone but also by other monetary and non-monetary characteristics of well-being, such as wealth, education, and health. In the measurement of economic inequality and poverty, several authors have pointed out that attributes beyond income should be included in the analysis. Consequently, they have introduced various multi-attribute measures of inequality and poverty; see Atkinson and Bourguignon (1982), Kolm (1977), Maasoumi (1986), Bourguignon and Chakravarty (2003), Tsui (1995, 1999).

Obviously, also the splitting of a society into groups is influenced by attributes besides income. The usual partition of the society into the poor, the middle class and the rich, which is based only on income, may be refined with other information on individuals or households, such as the level of education, wealth or health.

Davis and Huston (1992) have investigated the causes of lower and upper class membership by regression on many socio-economic attributes. But, to the best of our knowledge, there exists no attempt in the literature to measure polarization in many attributes. This paper presents a first inquiry into the multi-attribute measurement of polarization. Our approach follows the second strand of literature: multi-attribute polarization corresponds to splitting the society into groups that are well separated, inside homogeneous, and of comparable size.

We construct multivariate indices of polarization, using the decomposition by subgroups of certain indices of multivariate inequality. These indices can be decomposed into a 'within groups' component and a 'between groups' component of inequality. Based on them we introduce multivariate polarization indices that increase with respect to between groups inequality and decrease with respect to within groups inequality. Besides, the relative size of the groups matters. Therefore, we employ simple measures of relative groups size that indicate the deviation from equally sized groups and construct polarization indices which, additionally, decrease in these measures. Thus, our approach results in indices which are function of three elements: the measures of inequality between groups, of inequality within groups and of relative groups size.

Further, classical postulates on the measurement of univariate polarization are considered and extended to the multivariate setting. We then investigate how these postulates are satisfied by our polarization indices.

Two particular problems are intrinsic to the multivariate setting: First, while with income alone people naturally divide into two groups above and below the center, with more than one attribute an infinity of directions arise that point away from the center. Second, in the evaluation of a multiattribute distribution, possible interactions between the attributes have to be modelled. E.g., two attributes may be taken either as substitutes or as complements.

The paper is organized as follows: in Section 2 the general principle of construction is introduced, including special measures of groups size. Sections 3 and 4 study special indices of multiattribute inequality, their decompositions, and the polarization indices built on them. In Section 5 we study postulates on the measurement of univariate and multiattribute polarization and investigate how they are satisfied by our special indices. Section 6 is devoted to a discussion of value interaction among attributes. In Section 7 an empirical illustration is given. Section 8 concludes.

# 2 Index construction

Consider a population of $N$ individuals and their endowments in $K$ attributes. The distribution is notated by a matrix $\mathbf{X}$,

$$\mathbf{X} = \begin{bmatrix} x_{11} & \dots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{N1} & \dots & x_{NK} \end{bmatrix}_{N \times K},$$

where $x_{ik}$ denotes the endowment of individual $i$ with attribute $k$. $\mathcal{M}^{N \times K}$ is the set of all $N \times K$ matrices, and $\mathbb{R}_+^K$ is the non-negative orthant of the Euclidean $K$-space $\mathbb{R}^K$. The row $\mathbf{x}_i = (x_{i1}, ..., x_{iK})$ represents the endowment of the $i$-th individual, $i = 1, ..., N$, while the column $\mathbf{x}^k = (x_{1k}, ..., x_{Nk})^T$ describes the distribution of the $k$-th variable, $k = 1, ..., K$. With $\bar{x}_k$ we indicate the mean value of the $k$-th variable and with $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_K)$ the total mean vector.

## 2.1 Measuring polarization through inequality decomposition

As mentioned in the introduction, our concept of polarization is based on the idea that the population divides into groups which are, according to the given attributes, homogeneous inside and different to each other. Therefore, the more evident are these two phenomena, the more polarized is the society. Moreover, the more equal are the sizes of the different groups, the more increases the polarization level.

So, we propose polarization indices which are basically functions of three elements: inequality between groups, inequality within groups, and relative groups size. Given the groups, such an index has the form

$$P(\mathbf{X}) = \zeta(B(\mathbf{X}), W(\mathbf{X}), S(\mathbf{X})), \quad \mathbf{X} \in \mathcal{M}^{N\times K}, \tag{1}$$

where $B$ and $W$ are indices that measure inequality between and within groups, respectively, $S$ is an index of relative groups size, and $\zeta$ is a function $\mathbb{R}^3 \to \mathbb{R}$ that increases on $B$ and $S$, and decreases on $W$.

Concerning $B$, $W$ and $S$ we make the following *normalizing assumptions*: The three measures have infimum zero. $S$ takes its maximum $S = 1$ if there are two or more groups of equal size. $W = 0$ if all groups are internally homogeneous, that is, all individuals in a group have the same endowment. $B$ is be minimal, equal to 0, in absence of intergroup inequality.

## 2.2 Measures of groups size

As already noticed, an important component of our polarization measures is given by the relative size of groups. Also Esteban and Ray (1994) and D´Ambrosio (2001) underline that a polarization index has to register the differences in the frequencies among groups, so that the more similar are the clusters sizes, the more polarized is the population. We suppose that the size of a group is measured by its population share. We need a function which measures how equally populated are the groups, taking maximum value when the groups sizes are identical, and minimum value in case of a very unequal population distribution.

Let us assume that the population is split into $G$ groups and, without loss of generality, let us order them from above by their population size, so that $N_1 \geq N_2 \geq \ldots \geq N_G$ and $N = \sum_{g=1}^{G} N_g$.

In the case of two groups, a simple measure of relative groups size is

$$S_1(\mathbf{X}) = 1 - \left| \frac{N_1}{N} - \frac{N_2}{N} \right|. \tag{2}$$

The index $S_1$ has maximum equal to one if the two groups have the same size. It has minimum $\frac{2}{N}$ if one group is a singleton.

More generally, for $G \geq 2$, we propose *relative groups size measures* that are inverse concentration measures, more specifically, that are equivalent to indices of concentration. A *numbers equivalent* is an inverse measure of concentration, usually employed to measure the size of firms, in order to monitor the degree of concentration in a given industry. E.g. in case of three groups of unequal size, a numbers equivalent equal to 2 says that the given population partition has the same concentration as two groups of equal size. However, note that the numbers equivalent is not restricted to integer values. For details see Chakravarty and Weymark (1988).

Here we use numbers equivalents to measure the concentration of population among different groups. With $G$ groups, a numbers equivalent has values in the interval $]1, G]$. It reaches maximum value $G$ if the groups have equal size. The minimum value is close to 1; it is attained if one group contains $N - G + 1$ individuals and each of the remaining groups contain one individual.

Given an index of concentration $C$, we introduce the normalized numbers equivalent $S$ as a *measure of relative groups size*,

$$S(\mathbf{X}) = \frac{\phi^{-1}(C(\mathbf{X})) - 1}{G - 1}, \tag{3}$$

where $\phi^{-1}(C)$ is the numbers equivalent of $C$.

To obtain special measures $S$, we insert three common concentration indices into (3), the indices by Herfindahl ($C^H$) and Rosenbluth ($C^R$) and the Negative Entropy index ($C^{NE}$), as shown in Table 1. [1]

[Insert Table 1 here]

## 2.3 Special index types

Particular forms of the general polarization index (1) are

$$P_1(\mathbf{X}) = \phi\left(\frac{B(\mathbf{X})}{W(\mathbf{X}) + c}\right) \cdot S(\mathbf{X}), \tag{4}$$

$$P_2(\mathbf{X}) = \psi\left(B(\mathbf{X}) - W(\mathbf{X})\right) \cdot S(\mathbf{X}), \tag{5}$$

$$P_3(\mathbf{X}) = \tau\left(\frac{B(\mathbf{X})}{B(\mathbf{X}) + W(\mathbf{X}) + c}\right) \cdot S(\mathbf{X}). \tag{6}$$

These types of measures will be used later, in Section 3, with additively decomposable inequality indices. The constant $c$ must be positive and may depend on the choice of indices $B$ and $W$. The functions $\phi$, $\psi$ and $\tau$ are assumed to be continuous and strictly increasing, with $\phi(0) = \tau(0) = 0$. Consequently, $P_3$ increases strictly with $B$. $S$ shall be normalized with minimum 0 and maximum 1. Depending on the specific choice of $B$ and $W$, the functions $\phi$, $\psi$ and $\tau$ will be chosen in a way that the indices $P_i$ have infimum 0 and supremum 1.

Then, in case of groups having null intergroup inequality, indices $P_1$ and $P_3$ are minimum and equal to 0, regardless of the value of $W$. Measure $P_2$, instead, is minimum when intergroup inequality vanishes and inequality within is maximum. E.g., choosing $\psi(-\sup W) = 0$ makes $P_2$ normalized.

[1] The numbers equivalent concentration measures of $C^H$ and $C^{NE}$ belong to the general class of Hannay-Kay numbers equivalent concentration indices, defined as $HK_\alpha(\mathbf{X}) := \left(\sum_{g=1}^{G}\left(\frac{N_g}{N}\right)^\alpha\right)^{1/1-\alpha}$, for $\alpha \neq 1$, $HK_\alpha(\mathbf{X}) := \prod_{g=1}^{G}\left(\frac{N_g}{N}\right)^{-N_g/N}$, for $\alpha = 1$. $HK_1$ corresponds to the Negative Entropy, and $HK_2$ to the Herfindahl index; see Chakravarty and Weymark (1988).

The maximum value for $P_i, i = 1, 2, 3$, is attained when $B$ and $S$ are maximum and $W$ is minimum. However, the maxima of $B$ and $S$ are attained at different distributions. Here we firstly fix the groups, and consequently $S$, and then we maximize $B$. By assumption, the infimum of $W$ is 0 and the supremum of $S$ is 1. In this case,

$$P_1(\mathbf{X}) = \phi\left(\frac{B(\mathbf{X})}{c}\right), \quad P_2(\mathbf{X}) = \psi(B(\mathbf{X})), \quad P_3(\mathbf{X}) = \tau\left(\frac{B(\mathbf{X})}{B(\mathbf{X}) + c}\right).$$

Hence, given two or more groups of equal size and internal homogeneity, each of the three polarization indices is maximized if and only if the inequality between groups is maximized. E.g., in the case of a univariate income distribution, $B$ is maximum at the extremely right-skewed distribution that has one individual at the highest, and all others at the lowest income. Therefore, given the total income of the population, $B$ can be maximized among groups of equal size by increasing the distances among groups.

The multivariate indices $P_1, P_2$ and $P_3$ resemble well known univariate measures of income polarization. For example, the measure $P^{ZK}$ proposed in Zhang and Kanbur (2001) is given by the ratio of income inequality between and inequality within groups, like the measure $P_1$ in (4). Also Wolfson's measure can be rewritten, analogous to $P_2$ in (5), as a function of the difference between the Gini index $G_B$ between groups and the Gini index $G_W$ within groups[2],

$$P^W = \frac{2\bar{x}}{m}(G_B - G_W),$$

where $\bar{x}$ is the mean income and $m$ is the median income[3].

Rodríguez and Salas (2003) propose a polarization measure that generalizes the Wolfson index by including a sensitivity parameter $v$,

$$P^{RS} = G_B(v) - G_W(v), \quad v \in [2, 3],$$

where $G_B(v)$ and $G_W(v)$ are, respectively, the between-group and within-group extended Gini indices introduced in Donaldson and Weymark (1980). Measure $P^{RS}$ has the same structure of the index $P_2$, although without any groups size measure.

Slightly different from the class of indices $P_2$ is, instead, the index $P^{EGR}$ of Esteban et al. (2007), consisting of a difference between a term of between-groups polarization and a term of within-groups inequality,

$$P^{EGR} = \sum_{g=1}^{G}\sum_{h=1}^{G} \pi_g^{1+\alpha}\pi_h|\bar{x}^g - \bar{x}^h| - (\beta(G_I - G_B)), \tag{7}$$

where $G_I$ is the Gini index of the entire distribution, $\bar{x}^g$ is the mean income of group $g$, $\pi_g$ is the population share of group $g$ and the incomes are normalized with the mean, $\alpha \in [1, 1.6], \beta \geq 0$.

**NEW: In case groups defined according to variables different from income, Gradín (2000) proposes $P^G = P^{EGR} + \beta \geq 0$ with some constant $\beta$.**

[2] See Wolfson (1994), Esteban et al. (2007) and Rodríguez and Salas (2003).

[3] The ratio $\bar{x}/m$ may be considered as a measure of right-skewness of the income distribution.

In the subsequent Section 3 we shall study multivariate measures of inequality that can be additively decomposed by subgroups, obtaining specific polarization measures of the types $P_1, P_2$ and $P_3$. In Section 4 we consider, instead, inequality measures that are decomposable in non-additive ways and for them we propose other particular forms for $P$ in (1).

# 3 Polarization via additive inequality decomposition

In this section we survey several existing multivariate inequality indices that are additively decomposable.

Let us consider a multivariate inequality measure of type

$$I(\mathbf{X}) = f\left(\frac{1}{N}\sum_{i=1}^{N} h(s_i, \bar{s})\right), \qquad \mathbf{X} \in \mathcal{M}^{N\times K}. \tag{8}$$

Here $s_i = s_i(x_{i1}, \ldots, x_{iK})$ signifies an individual evaluation function, $\bar{s}$ denotes a proper average either of the individual values $s_i$ or of the attribute means $\bar{x}_k$, and $f$ and $h$ are continuous functions, $f$ strictly increasing. We assume that, for some choice of $f$, $h$ and $s_i$, $I(\mathbf{X})$ has an *additive decomposition by subgroups*,

$$I(\mathbf{X}) = B(\mathbf{X}) + W(\mathbf{X}) = B(\mathbf{X}) + \sum_{g=1}^{G} w_g I_g(\mathbf{X}), \tag{9}$$

where the inequality between groups and inside a group $g$ are given, respectively, by

$$B(\mathbf{X}) \;=\; f\left(\sum_{g=1}^{G} \frac{N_g}{N}\cdot h(\bar{s}^g, \bar{s})\right), \tag{10}$$

$$I_g(\mathbf{X}) \;=\; f\left(\frac{1}{N_g}\sum_{i\in g} h(s_i, \bar{s}^g)\right). \tag{11}$$

Here $\bar{s}^g$ is a mean like $\bar{s}$ that refers to group $g$, and $w_g$ is a weight of group $g$.

¿From a multivariate inequality measure like this, polarization indices (4) to (6) are obtained. Table 2 lists five special decomposable measures that satisfy (8) to (11).

1. *Multivariate generalized entropy by Maasoumi.*

   As an index of multivariate inequality, Maasoumi (1986) proposed the following generalized entropy measure (henceforth, GEM):

$$GEM_\gamma(\mathbf{X}) \;=\; \frac{1}{\gamma(1+\gamma)}\frac{1}{N}\sum_{i=1}^{N}\left[\left(\frac{s_i}{\bar{s}}\right)^{1+\gamma} - 1\right], \quad \gamma \neq -1, 0, \tag{12}$$

$$GEM_{-1}(\mathbf{X}) \;=\; \frac{1}{N}\sum_{i=1}^{N} \log\left(\frac{\bar{s}}{s_i}\right), \tag{13}$$

$$GEM_0(\mathbf{X}) \;=\; \frac{1}{N}\sum_{i=1}^{N} \frac{s_i}{\bar{s}} \log\left(\frac{s_i}{\bar{s}}\right). \tag{14}$$

The attributes of each person, which have to be non-negative, i.e. $\mathbf{x}_i \in \mathbb{R}_+^K$, are aggregated through $s_i = (\sum_{k=1}^K \delta_k x_{ik}^{-\beta})^{-1/\beta}$, with $\delta_k \in [0,1]$ and $\sum_{k=1}^K \delta_k = 1$. $\delta_k$ represents the weight of the $k$-th attribute and $\beta$ is a constant that reflects the elasticity of substitution between attributes.

As proved in Maasoumi (1986), the GEM is additively decomposable in the sense of (9) to (11). The values of this index range from 0 to infinity; its components are shown in Table 2, where $\bar{s}$ is the arithmetic mean of the functions $s_i$ over all $N$ individuals, and $\bar{s}^g$ is the arithmetic mean of $s_i$ over the individuals in subgroup $g$.

2. *Multivariate generalized entropy measure by Tsui.*

   Another multivariate extension of the entropy measure (in the following, GET) has been introduced by Tsui (1999):

$$GET(\mathbf{X}) = \frac{\rho}{N} \sum_{i=1}^{N} \left( \prod_{k=1}^{K} \left( \frac{x_{ik}}{\bar{x}_k} \right)^{c_k} - 1 \right). \qquad (15)$$

   Such index imposes a restriction on the matrix $\mathbf{X}$: $x_{ik} > 0 \ \forall i, k$. The elements which constitute the GET measure are shown in Table 2. Here, the constants $\rho$ and $c_1, \ldots, c_K$ must satisfy particular conditions that are specified in Tsui (1999)[4]. This measure has its minimum at 0 and its supremum at infinity.

   It is easily seen that, with group weights $w_g$ given in Table 2, GET is an additively decomposable measure.

3. *Multivariate Kolm measure by Tsui.*

   A third multivariate measure which can be additively decomposed by subgroups is a generalization of Kolm's measure (in the following, KT), that has been introduced by Tsui (1995) and is given by:

$$KT(\mathbf{X}) = \frac{1}{\sum_{k=1}^K c_k} \ln \left\{ \frac{1}{N} \sum_{i=1}^{N} \exp \left\{ \sum_{k=1}^{K} c_k (\bar{x}_k - x_{ik}) \right\} \right\}. \qquad (16)$$

   However, the decomposition of KT differs slightly from that in the previous cases. It resembles the decomposition given by Blackorby et al. (1981) for the univariate Kolm index: the total inequality measure is the sum of the following within and between groups components:

$$\begin{aligned} W(\mathbf{X}) &= \sum_{g=1}^{G} \frac{N_g}{N} \left( \frac{1}{\sum_{k=1}^K c_k} \ln \left\{ \frac{1}{N_g} \sum_{i \in g} \exp \left\{ \sum_{k=1}^{K} c_k (\bar{x}_k^g - x_{ik}) \right\} \right\} \right), \\ B(\mathbf{X}) &= \frac{1}{\sum_{k=1}^K c_k} \ln \left\{ \sum_{g=1}^{G} \frac{N_g}{N} \exp \left\{ \sum_{k=1}^{K} c_k \left( \sum_{g} \frac{N_g}{N} \xi_g - \xi_g \right) \right\} \right\}, \end{aligned}$$

[4] In case of $K = 2$, the conditions on $\rho$ and $c_k$ are the following: $\rho c_1(c_1 - 1) > 0, c_1 c_2 (1 - c_1 - c_2) > 0, \rho c_1 c_2 > 0$; they imply that $\rho > 0, c_1, c_2 < 0$.

where

$$\xi_g = -\frac{1}{\sum_{k=1}^K c_k} \ln \left\{ \frac{1}{N_g} \sum_{i \in g} \exp \left\{ - \sum_{k=1}^{K} c_k x_{ik} \right\} \right\}$$

is an equivalent equally-distributed endowment of subgroup $g$, and $c_k$ is a constant regarding the $k$-th attribute. For details, see Tsui (1995).

The total inequality $I$ and the within groups inequality $I_g$ have the form (8) and (11), respectively, with $s_i, \bar{s}$ and $\bar{s}^g$ shown in Table 2. The between component is, different from (10), not a function of $\bar{s}^g$ and $\bar{s}$, but of the $\xi_g$:

$$B^{KT}(\mathbf{X}) = f\left(\sum_{g=1}^{G} \frac{N_g}{N} \cdot h(\bar{\chi}^g, \bar{\chi})\right), \tag{17}$$

with $\bar{\chi}^g = \sum_{k=1}^K c_k \xi_g$ and $\bar{\chi} = \sum_{k=1}^K c_k \sum_{g=1}^G \frac{N_g}{N} \xi_g$. The values of this measure range from 0 to $\frac{\sum_k c_k \bar{x}_k}{\sum_k c_k}$.

Note that the index $KT$ allows also for negative values of the attributes, e.g. for negative wealth due to liabilities.

[Insert Table 2 here]

# 4 Polarization via other inequality decompositions

The inequality indices considered so far are additively decomposable. In the sequel we study indices that can be multiplicatively decomposed and the multivariate Gini mean difference.

## 4.1 Multiplicative decomposition

Two special inequality measures that have a multiplicative decomposition are multivariate extensions of Atkinson's measure and have been introduced by Maasoumi (1986) (henceforth, AM) and Tsui (1995) (henceforth, AT). Tsui, in particular, proposes a double generalization, that will be indicated here with $AT_1$ and $AT_2$.

$$AM_v(\mathbf{X}) = 1 - \left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{s_i}{\bar{s}}\right)^{1-v}\right)^{1/(1-v)}, \quad v > 0, \quad v \neq 1, \tag{18}$$

$$AM_1(\mathbf{X}) = 1 - \exp\left(\frac{1}{N}\sum_{i=1}^{N}\log\left(\frac{s_i}{\bar{s}}\right)\right). \tag{19}$$

$$AT_1(\mathbf{X}) = 1 - \left(\frac{1}{N}\sum_{i=1}^{N}\prod_{k=1}^{K}\left(\frac{x_{ik}}{\bar{x}_k}\right)^{r_k}\right)^{1/\sum_k r_k}, \tag{20}$$

$$AT_2(\mathbf{X}) = 1 - \exp\left(\frac{1}{N}\sum_{i=1}^{N}\log\left(\prod_{k=1}^{K}\left(\frac{x_{ik}}{\bar{x}_k}\right)^{r_k/\sum_j r_j}\right)\right). \tag{21}$$

Both assume values between 0 and 1 and have the form

$$I(\mathbf{X}) = 1 - A(\mathbf{X})\,,$$

where $A$ is a multivariate similarity measure of the type

$$A(\mathbf{X}) = f\left(\frac{1}{N}\sum_{i=1}^{N} h(s_i, \bar{s})\right), \tag{22}$$

with $h$, $s_i$ and $\bar{s}$ as in (8) and $f$ continuous and strictly monotone function.

For the particular functions $f$, $h$ and $s_i$, chosen by Maasoumi (1986) and Tsui (1995), and following the approach of Lasso de la Vega and Urrutia (2003), the similarity measure $A$ in (22) can be multiplicatively decomposed into $A = A_B \cdot A_W$ or, equivalently,

$$\ln A = \ln A_B + \ln A_W\,,$$

where $A_B$ and $A_W$ are similarity measures, respectively, between and within groups, given by

$$A_B(\mathbf{X}) \;=\; f\left(\sum_{g=1}^{G} \frac{N_g}{N} \cdot h(\bar{s}^g, \bar{s})\right), \tag{23}$$

$$A_W(\mathbf{X}) \;=\; \left(\sum_{g=1}^{G} w_g \left(A_g(\mathbf{X})\right)^{\epsilon}\right)^{1/\epsilon}, \text{ or} \tag{24}$$

$$A'_W(\mathbf{X}) \;=\; \left(\prod_{g=1}^{G} A_g(\mathbf{X})\right)^{w_g}. \tag{25}$$

The first type of similarity measure within groups, $A_W$, is a weighted mean of order $\epsilon$ of the similarity measure inside each group, $A_g$, which is given by

$$A_g(\mathbf{X}) \;=\; f\left(\frac{1}{N_g}\sum_{i\in g} h(s_i, \bar{s}^g)\right).$$

This holds for measure $AM_v$ with $v \neq 1$ and for the first measure of Tsui, $AT_1$. The second type of similarity within groups, $A'_W$ in (25), holds for the measure $AM_1$ and for the second measure of Tsui, $AT_2$.

Table 3 shows the particular components of the measures proposed both by Maasoumi, with parameters $\delta_k \in [0,1]$, $\sum_{k=1}^{K} \delta_k = 1$, and by Tsui, where the parameter $r_k$ has to satisfy particular restrictions specified in Tsui (1995)[5]. Restrictions on matrix $\mathbf{X}$ are required by both the measures: $x_{ik} \geq 0$ for AM and $x_{ik} > 0$ for $AT_1$ and $AT_2$, $\forall i, k$.

[5]For $AT_1$, in case of $K = 2, r_1 \in (0,1)$ and $r_2 < 1 - r_1$; for $AT_2$, $r_k > 0$ for all $k = 1, ..., K$.

In case of multiplicative decomposition, we construct particular forms of (1) which are parallel to $P_1, P_2, P_3$:

$$
\begin{aligned}
P_4(\mathbf{X}) &= \phi\Big(\frac{\ln A_B(\mathbf{X})}{\ln A_W(\mathbf{X}) + c}\Big) \cdot S(\mathbf{X}); && (26)\\
P_5(\mathbf{X}) &= \psi\Big(\ln A_W(\mathbf{X}) - \ln A_B(\mathbf{X})\Big) \cdot S(\mathbf{X}); && (27)\\
P_6(\mathbf{X}) &= \tau\Big(\frac{\ln A(\mathbf{X}) + c}{\ln A_W(\mathbf{X}) + c}\Big) \cdot S(\mathbf{X}), && (28)
\end{aligned}
$$

with properly chosen $\phi, \psi$ and $\tau$.

[Insert Table 3 here]

## 4.2 Gini decomposition

The last inequality measure we consider here is a multivariate generalization of the Gini mean difference, the distance-Gini mean difference (Koshevoy and Mosler (1997)), shortly GMD. It is given by

$$\Delta(\mathbf{X}) = \frac{1}{2KN^2} \sum_{i=1}^{N} \sum_{j=1}^{N} ||\mathbf{x}_i - \mathbf{x}_j||,$$

where $||\cdot||$ indicates the Euclidean distance in $\mathbb{R}^K$. The distance-Gini mean difference is bounded between 0 and $\frac{1}{K}\sum_{k=1}^{K} \bar{x}_k$ and is defined also for negative endowments, $\mathbf{x}_i \in \mathbb{R}^K$; see Koshevoy and Mosler (1997).

To decompose the multivariate GMD, we follow the approach of Bhattacharya and Mahalanobis (1967) given for the univariate measure. By straightforward calculation we obtain

$$
\begin{aligned}
\Delta(\mathbf{X}) &= \sum_{g=1}^{G} \left(\frac{N_g}{N}\right)^2 \Delta_g + \frac{1}{2K} \sum_{g=1}^{G} \sum_{h \neq g}^{G} \frac{N_g N_h}{N^2} ||\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h|| \\
&+ \frac{1}{2K} \sum_{g=1}^{G} \sum_{h \neq g}^{G} \frac{N_g N_h}{N^2} \left\{ \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} ||\mathbf{x}_i - \mathbf{x}_j|| - ||\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h|| \right\} \\
&= \Delta_W(\mathbf{X}) + \Delta_B(\mathbf{X}) + \Delta_{OV}(\mathbf{X}) \,.
\end{aligned}
$$

In the previous equation, $\Delta_g, \Delta_W$ and $\Delta_B$ represent the distance-Gini mean difference, respectively, inside group $g$, within all the groups and between groups. The residual component $\Delta_{OV}$ corresponds to the univariate overlap component (see the following remarks) and is equal to zero if $\sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} ||\mathbf{x}_i - \mathbf{x}_j|| = ||\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h||$ for every two groups $g$ and $h$.

**Remarks**

- The multivariate overlap component $\Delta_{OV}$ is always non-negative, which is seen from

$$\Delta_{OV} = \sum_{g=1}^{G} \sum_{h \neq g}^{G} \frac{N_g N_h}{2KN^2} \left\{ \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} ||\mathbf{x}_i - \mathbf{x}_j|| - \left|\left| \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} (\mathbf{x}_i - \mathbf{x}_j) \right|\right| \right\}$$

  and the triangle inequality.

- Necessary for $\Delta_{OV} = 0$ is that the groups have no 'geometric overlap', in the sense that their convex hulls do not intersect.

- With one attribute only, $\Delta_{OV} = 0$ if and only if there is no geometric overlap between the groups, that is, the groups are restricted to separate intervals. With more than one attribute the 'if' implication does not hold in general: Figure 1 shows an example with $N = 4$ and $K = G = 2$. The first group consists of endowment vectors $(1, 7)$ and $(3, 1)$, the second of $(4, 5)$ and $(6, 7)$. The two groups can be separated by a straight line, hence have no geometric overlap, but there holds $\Delta_{OV} = 0.157 > 0$.

- Sufficient for $\Delta_{OV} = 0$ is that there exists no inequality within groups. Then, all the individuals a group have endowment vector equal to the group mean; therefore:

$$\sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} ||\mathbf{x}_i - \mathbf{x}_j|| = \sum_{i=1}^{N_g} \sum_{j=1}^{N_h} \frac{1}{N_g N_h} ||\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h|| = ||\bar{\mathbf{x}}^g - \bar{\mathbf{x}}^h||.$$

  Another sufficient condition for $\Delta_{OV} = 0$ is that the endowment vectors $\mathbf{x}_i$ of all individual lie on a straight line, (i.e. the situation is essentially univariate) and, in addition, there is no geometric overlap among groups.

[Insert Figure 1 here]

The polarization measures based on the distance-Gini mean difference are of the following types:

$$\begin{aligned}
P_7(\mathbf{X}) &= \phi\Big(\frac{\Delta_B(\mathbf{X})}{\Delta_W(\mathbf{X}) + \Delta_{OV}(\mathbf{X}) + c}\Big) \cdot S(\mathbf{X})\,, \\
P_8(\mathbf{X}) &= \psi\Big(\Delta_B(\mathbf{X}) - \Delta_W(\mathbf{X}) - \Delta_{OV}(\mathbf{X})\Big) \cdot S(\mathbf{X})\,, \\
P_9(\mathbf{X}) &= \tau\left(\frac{\Delta_B(\mathbf{X})}{\Delta(\mathbf{X}) + c}\right) \cdot S(\mathbf{X}),
\end{aligned}$$

with functions $\phi, \psi$ and $\tau$ continuous and strictly increasing, properly chosen, and constant $c$ positive.

# 5 Properties for polarization indices

For univariate polarization measurement, a number of postulates or axioms have been presented in the literature. Part of them are continuity and invariance properties, others concern the minima and maxima of polarization indices and their monotonicity with respect to certain changes of the distribution. In this section we extend some of the properties to the multivariate setting and discuss whether they are satisfied with the multivariate indices introduced in Sections 3 and 4.

## 5.1 Invariance and continuity properties

A first group of properties concerns the continuity of a multivariate polarization index $P$ and its invariance with respect to certain transformations of the matrix $\mathbf{X}$. In our setting, such properties are generally inherited from the same properties of the indices $B$, $W$, $S$, and $I$ on which $P$ is based.

1. **Continuity** $P$ is continuous as a function of $\mathbf{X} \in \mathcal{M}^{N \times K}$.

2. **Anonymity** $P$ is invariant to the individual labels. Formally, for any $N \times N$ permutation matrix $\Pi$, the postulate requires that $P(\mathbf{X}) = P(\Pi\mathbf{X})$.

3. **Replication Invariance** The index depends on the frequency distribution of endowments only. Formally, let $\mathbf{Y}$ be the matrix obtained by repeating $\mathbf{X}$ matrix $H$ times, such that the number of columns of $\mathbf{Y}$ is $K$ and the number of rows is $N \times H$. The property requires that $P(\mathbf{Y}) = P(\mathbf{X})$. It means that replicating the population, without changing the distribution of the variables, does not influence polarization.

4. **Weak Scale Invariance** The index does not depend on a common scale factor. Formally, $P(\lambda\mathbf{X}) = P(\mathbf{X})$ for all $\lambda > 0$.

5. **Strong Scale Invariance** The index does not depend on the units of measurement of the attributes, it is a *relative index*. Formally, $P(\mathbf{X}\Lambda) = P(\mathbf{X})$ if $\Lambda = diag(\lambda_1, ..., \lambda_K)$, with $\lambda_i > 0, i = 1, \ldots K$.

6. **Translation Invariance** The index does not change when each individual receives the same additional vector of endowments. Formally, $P(\mathbf{X} + \Lambda) = P(\mathbf{X})$ if $\Lambda$ is an $N \times K$ matrix with all identical rows. A translation invariant index is also called an *absolute index*.

For each of these properties holds: A multivariate polarization index $P$ of type (1) satisfies the property if the indices ($B, W, S$, and $I$) on which it is based do. By this, all special indices introduced in Section 3 satisfy Anonymity and Replication Invariance.

About Continuity, observe that the relative groups size measures $S$ are continuous function of $\mathbf{X}$ as long as the number of groups is kept constant.

Obviously, the size indices (2) and (3) of Subsection 2.2 are scale and translation invariant. Moreover, among the multivariate inequality measures considered, GEM, GET, AM and AT are scale invariant[6]. Therefore, also the corresponding polarization measures $P$ are scale invariant. But, the polarization indices constructed from the distance-Gini mean difference (GMD) and the KT index are absolute indices, as the underlying inequality measures satisfy Translation Invariance.

## 5.2 Polarization properties

A second group of properties pertains properly to the polarization concept, i.e. to the double tendency of the groups to be internally homogeneous and externally heterogeneous.

1. **Maximum Polarization** In univariate polarization measurement (Esteban and Ray (1994), Wolfson (1997), Milanovic (2000)), the following situation is regarded as the extreme case in which the society is perfectly polarized: the society divides into two groups of identical size (the rich and the poor), and the groups are completely homogeneous inside (i.e. without any internal inequality) and at maximum distance to each other, given the income endowment of the entire society.

   Analogously, in the multivariate context, we postulate that a two-groups society shows **maximum polarization** if it consists of two equally large groups, the individuals in each group have the same endowment vector and the mean vectors of the two groups are at maximum distance. Different from Esteban and Ray (1994), in our scenario an empty group is not considered as a group. Given $G > 2$ non-empty groups, maximum polarization cannot be reached when population is equally split into two extreme groups and the remaining G-2 groups are empty, but rather when the G groups are equally sized, internally homogeneous and the group mean vectors show maximum disparity as measured by a proper inequality index.

   However, for univariate distributions, polarization measures (1) increase also when moving from a distribution that is uniform on the supports of $G > 2$ equally sized groups to a symmetric bimodal distribution with two groups and population bipolarized at the extremes of the joint support, as postulated in Esteban and Ray (1994). In this situation, between-group inequality strictly increases from the uniform to the bimodal distribution, while within-group inequality and the measure $S$ remain constant (respectively, equal to 0 an to 1), since the first distribution is based on $G$ groups and the second distribution on two groups. Thus, polarization as measured by (1) increases.

   The property extends immediately to the multivariate case as follows:

   **Proposition 1.** *Consider a d-variate distribution* $\mathbf{X}$ *that is uniform on the supports* $A_1, \ldots, A_G$ *of* $G \geq 2$ *equally sized groups and another distribution* $\mathbf{Y}$ *having support on the extreme points of the convex hull of the joint support* $\bigcup_{g=1}^{G} A_g$*. Then, for any* $P$ *that has form (1) holds*

$$P(\mathbf{Y}) \geq P(\mathbf{X})\,.$$

[6]GEM fulfils the weak version only, while the others also satisfy the strong version.

For proof consider the distribution $\mathbf{Z}$ that arises from $\mathbf{X}$ by concentrating each group at its center. Obviously, as $S(\mathbf{Z}) = S(\mathbf{X})$, $B(\mathbf{Z}) = B(\mathbf{X})$, and $W(\mathbf{X}) \geq W(\mathbf{Z}) = 0$, we obtain $P(\mathbf{Z}) \geq P(\mathbf{X})$. Note that $\mathbf{Y}$ is a dilation[7] of $\mathbf{Z}$, and therefore[8] $B(\mathbf{Y}) \geq B(\mathbf{Z})$. Since $W(\mathbf{Z}) = W(\mathbf{X}) = 0$ and $S(\mathbf{Z}) = S(\mathbf{X})$, we conclude $P(\mathbf{Y}) \geq P(\mathbf{Z}) \geq P(\mathbf{X})$.

Consequently, any bipolar or multipolar distribution $\mathbf{Y}$ that is supported by the extreme points of the convex hull yields a larger value of the polarization index than $\mathbf{X}$.

Focusing on univariate distributions, we now discuss several axioms and examples from Esteban and Ray (1994). Most of them are satisfied also by polarization measures (1).

**A1** Consider a distribution involving four groups concentrated at incomes $0, x, (x+y)/2, y$ with population shares $p, q-\epsilon/2, \epsilon, q-\epsilon/2$, respectively, $0 \leq \epsilon \leq 2q$, $p, q > 0$. As $\epsilon$ becomes zero, three groups with sizes $p, q, q$ are obtained; and as $\epsilon$ becomes $2q$, there are two groups with sizes $p$ and $2q$. The Axiom A1 requires that polarization increases when moving from the first to the second distribution. The polarization indices (1) satisfy the axiom if $2q < p$.[9]

**A2** Let a distribution divide into three groups concentrated at incomes $0, x, y$ with population shares $p, q, r > 0$, respectively, where $p < r$ and $x > |y-x|$. The axiom requires polarization to increase when moving $x$ to the right towards $y$. Axiom 2 is not satisfied by a polarization measure (1), as the component $S$ remains unchanged, while the between-group inequality decreases. This is a case in which polarization is completely driven by between-group inequality.[10]

**A3** Consider a distribution involving three groups at income values $0, d, 2d$ having population shares $p+\Delta, q-2\Delta, p+\Delta$, respectively, with $0 < p < q$, $0 \leq \Delta \leq \frac{q}{2}$. The axiom postulates that polarization increases when the central mass of the population is shifted in equal parts to the two lateral masses, i.e. when we move from $\Delta = 0$ to $\Delta > 0$. Our measures (1) satisfy Axiom 3 for any $\Delta \in ]0, \frac{q-p}{2}[$, since $S$ increases and $B$ remains constant. Again, for $\Delta = \frac{q}{2}$ the axiom is satisfied, as $B$ is larger and $S$ is constant.

**A4** Consider a distribution involving three groups at $0, x, y$ with population shares $p, q, r$, respectively, with $r + p < q$. The axiom requires polarization to increase when transferring population mass from the $p$ mass to the $r$ mass. Axiom A4 is satisfied by (1) if the entire mass $p$ is transferred to the $r$ mass.

2. **Minimum Polarization** The 'normalization axiom' of univariate polarization measurement (Wang and Tsui (2000), Chakravarty and Majumder (2001)) states that

[7] That is, $\mathbf{Z} = B\,\mathbf{Y}$ holds for some bistochastic matrix $B$.

[8] As a postulate, any measure of multivariate inequality must increase on dilations.

[9] Example 4 of Esteban and Ray (1994) may be similarly discussed.

[10] We find, however, that such axiom does not express a peculiar aspect of polarization, as both the cohesion within the groups and the population share remain unchanged, while only the between-group inequality is modified. Another case in which polarization coincides with between-group inequality is illustrated in Example 2 of Esteban and Ray (1994); in such example our indices (1) are in accordance with the index of Esteban and Ray (1994).

polarization reaches its minimum value (= 0) when all the individuals have the same income, i.e. in the case of an egalitarian distribution.

Gradín (2000) postulates, instead, that polarization is minimized if there is both perfect equality between groups and maximum intra-group disparity; in particular, minimum polarization arises if the groups which constitute the population have null intergroup inequality and, inside each groups, inequality is maximum. For our indices $P$ it is obvious from the formula (1) that **minimum polarization** is obtained when $B$ and $S$ are minimized and $W$ is maximized, that is, when the population is constituted by only one group and inequality is maximum. Hence, $P$ satisfies Gradín's postulate, but not the above 'normalization axiom'.

3. **Increased Spread** The 'increased spread' property of univariate polarization measurement (Wang and Tsui (2000), Chakravarty and Majumder (2001)) establishes that, given two groups, if any individual of one group moves further from the other group, polarization increases.

To extend this notion to the multivariate case, we consider shifts of two or more groups that increase the dispersion of their group means. A group $g$ is shifted by some $\mathbf{c}^g \in \mathbb{R}^K$ if the endowment vector $X_i$ of each member $i \in g$ is shifted to $Y_i = X_i + \mathbf{c}^g$. Consequently the mean $\bar{\mathbf{x}}^g$ of group $g$ is shifted to $\bar{\mathbf{y}}^g = \bar{\mathbf{x}}^g + \mathbf{c}^g$. To describe increasing dispersion of group means, we employ four different notions of multivariate majorization.

Consider matrices $\mathbf{U}$ and $\mathbf{V}$ that have format $M \times K$. Each of the following six notions reduces to univariate Pigou-Dalton majorization[11] when $K = 1$:

(a) $\mathbf{U} \prec_T \mathbf{V}$ if $\mathbf{U} = A\mathbf{V}$ with $A =$ finite products of $T$- matrices, where $T = \lambda I + (1-\lambda)Q, \lambda \in [0,1]$, $I$ is the identity matrix, and $Q$ a permutation matrix that interchanges only two coordinates;

(b) $\mathbf{U} \prec_B \mathbf{V}$ if $\mathbf{U} = B\mathbf{V}$ where $B$ is an $M \times M$ bistochastic matrix;

(b') $\mathbf{U} \prec_c \mathbf{V}$ if $\mathbf{U} \in conv\{\Pi\mathbf{V} : \Pi \text{ is a } M \times M \text{ permutation matrix}\}$;

(c) $\mathbf{U} \prec_p \mathbf{V}$ if $\mathbf{U}\mathbf{p}^T = B_k \cdot \mathbf{V}\mathbf{p}^T,\ \mathbf{p} \in \mathbb{R}^K$ and $B_k = M \times M$ bistochastic matrix specific for $k = 1, ..., K$;

(c') $\mathbf{U} \prec_k \mathbf{V}$ if $\mathbf{u}^k = B_k\mathbf{v}^k$, with $\mathbf{v}^k = k$-th column of $\mathbf{V}$ and $B_k = M \times M$ bistochastic matrix, $\forall k = 1, ..., K$;

(d) $\mathbf{U} \prec_L \mathbf{V}$ if $LZ(\mathbf{U}) \subset LZ(\mathbf{V})$, with $LZ(\mathbf{U}) =$ Lorenz zonoid of distribution $\mathbf{U}$.

However, the six notions are not equivalent; in fact: (a) $\Rightarrow$ (b)$\Leftrightarrow$ (b') $\Rightarrow$ (c) $\Leftrightarrow$ (c')$\Rightarrow$ (d). For details, see Mosler (1994) and Marshall and Olkin (1979).

We propose the following **multidimensional increased spread property**: whenever two or more groups are shifted such that their means become more dispersed in terms of majorization (a), (b), (c) or (d), then polarization increases.

[11] that is, $\mathbf{U} \succ_{PD} \mathbf{V}$ if $\mathbf{U}$ is obtained from $\mathbf{V}$ by a finite sequence of Pigou-Dalton transfers.

Neither the inequality $W$ within groups nor the groups size measure $S$ are modified by a majorization movement of the group centers; the only component of the measure $P$ that is involved is the inequality between groups $B$. That is, the polarization measure (1) satisfies the multidimensional increased spread property (a), (b), (c) or (d) if and only if the between groups inequality measure increases under majorization (a), (b), (c) or (d), respectively.

Every multivariate inequality measure used in Sections 3 and 4 increases with one of these majorizations. In particular, the measures GEM, GET, KT, AM, and AT satisfy the property with (b), while GMD is increasing with (d) (and the majorizations that imply these). Therefore, all polarization measures obtained from these inequality indices fulfil the property.

4. **Increased Polarity** The univariate version of this property (often called 'increased bipolarity'; see Wang and Tsui (2000), Chakravarty and Majumder (2001)) requires that a Pigou-Dalton transfer within one or more groups increases polarization. It means that if, inside a group, one distribution is obtained from the other by univariate Pigou-Dalton majorization, then the polarization in the first distribution is higher than in the second.

   In the multidimensional case, we say that the **increasing polarity** property of type (a), (b), (c) or (d) holds if polarization increases whenever the population in one of the groups is exchanged against a majorizing population of type (a), (b), (c) or (d), respectively. For the polarization measure (1) the property is satisfied if and only if the within inequality $W$ decreases with a majorization (a), (b), (c) or (d).

   Obviously, each multivariate inequality measures considered in Sections 3 and 4 satisfies one of these notions. In particular, the measures GEM, GET, KT, AM, AT respect majorization (b), while GMD majorization (d) (and as well the majorizations that imply these). By this, all polarization measures obtained from these inequality indices fulfil the increased polarity property in one of the four versions.

# 6 Interaction among attributes

We further have to take into account what kind of interaction among the variables is evaluated by the researcher (or by society). Multivariate inequality increases when the variability of an attribute increases. It also increases when the correlation between variables rises and the variables are substitutes; it decreases when they are complements. Consequently, the results of polarization measurement are different.

The importance of considering the interaction between the attributes has been underlined in Maasoumi and Nickelsburg (1988) and in Bourguignon and Chakravarty (2003). In these papers, an appropriate parameter is introduced that reflects the evaluation of the researcher, or of the society, on the relationship between the variables.

Some of the inequality measures considered above are so flexible to allow for different kinds of association between attributes; they are the GEM and AM measures. The aggregative function $s_i$, introduced by Maasoumi (1986), is, in fact, based on the parameter $\beta$, which expresses the degree of substitution between attributes, such that $\beta = (1/\sigma) - 1$, where $\sigma$ is a constant elasticity of substitution. So, if two attributes are substitutes, $\sigma$ tends to infinity and, correspondingly, $\beta \to -1$. If they are complements, $\sigma \to 0$ and $\beta \to \infty$. $\sigma = 1$ and $\beta = 0$ means an intermediate situation with a certain degree of substitution between attributes.

The other inequality measures of Sections 3 and 4 do not possess such flexibility: the GET measure regards all goods as substitutes, ignoring the case of complements; the measures AT, KT and GMD, instead, do not consider this aspect of evaluation.

**Correlation increasing majorizations** The last type of properties we consider is peculiar to multivariate analysis, as it takes into account the interaction between the different variables involved in the analysis. In particular, we study the effect on the polarization measure of transfers that increase the correlation between the attributes; see, e.g., Tsui (1999).

As our multivariate polarization measures are based on inequality between groups and inequality within groups, which may point in opposite directions, it seems necessary to study correlation increasing transfers separately between and within groups. For that reason we introduce two notions of correlation increasing majorizations for polarization measures, one related to the between-group transfers and the other to the within-group transfers.

**Definition 6.1** (Between groups correlation increasing transfers)**.** *Matrix* $\mathbf{Y}$ *is obtained from* $\mathbf{X}$*, with* $\mathbf{X}, \mathbf{Y} \in \mathcal{M}^{N\times K}$*, by a between groups correlation increasing transfer (BCIT) if* $\mathbf{X} \neq \mathbf{Y}$*,* $\mathbf{X}$ *is not a permutation of* $\mathbf{Y}$ *and there exist equally sized groups* $g, h \in \{1, 2, \ldots, G\}$ *with null within group disparity such that* $\bar{\mathbf{y}}^g = \bar{\mathbf{x}}^g \wedge \bar{\mathbf{x}}^h$*,* $\bar{\mathbf{y}}^h = \bar{\mathbf{x}}^g \vee \bar{\mathbf{x}}^h$ *and* $\bar{\mathbf{y}}^\ell = \bar{\mathbf{x}}^\ell$*,* $\forall \ell \notin \{g, h\}$*.*

A *BCIT* is a transfer between individuals of different groups that increases the correlation between attributes regarding the centers of the groups. It is only considered for distributions with equally sized groups and no within-group inequality.

Now we are able to introduce the first property regarding the interaction between attributes. $P$ satisfies the **between groups correlation increasing majorization property** if $P(\mathbf{Y}) \geq (\leq) P(\mathbf{X})$ whenever $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a *BCIT* and the attributes are substitutes (complements).[12]

Substitutability means some proximity in nature of the attributes, so that the utility provided by one attribute may be as well obtained by the other attribute.

A between groups correlation increasing transfer means that a group with higher average amount of one attribute gets higher average amount of the other; if attributes are close to each other, i.e. are substitutes, such transfer should increase the heterogeneity among the groups, augmenting the polarization.

Consider now a transfer which increases the correlation between the attributes only for individuals inside a given group.

[12]Such trend is due to the fact that polarization measures are increasing functions of between-groups inequality.

**Definition 6.2** (Within groups correlation increasing transfers)**.** *Matrix* $\mathbf{Y}$ *is obtained from* $\mathbf{X}$*, with* $\mathbf{X}, \mathbf{Y} \in \mathcal{M}^{N \times K}$*, by a within groups correlation increasing transfer (WCIT) if* $\mathbf{X} \neq \mathbf{Y}$*,* $\mathbf{X}$ *is not a permutation of* $\mathbf{Y}$ *and there exist a group* $g \in \{1, \ldots G\}$ *such that, for some* $i, j \in \{1, 2, \ldots, N_g\}$*,* $\mathbf{y}_i = \mathbf{x}_i \wedge \mathbf{x}_j$*,* $\mathbf{y}_j = \mathbf{x}_i \vee \mathbf{x}_j$ *and* $\mathbf{y}_h = \mathbf{x}_h \forall h \notin \{i, j\}$*.*

$P$ satisfies the **within groups correlation increasing majorization property** if $P(\mathbf{Y}) \leq (\geq) P(\mathbf{X})$ whenever $\mathbf{Y}$ is obtained from $\mathbf{X}$ by a $WCIT$ and the attributes are substitutes (complements).[13]

Among the measures considered in Sections 3 and 4, the only inequality index that satisfies the correlation increasing majorization is GET. Such measure considers, however, the attributes only as perfect substitutes. In presence of correlation increasing transfers, in fact, GET can only increase. Therefore, the polarization measures $P_i, i = 1, 2, 3$ obtained from GET increase, in presence of $BCIT$, and decrease, in case of $WCIT$.

# 7 Application to German microdata

In order to illustrate the proposed measures, we now present an empirical application that analyzes the degree of polarization in the bivariate distribution of income and education between the West Germany and the East Germany for the years 1994 to 2002.

The data set employed is the English Language Public Use File of the the German Socio-Economic Panel (GSOEP). The unit of our analysis is the household, defined as the income sharing unit, i.e. married couples, singles and cohabitants, with or without children.

As income variable we consider the household net income, deflated and transformed, in order to be equivalent, with the old OECD equivalent scale.

Education of the household is measured in terms of the number of year of education of the householder.

The polarization indices proposed in the previous sections are based on an inequality decomposition that holds for attributes, which can be considered transferable among people, such as income and wealth. However, according to our opinion, other attributes, such as health and education, can be employed for the multidimensional measurement of polarization, since their distribution is comparable to the distribution of income or wealth, in sense that it can be influenced by the government through public health or education programs, insurance systems or through standard accessibility of medical and educational services. We prefer instead not to incorporate in our measurement of multidimensional polarization individual characteristics such as sex, religion and gender, which are typically not transferable, but we can use them for building groups.

According to the particular inequality measures used in the construction of the polarization measures of class $P$ in (1), specific measures $P_i, i = 1, \ldots, 9$ are computed for all the years.

[13] The direction of the dominance is due to the fact that polarization measures are inverse functions of the within-groups inequality.

For simplicity of notation, in the rest of the section we will call *ratio-based measures* the polarization measures based on the ratio of inequality between groups over inequality within groups, which are $P_1$, for the additively decomposable measures $GEM$ and $KT$, $P_4$ for the multiplicatively decomposable indices $AM$ and $AT_1$ and $P_7$ for $DGMD$. Analogously, the polarization indices, based on the difference between the inequality between groups and the inequality within groups, will be nominated *difference-based measures*, given by $P_2$ for $GEM$ and $KT$, $P_5$ for $AM$ and $AT_1$ and $P_8$ for $DGMD$. In this illustrative example we do not report results for the measures $P_3, P_6$ and $P_9$. Table 4 illustrates the polarization levels measured by such kind of indices for every year.

[Insert Table 4 here]

For the polarization measures based on the inequality indices GEM and AM we choose the substitution parameter $\beta = -1$, when income and education are regarded as perfect substitutes, and increasing values of $\beta$ (e.g. $\beta$ = -0.5 and 9), if the degree of substitution diminishes. We can see, from Table 4, that the trend of polarization is not substantially different in case of substitute or complement attributes.

Table 4 shows also the role played by the attributes' weights: for the polarization measures based on GEM[14], if we put more weight to income (i.e. $\delta = (\delta_1, \delta_2) = (0.8; 0.2)$), polarization values are higher than if we weight more education (i.e. $\delta = (\delta_1, \delta_2) = (0.2; 0.8)$), according to the ratio-based measures, while the reverse holds for the difference-based indices. The case of equally weighted attributes ($d = (0.5; 0.5)$) assumes, in both the cases, intermediate values.

Figure 2 illustrates the levels of multivariate polarization over the years. Both the ratio-based indices and the difference-based measures show a similar tendency, according to which polarization has slightly reduced during the entire period of time 1994-2002; in particular polarization decreases from 1994 to 1997, then increases until 1999, decreases in 2000 and since then it increases again.

[Insert Figure2 here]

Due to their peculiar structure, the multivariate polarization measures defined in (1) can be decomposed into the three components $B, W$ and $S$. Table 5 shows that the percentage of the households in East and in West Germany remains quite constant, i.e. the relative groups size measure $S$ remains stable, over the years, while the within groups inequality is much higher than the between groups inequality. Moreover, W increases from 1994 to 2002, according to almost all the measures, while B decreases. The slight decrease in polarization is therefore due mainly to the changes in the within groups inequality. Such kind of multidimensional polarization indices has therefore the advantage to be decomposable into the principal components that may influence the polarization trend. Figure 3 shows the trend of one inequality index, GEM, as representative for most of the indices of Table 5.

[14]The same results are obtained for all the other inequality measures considered.

[Insert Table 5 and Figure 3 here]

It could be interesting to compare the multivariate polarization measures with the measures of marginal polarization to see whether differences exist between a uni-dimensional and a multi-dimensional analysis. Since in this example the groups are exogenously defined, the most appropriate univariate polarization is the measure $P^G$ proposed by Gradín (2000). Figure 4 shows that both income and education slightly decrease over the years, with a rapid drop for education in the recent years.

[Insert Figure 4 here]

Comparing Figure 2 and Figure 4, it seems that the polarization trend of the joint distribution of income and education quite diverge from the polarization trend of the marginal distributions.

Finally, we should underline that, in this empirical example, the definition of the groups is exogenous in sense that the whole population is partitioned according to an attribute different from the ones involved in the analysis of polarization (i.e. geographical area, on one side, and income and education, on the other side); however, the measures proposed in this paper are general and allow also for an endogenous definition of the groups, based, e.g., on statistical tools aimed to identify clusters, according to the similarity among the individual attributes.

# 8 Concluding remarks

We have proposed a multidimensional approach to polarization measurement, in order to include monetary and non-monetary attributes besides income. Our point of view on polarization focuses on the presence of two or more groups in the society, which are similar inside, distant to each other and equal in size.

We have proposed a new class of multivariate polarization indices, which are functions of three components: inequality between groups, inequality within groups and groups size. We have introduced indices of groups size, which measure the degree of similarity in population shares among the clusters. Exploiting the decomposition by groups of certain multivariate inequality measures, we have then used the two components of between and within inequality, in order to obtain a general class of multivariate polarization measures.

The new indices are general, in sense that they apply to any grouped distribution and require no fixed relative groups sizes. They evaluate the total data and their grouping as well; moreover, they may also be used to compare alternative groupings.

Many properties have been investigated, which are multidimensional extensions of the classical univariate polarization axioms, and the conditions, under which our indices satisfy them, have been analyzed.

In the multi-attribute analysis, interactions between attributes have to be taken into account. We have handled this problem from an evaluative point of view, considering their association in terms of substitutional or complementary goods. If one ignores such aspect, all above multivariate inequality measures can be used to construct a polarization measure of form (1). However, if interactions are considered as relevant, the range of choices in our approach is reduced to those inequality measures which allow for such evaluation; in particular, the multivariate extensions of the generalized entropy measure and of the Atkinson's index proposed by Maasoumi (1986) are appropriate for such intent.

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Table 1: Special indices (3) of relative groups size.

| Index based on | $S(\mathbf{X})$ | $C(\mathbf{X})$ | $\phi^{-1}(C)$ |
|---|---|---|---|
| Herfindahl | $\frac{\left(\sum_{g=1}^{G}\left(\frac{N_g}{N}\right)^2\right)^{-1}-1}{G-1}$ | $\sum_{g=1}^{G}\left(\frac{N_g}{N}\right)^2$ | $\frac{1}{C}$ |
| Rosenbluth | $\frac{\left(2\sum_{g=1}^{G}\frac{N_g g}{N}\right)-2}{G-1}$ | $\left(\left(2\sum_{g=1}^{G}\frac{N_g g}{N}\right)-1\right)^{-1}$ | $\frac{1}{C}$ |
| Entropy | $\frac{\left(\prod_{g=1}^{G}\left(\frac{N_g}{N}\right)^{-\frac{N_g}{N}}\right)-1}{G-1}$ | $\sum_{g=1}^{G}\frac{N_g}{N}\log\left(\frac{N_g}{N}\right)^{*}$ | $\exp\{-C\}$ |

$^{*}$ with $0\cdot\log 0 = 0$

Table 2: Additively decomposable inequality measures $I(\mathbf{X})$ of type (8).

| Index | $f(y)$ | $s_i$ | $\bar{s}$ | $\bar{s}^g$ | $h(t;\bar{t})$ | $w_g$ |
|---|---|---|---|---|---|---|
| $GEM_{\gamma\neq -1,0}$ | $\frac{y}{\gamma(1+\gamma)}$ | $\left(\sum_{k=1}^K \delta_k x_{ik}^{-\beta}\right)^{-\frac{1}{\beta}}$ | $\frac{1}{N}\sum_{i=1}^N s_i$ | $\frac{1}{N_g}\sum_{i\in g} s_i$ | $(t/\bar{t})^{1+\gamma}-1$ | $\frac{N_g}{N}\left(\frac{\bar{s}^g}{\bar{s}}\right)^{1+\gamma}$ |
| $GEM_{-1}$ | $y$ | $\left(\sum_{k=1}^K \delta_k x_{ik}^{-\beta}\right)^{-\frac{1}{\beta}}$ | $\frac{1}{N}\sum_{i=1}^N s_i$ | $\frac{1}{N_g}\sum_{i\in g} s_i$ | $\log(\bar{t}/t)$ | $\frac{N_g}{N}$ |
| $GEM_0$ | $y$ | $\left(\sum_{k=1}^K \delta_k x_{ik}^{-\beta}\right)^{-\frac{1}{\beta}}$ | $\frac{1}{N}\sum_{i=1}^N s_i$ | $\frac{1}{N_g}\sum_{i\in g} s_i$ | $(t/\bar{t})\log(t/\bar{t})$ | $\frac{N_g\bar{s}^g}{N\bar{s}}$ |
| GET | $\rho\cdot y$ | $\prod_{k=1}^K (x_{ik})^{c_k}$ | $\prod_{k=1}^K (\bar{x}_k)^{c_k}$ | $\prod_{k=1}^K (\bar{x}_k^g)^{c_k}$ | $(t/\bar{t})-1$ | $\frac{N_g}{N}\prod_{k=1}^K\left(\frac{\bar{x}_k^g}{\bar{x}_k}\right)^{c_k}$ |
| KT | $\frac{\ln(y)}{\sum_k c_k}$ | $\sum_{k=1}^K c_k x_{ik}$ | $\sum_{k=1}^K c_k\bar{x}_k$ | $\sum_{k=1}^K c_k\bar{x}_k^g$ | $\exp\{\bar{t}-t\}$ | $\frac{N_g}{N}$ |

Table 3: Multiplicatively decomposable similarity measures (22).

| Index | $f(y)$ | $s_i$ | $\bar{s}$ | $\bar{s}^g$ | $h(t,\bar{t})$ | $w_g$ | $\epsilon$ |
|---|---|---|---|---|---|---|---|
| $AM_{v\neq 1}$ | $y^{\frac{1}{(1-v)}}$ | $\left(\sum_{k=1}^{K}\delta_k x_{ik}^{-\beta}\right)^{-\frac{1}{\beta}}$ | $\frac{1}{N}\sum_{i=1}^{N}s_i$ | $\frac{1}{N_g}\sum_{i\in g}s_i$ | $(t/\bar{t})^{1-v}$ | $\frac{\frac{N_g}{N}\left(\frac{\bar{s}^g}{\bar{s}}\right)^{1-v}}{\sum_{g=1}^{G}\frac{N_g}{N}\left(\frac{\bar{s}^g}{\bar{s}}\right)^{1-v}}$ | $1-v$ |
| $AM_1$ | $\exp(y)$ | $\left(\sum_{k=1}^{K}\delta_k x_{ik}^{-\beta}\right)^{-\frac{1}{\beta}}$ | $\frac{1}{N}\sum_{i=1}^{N}s_i$ | $\frac{1}{N_g}\sum_{i\in g}s_i$ | $\log(t/\bar{t})$ | $\frac{N_g}{N}$ | – |
| $AT_1$ | $y^{\frac{1}{\sum_k r_k}}$ | $\prod_{k=1}^{K}(x_{ik})^{r_k}$ | $\prod_{k=1}^{K}(\bar{x}_k)^{r_k}$ | $\prod_{k=1}^{K}(\bar{x}_k^g)^{r_k}$ | $t/\bar{t}$ | $\frac{\frac{N_g}{N}\prod_k\left(\frac{\bar{x}_k^g}{\bar{x}_k}\right)^{r_k}}{\sum_{g=1}^{G}\frac{N_g}{N}\prod_k\left(\frac{\bar{x}_k^g}{\bar{x}_k}\right)^{r_k}}$ | $\sum_{k=1}^{K}r_k$ |
| $AT_2$ | $(\exp(y))^{\frac{1}{\sum_k r_k}}$ | $\prod_{k=1}^{K}(x_{ik})^{r_k}$ | $\prod_{k=1}^{K}(\bar{x}_k)^{r_k}$ | $\prod_{k=1}^{K}(\bar{x}_k^g)^{r_k}$ | $\log(t/\bar{t})$ | $\frac{N_g}{N}$ | - |

Table 4: Polarization measures of income and education.

| Measure P based on | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 |
|---|---|---|---|---|---|---|---|---|---|
| | | | | RATIO-BASED MEASURES | | | | | |
| GMD | 1.0e-5 | 8.6e-6 | 8.4e-6 | 7.2e-6 | 7.5e-6 | 7.2e-6 | 7.9e-6 | 8.5e-6 | 8.9e-6 |
| KT c=(0.01;0.01) | 6.7e-8 | 1.3e-7 | 4.2e-7 | 3.0e-7 | 1.6e-8 | 3.2e-7 | 1.3e-9 | 1.0e-8 | 2.5e-7 |
| $GEM_{1.5}, \beta$=-1* | 0.00127 | 0.00093 | 0.00090 | 0.00068 | 0.00067 | 0.00065 | 0.00082 | 0.00092 | 0.00109 |
| $GEM_{1.5}, \beta$=-0.5* | 0.00120 | 0.00087 | 0.00084 | 0.00063 | 0.00063 | 0.00060 | 0.00077 | 0.00086 | 0.00103 |
| $GEM_{1.5}, \beta$=9* | 0.00003 | 0.00003 | 0.00004 | 0.00005 | 0.00003 | 0.00003 | 0.00000 | 0.00001 | 0.00000 |
| $GEM_{1.5}, \delta$=(0.8;0.2) ** | 0.00125 | 0.00092 | 0.00088 | 0.00067 | 0.00066 | 0.00064 | 0.00081 | 0.00090 | 0.00107 |
| $GEM_{1.5}, \delta$=(0.2;0.8) ** | 0.00099 | 0.00072 | 0.00069 | 0.00051 | 0.00051 | 0.00049 | 0.00065 | 0.00072 | 0.00088 |
| $AM_{0.5}, \beta$=-1* | 0.00081 | 0.00060 | 0.00057 | 0.00041 | 0.00041 | 0.00039 | 0.00051 | 0.00055 | 0.00071 |
| $AM_{0.5}, \beta$=9* | 0.00001 | 0.00002 | 0.00002 | 0.00002 | 0.00001 | 0.00001 | 0.00000 | 0.00000 | 0.00000 |
| $AM_2, \beta$=-1* | 0.00202 | 0.00131 | 0.00103 | 0.00088 | 0.00101 | 0.00116 | 0.00112 | 0.00140 | 0.00180 |
| $AM_2, \beta$=-0.5* | 0.00189 | 0.00122 | 0.00094 | 0.00081 | 0.00095 | 0.00113 | 0.00109 | 0.00132 | 0.00168 |
| $AT_1$ r=(0.5;0.4) | 0.00057 | 0.00044 | 0.00043 | 0.00034 | 0.00031 | 0.00029 | 0.00032 | 0.00035 | 0.00045 |
| | | | | DIFFERENCE-BASED MEASURES | | | | | |
| GMD | 0.2198 | 0.2148 | 0.2138 | 0.2128 | 0.2175 | 0.2152 | 0.2191 | 0.2233 | 0.2223 |
| KT c=(0.01;0.01) | 0.0121 | 0.0117 | 0.0121 | 0.0122 | 0.0120 | 0.0143 | 0.0107 | 0.0118 | 0.0131 |
| $GEM_{1.5}$, $\beta$=-1* | 0.1598 | 0.1542 | 0.1537 | 0.1568 | 0.1615 | 0.1613 | 0.1597 | 0.1670 | 0.1572 |
| $GEM_{1.5}$ $\beta$=-0.5* | 0.1614 | 0.1561 | 0.1556 | 0.1583 | 0.1630 | 0.1626 | 0.1615 | 0.1683 | 0.1594 |
| $GEM_{1.5}$, $\beta$=9* | 0.1817 | 0.1793 | 0.1782 | 0.1779 | 0.1812 | 0.1793 | 0.1828 | 0.1852 | 0.1857 |
| $GEM_{1.5}$, $\delta$=(0.8;0.2)** | 0.1602 | 0.1547 | 0.1541 | 0.1571 | 0.1618 | 0.1616 | 0.1602 | 0.1673 | 0.1577 |
| $GEM_{1.5}$, $\delta$=(0.2;0.8)** | 0.1656 | 0.1609 | 0.1604 | 0.1622 | 0.1666 | 0.1658 | 0.1659 | 0.1715 | 0.1647 |
| $AM_{0.5}$, $\beta$=-1 * | 0.1767 | 0.1735 | 0.1726 | 0.1725 | 0.1761 | 0.1745 | 0.1772 | 0.1803 | 0.1795 |
| $AM_{0.5}$ $\beta$=9 * | 0.1833 | 0.1808 | 0.1796 | 0.1794 | 0.1828 | 0.1810 | 0.1843 | 0.1868 | 0.1873 |
| $AM_2$, $\beta = -1$ * | 0.1016 | 0.0854 | 0.0656 | 0.0820 | 0.1033 | 0.1278 | 0.0882 | 0.1070 | 0.1059 |
| $AM_2$, $\beta = -0.5$ * | 0.1034 | 0.0873 | 0.0650 | 0.0828 | 0.1071 | 0.1358 | 0.0937 | 0.1100 | 0.1066 |
| $AT_1$ r=(0.5;0.4) | 0.1807 | 0.1778 | 0.1768 | 0.1766 | 0.1801 | 0.1783 | 0.1814 | 0.1841 | 0.1840 |

*$\delta$=(0.5;0.5). ** $\beta = -0.5$.

Table 5: Relative groups size (S), between- and within-groups inequality measures of income and education.

| | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 |
|---|---|---|---|---|---|---|---|---|---|
| MEASURE S | 0.3694 | 0.3646 | 0.3624 | 0.3617 | 0.3684 | 0.3645 | 0.3716 | 0.3765 | 0.3775 |
| | BETWEEN GROUP INEQUALITY | | | | | | | | |
| GMD | 184.08 | 156.15 | 154.66 | 132.04 | 128.04 | 128.08 | 151.14 | 158.67 | 182.65 |
| KT c=(0.01;0.01) | 4.4593 | 8.7152 | 27.8416 | 19.8079 | 0.9960 | 21.1264 | 0.0934 | 0.7299 | 18.0606 |
| $GEM_{1.5}, \beta = -1^*$ | 0.0042 | 0.0032 | 0.0031 | 0.0023 | 0.0022 | 0.0021 | 0.0027 | 0.0029 | 0.0037 |
| $GEM_{1.5}, \beta = -0.5^*$ | 0.0039 | 0.0030 | 0.0029 | 0.0021 | 0.0020 | 0.0019 | 0.0025 | 0.0027 | 0.0034 |
| $GEM_{1.5}, \beta = 9^*$ | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| $GEM_{1.5}\delta = (0.8; 0.2)^{**}$ | 0.0041 | 0.0031 | 0.0030 | 0.0022 | 0.0021 | 0.0021 | 0.0027 | 0.0028 | 0.0036 |
| $GEM_{1.5}\delta = (0.2; 0.8)^{**}$ | 0.0031 | 0.0023 | 0.0022 | 0.0016 | 0.0016 | 0.0015 | 0.0020 | 0.0022 | 0.0028 |
| $AM_{0.5}, \beta = -1^*$ | 0.0023 | 0.0018 | 0.0017 | 0.0012 | 0.0012 | 0.0011 | 0.0015 | 0.0016 | 0.0020 |
| $AM_{0.5}, \beta = 9^*$ | 0.0000 | 0.0000 | 0.0001 | 0.0001 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $AM_2, \beta = -1^*$ | 0.0102 | 0.0076 | 0.0073 | 0.0052 | 0.0050 | 0.0048 | 0.0063 | 0.0067 | 0.0087 |
| $AM_2, \beta = -0.5^*$ | 0.0094 | 0.0069 | 0.0067 | 0.0048 | 0.0046 | 0.0044 | 0.0058 | 0.0062 | 0.0081 |
| $AT_1$ r=(0.5;0.4) | 0.0016 | 0.0012 | 0.0012 | 0.0010 | 0.0009 | 0.0008 | 0.0009 | 0.0010 | 0.0012 |
| | WITHIN GROUP INEQUALITY | | | | | | | | |
| GMD | 1283.68 | 1317.50 | 1315.23 | 1315.38 | 1262.00 | 1284.56 | 1356.78 | 1325.54 | 1440.35 |
| KT c=(0,01;0,01) | 4776.24 | 4685.29 | 4741.23 | 4714.24 | 4630.18 | 4686.12 | 4925.59 | 4958.22 | 5032.95 |
| $GEM_{1.5}, \beta = -1^*$ | 0.2193 | 0.2498 | 0.2462 | 0.2145 | 0.1982 | 0.1850 | 0.2266 | 0.1823 | 0.2724 |
| $GEM_{1.5}, \beta = -0.5^*$ | 0.2044 | 0.2322 | 0.2283 | 0.2001 | 0.1850 | 0.1732 | 0.2108 | 0.1710 | 0.2529 |
| $GEM_{1.5}, \beta = 9^*$ | 0.0255 | 0.0258 | 0.0262 | 0.0257 | 0.0255 | 0.0252 | 0.0254 | 0.0253 | 0.0252 |
| $GEM_{1.5}, \delta = (0.8; 0.2)^{**}$ | 0.2158 | 0.2456 | 0.2420 | 0.2111 | 0.1951 | 0.1822 | 0.2229 | 0.1796 | 0.2677 |
| $GEM_{1.5}, \delta = (0.2; 0.8)^{**}$ | 0.1675 | 0.1888 | 0.1845 | 0.1645 | 0.1522 | 0.1440 | 0.1722 | 0.1427 | 0.2052 |
| $AM_{0.5}, \beta = -1^*$ | 0.0683 | 0.0745 | 0.0736 | 0.0713 | 0.0674 | 0.0661 | 0.0713 | 0.0659 | 0.0757 |
| $AM_{0.5}, \beta = 9^*$ | 0.0117 | 0.0123 | 0.0138 | 0.0124 | 0.0117 | 0.0111 | 0.0121 | 0.0120 | 0.0122 |
| $AM_2 \beta = -1^*$ | 0.5788 | 0.6711 | 0.7924 | 0.6878 | 0.5638 | 0.4009 | 0.6635 | 0.5558 | 0.5651 |
| $AM_2 \beta = -0.5^*$ | 0.5672 | 0.6587 | 0.7961 | 0.6821 | 0.5398 | 0.3478 | 0.6291 | 0.5372 | 0.5605 |
| $AT_1$ r=(0.5;0.4) | 0.0354 | 0.0389 | 0.0386 | 0.0372 | 0.0350 | 0.0345 | 0.0372 | 0.0346 | 0.0396 |

$^*\delta = (0.5; 0.5)$. $\quad$ $^{**}\ \beta = -0.5$.

Figure 1: Example: No geometric overlap (groups divided by a straight line), but $\Delta_{OV} > 0$.

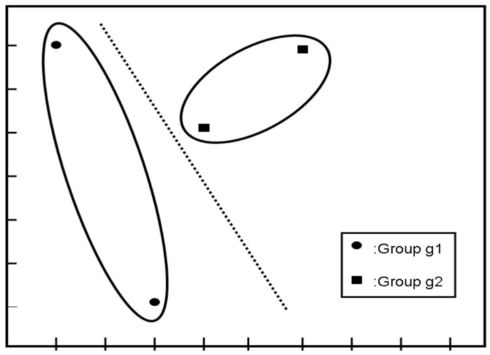



Figure 2: Polarization measures of income and education.

(a) Ratio-based measures

(b) Difference-based measures

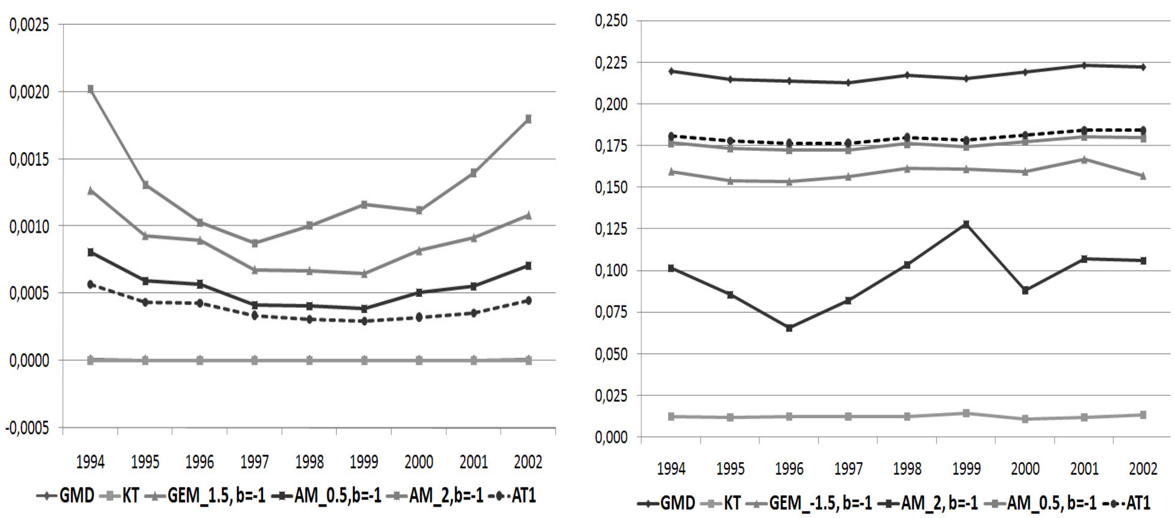


Figure 3: Relative groups size (S), between- and within-groups inequality measures of income and education.

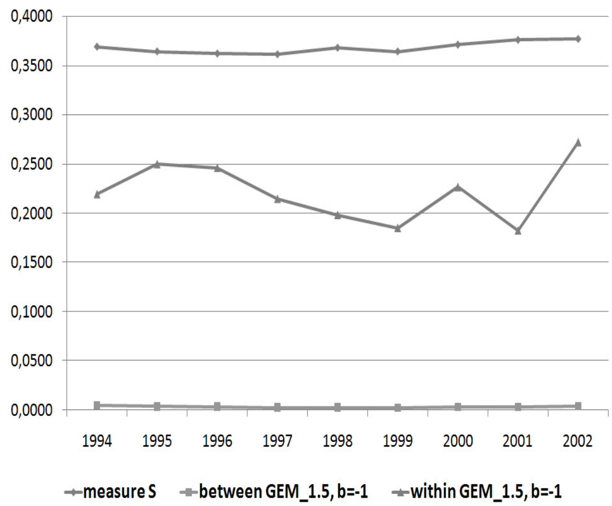
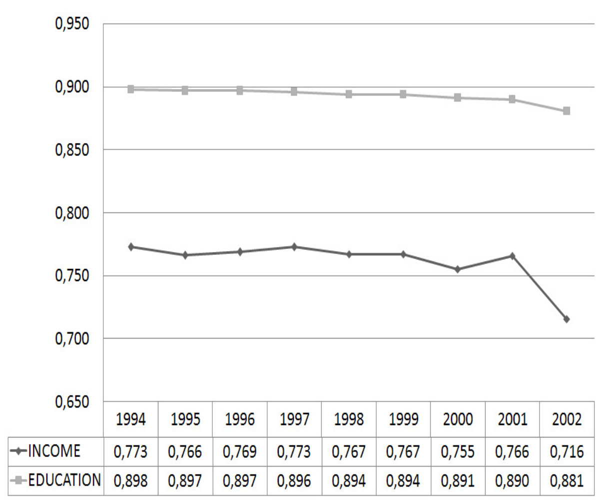



Figure 4: Univariate polarization measure $P^G$ income and of education.




| | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 |
|---|---|---|---|---|---|---|---|---|---|
| INCOME | 0,773 | 0,766 | 0,769 | 0,773 | 0,767 | 0,767 | 0,755 | 0,766 | 0,716 |
| EDUCATION | 0,898 | 0,897 | 0,897 | 0,896 | 0,894 | 0,894 | 0,891 | 0,890 | 0,881 |