Multivariate Lorenz dominance based on zonoids*

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Abstract

The classical Lorenz curve visualizes and measures the disparity of items which are characterized by a single variable: The more the curve bends, the more scatter the data. Recently a general approach has been proposed and investigated that measures the disparity of multidimensioned items regardless of their dimension.

This paper surveys various generalizations of Lorenz curve and Lorenz dominance for multidimensional data. Firstly, the Lorenz zonoid of multivariate data and, more general, of a random vector is introduced. Then three multivariate extensions of univariate Lorenz dominance are surveyed and contrasted, the set inclusion of lift zonoids, the scaled convex order, and the price Lorenz order. The latter is based on the set inclusion of extended Lorenz zonoids. Finally, a decomposition of the multivariate volume-Gini mean difference is given.

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1 Introduction

Max O. Lorenz [27] has introduced a special tool to visualize data. He represented the distribution of relative data (that is, data over their mean) by a curve, his celebrated Lorenz curve. We stress three of Lorenz' achievements: (1) The Lorenz curve is visual: it indicates the degree of disparity 'the more the curve is bent'. (2) Up to a scale parameter, the Lorenz curve fully describes the underlying distribution. (3) The pointwise ordering of Lorenz curves provides an ordering of disparity, the Lorenz dominance.

Originally designed to visualize data on household incomes and compare economic inequality, the Lorenz curve has been employed in virtually every field of statistical application. In the economic and social sciences Lorenz dominance has been used to measure economic inequality and poverty [11, 2, 37], industrial concentration [9], social segregation [1], the distribution of publishing activities [8], ecological diversity [31], and many others. The Gini index, which is twice the area between the Lorenz curve and the diagonal of the unit square, serves as the most popular index of income inequality. A comprehensive guide to the widely dispersed literature on Lorenz dominance up to 1993 is provided by [28]; for more recent applications, see [35].

Generally, the Lorenz curve measures sort of scatter or dispersion with respect to a single variable. In many of these applications it is natural to consider several variables instead of a single one and to measure their dispersion in a multivariate setting. E.g., economic inequality with respect to income and wealth, poverty in terms of income, education and health.

In comparing two distributions of multiattribute endowments, we may check each of the attributes separately. However, looking only at the marginal distributions neglects possible dependencies in the value of attributes. E.g., income may be substituted by wealth. In such a case, joint distributions have to be investigated and compared.

Multivariate comparisons are either based on real-valued indexes, which yield complete orderings of distributions, or on partial orderings. The usual Lorenz dominance is a partial ordering in dimension one. Special indexes of multivariate economic inequality have been proposed by [26, 39], and others. Also, partial orderings of multivariate inequality have been introduced by various authors, among them [12, 3, 4, 30]. Kolm [12] orders distributions of multivariate endowments by matrix majorization, which amounts to the convex scaled order mentioned below; in dimension one, it is the usual Lorenz dominance.

This paper surveys several extensions of Lorenz dominance and Lorenz curve to multidimensional data. Each of the multivariate ordering relations discussed in the sequel implies Lorenz dominance of the marginals. From given data, marginal Lorenz dominance can be checked by inspection of the Lorenz curves; this provides a simple necessary condition for multivariate dominance. A sufficient condition for the multivariate relations, which is also necessary for scaled convex order and easy to calculate, will be given below.

In the first part of the paper (Sections 2 and 3) we recall the multivariate generalization of the Lorenz curve, the Lorenz zonoid, and the basic properties of this generalization. The Lorenz zonoid has been introduced by Koshevoy [13, 14] for empirical distributions and by Mosler [29] for general probability distributions; it has been investigated in detail in [18].

Based on this generalization, one can define the multivariate Lorenz dominance via inclusions of the Lorenz zonoids (Section 4). One of several equivalent characterizations of this ordering states that a d-dimensional random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ is Lorenz dominated by another random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ if for all $p_1, \ldots, p_d \in \mathbb{R}$, the random variable $\sum_{j=1}^d p_j X_j$ is Lorenz dominated by the random variable $\sum_{j=1}^d p_j Y_j$. Another characterization is through convex-linear functions. We contrast the Lorenz dominance with the scaled convex order, which is similarly characterized by convex functions (Section 5).

In the economic setup, the direction $\boldsymbol{p}$ may be interpreted as a price vector. However, some or all components of $\boldsymbol{p}$ may be negative. If we restrict to nonnegative prices, the price Lorenz dominance is obtained (Section 6). It is also characterized by set inclusions. The Lorenz zonoid is augmented by adding a polar price cone to it. ([15]). The boundary of this sum contains the graph of the inverse Lorenz function ([22, 32]), which has an economic interpretation. The pointwise ordering of the inverse Lorenz functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$ then is equivalent to the usual Lorenz dominance between the random variables $\sum_{j=1}^d p_j X_j$ and $\sum_{j=1}^d p_j Y_j$ for all non-negative p_j's. This ordering has been also applied to the measurement of industrial concentration; see [23], where an empirical application is included.

The second part (Section 7) is devoted to the study of a decomposition of the multivariate Gini mean difference that is based on the volume of the Lorenz zonoid. The results build on the additivity property of zonoids and Minkowski's theorem on the volumes of the sum of sets. The paper concludes with an outlook for further applications of the lift zonoid approach and for future research on multidimensioned economic inequality.

2 Lift zonoid and Lorenz zonoid of multivariate data

Here we provide the definitions of the lift zonoid and the Lorenz zonoid, which are basic to our multivariate notions of Lorenz dominance and Gini index. Consider a population of n economic units, say households, which are endowed with quantities of d commodities or other attributes of well-being. Let $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{id}) \in \mathbb{R}^d$ denote the endowment of household i, $i = 1, \ldots, n$, and the matrix

$$\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)' = \begin{pmatrix} x_{11} & \ldots & x_{1d} \\ \vdots & & \vdots \\ x_{n1} & \ldots & x_{nd} \end{pmatrix}$$

describe an *empirical distribution* $F_{\boldsymbol{X}}$ of endowments among households[1]. Assume that, in every attribute j, the mean endowment is positive. For each j we may consider the classical Lorenz curve. It refers to relative data, i.e., data over their mean,

$$\widetilde{x}_{ij} = \frac{n\ x_{ij}}{\sum_{k=1}^{n} x_{kj}}\,, \qquad \widetilde{\boldsymbol{x}}_i = (\widetilde{x}_{i1}, \ldots, \widetilde{x}_{id})\,, \qquad \tilde{\boldsymbol{X}} = (\widetilde{\boldsymbol{x}}_1, \ldots, \widetilde{\boldsymbol{x}}_n)'\,.$$

Note that each attribute is measured on a metric scale so that its mean is defined but (e.g. with health status) not necessarily its total value. (In Sections 2 to 5 of this paper we allow for negative values of the attributes, but all definitions and results may as well be restricted to nonnegative endowments.)

[1]Note that we use the same symbol $\boldsymbol{X}$ for a random variable in $\mathbb{R}^d$ and an $n \times d$ data matrix, as the latter corresponds to a random vector having an empirical distribution.

In case of only one attribute, say income, the endowments are real numbers $x_1, \ldots, x_n$ and can be ordered. Let $\widetilde{x}_{(1)}, \ldots, \widetilde{x}_{(n)}$ be the $\widetilde{x}_i$'s in ascending order. The *Lorenz curve* is the piecewise linear connection of points

$$\left(\frac{k}{n}, \frac{1}{n} \sum_{i=1}^{k} \widetilde{x}_{(i)} \right), \quad k = 0, \ldots, n, \tag{1}$$

in the unit square. The *generalized Lorenz curve* is the Lorenz curve with $\widetilde{x}_{(i)}$ replaced by $x_{(i)}$.

Example 1: Consider two households which receive $x_1 = 2400$ and $x_2 = 5600$ dollars of income, respectively. In commemorating Lorenz' work, we use his original version of his curve, which is above the square's diagonal and concave. The bold lines in Figure 1 show the Lorenz curve (a) and the generalized Lorenz curve (b) of this two-point distribution. Each point z_0, z_1 on the Lorenz curve indicates that the poorer $z_0 \cdot 100$ per cent of the population receives $z_1 \cdot 100$ per cent of total income. Note that the first coordinate is depicted in vertical direction.

The data $X = (x_1, \ldots, x_n)'$ of a single attribute is *Lorenz dominated* by some other data $Y = (y_1, \ldots, y_m)'$, $X \preceq_L^1 Y$, if the Lorenz curve of X lies, in Lorenz' orientation of the coordinate system, below the Lorenz curve of Y; see Figure 1. This is usually interpreted that X contains less inequality than Y. If $n = m$, Lorenz dominance is equivalent to *majorization* of n-vectors; $X \preceq_L^1 Y$ if and only if

$$\sum_{i=1}^{k} \widetilde{x}_{(i)} \geq \sum_{i=1}^{k} \widetilde{y}_{(i)} \quad \text{for } k = 1, \ldots, n-1\,. \tag{2}$$

Definition (1) and restriction (2) do not easily carry over to more than one attribute, since there is no natural complete order of d-dimensional space when $d > 1$. To avoid the ordering of vectors, the principal idea is: Regard each Lorenz curve as the boundary of a properly chosen set and order these sets by inclusion. But how should such a set be constructed? A natural choice is to employ a centrally symmetric set. Then, in case $d = 1$, the set is north-west bordered by the Lorenz curve and south-east bordered by the curve symmetric to it, that is, the piecewise linear connection of points

$$\left(\frac{n-k}{n}, 1 - \frac{1}{n} \sum_{i=1}^{k} \widetilde{x}_{(i)} \right), \quad k = 0, \ldots, n\,.$$

This curve is named the *dual Lorenz curve*. In Figure 1(a), it corresponds to the lower border of the hatched area.

For general $d \geq 1$, the *lift zonoid* of the data $\boldsymbol{X} = [\boldsymbol{x}_1 \dots \boldsymbol{x}_n]'$ is a set in $(d+1)$-space, defined by

$$\widehat{Z}(\boldsymbol{X}) = \frac{1}{n} \sum_{i=1}^{n} [(0, \mathbf{0}), (1, \boldsymbol{x}_i)] \; . \tag{3}$$

Here $[(0, \mathbf{0}), (1, \boldsymbol{x}_i)]$ denotes the line segment that extends from the origin $(0, \mathbf{0})$ to the point $(1, \boldsymbol{x}_i)$ in $\mathbb{R}^{d+1}$. The sum is the Minkowski sum, that is, the sets are added point by point. (For example the Minkowski sum of the two segments $[(0,0),(1,0.6)]$ and $[(0,0),(1,1.4)]$ in $\mathbb{R}^2$ amounts to the parallelogram spanned by the vectors $(1, 0.6)$ and $(1, 1.4)$.) As the zonoid $\widehat{Z}(\boldsymbol{X})$ in (3) is a convex polytope, it is also called the *lift zonotope*. We define the *Lorenz zonoid* of $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ as the lift zonoid of the relative data,

$$LZ(\boldsymbol{X}) = \widehat{Z}(\tilde{\boldsymbol{X}}) \, .$$

In Example 1 of two households, with $d = 1$, obtain the Lorenz zonoid

$$LZ(2400, 5600) = \widehat{Z}(0.6, 1.4) = \frac{1}{2}[(0,0),(1,0.6)] + \frac{1}{2}[(0,0),(1,1.4)] \, ,$$

which is depicted in Figure 1(a). Here, the upper border (in boldface) describes the Lorenz curve, while the lower border, the dual Lorenz curve, is rotation symmetric to it.

Example 2 (Egalitarian distribution): For general $d \geq 1$, let $\boldsymbol{E_c} = (\boldsymbol{c}, \dots, \boldsymbol{c})$ be an *egalitarian distribution*, where every household i has the same endowment $\boldsymbol{c} = (c_1, \dots, c_d)$. The lift zonoid of this distribution is

$$\widehat{Z}(\boldsymbol{E_c}) = \frac{1}{n} \sum_{i=1}^{n} [(0, \mathbf{0}), (1, \boldsymbol{c})] = [(0, \mathbf{0}), (1, \boldsymbol{c})] \; ,$$

the line segment connecting the origin with the point $(1, c_1, \dots, c_d)$, and the Lorenz zonoid is the line segment that extends from the origin to the point point $(1, 1, \dots, 1) \in \mathbb{R}^{d+1}$.

Example 3: As an example with $d = 2$ attributes consider three households, one having wealth 90000 and income 0, the other two having wealth 0 and income 3000 dollars, hence

$$\boldsymbol{X} = \begin{bmatrix} 90000 & 0 \\ 0 & 3000 \\ 0 & 3000 \end{bmatrix}, \qquad \tilde{\boldsymbol{X}} = \begin{bmatrix} 3 & 0 \\ 0 & 1.5 \\ 0 & 1.5 \end{bmatrix},$$

Then (see Figure 2)

$$LZ(\boldsymbol{X}) = \widehat{Z}(\tilde{\boldsymbol{X}}) = \frac{1}{3}[(0,0,0),(1,3,0)] + \frac{2}{3}[(0,0,0),(1,0,1.5)]\,.$$

3 Lift zonoid and Lorenz zonoid of a random vector

In many applications it is convenient to employ random variables and general probability distributions rather than empirical distributions only. Already Max Lorenz discussed his curve for continuous distributions. The Lorenz curve of a random variable X having finite expectation $\mu > 0$ is the graph of the function

$$t \mapsto \frac{1}{\mu}\int_0^t F_X^{-1}(s)\, ds\,, \quad 0 \le t \le 1\,,$$

where F_X^{-1} is the quantile function of X, $F_X^{-1}(t) = \min\{x : F(x) \ge t\}, 0 < t \le 1$.

The *lift zonoid* of a general d-variate random vector is defined as follows: Consider the set $\mathcal{X}^d$ of random vectors in $\mathbb{R}^d$ that have finite expectation, the subset $\mathcal{X}^{d+} \subset \mathcal{X}^d$ of those vectors that have positive (in each component) expectation, and the subset $\mathcal{X}_+^{d+} \subset \mathcal{X}^{d+}$ of those that have, in addition, support in $\mathbb{R}_+^d$. Define, for $\boldsymbol{X} \in \mathcal{X}^d$, the *lift zonoid*

$$\widehat{Z}(\boldsymbol{X}) = \Big\{ \mathrm{E}\left[(g(\boldsymbol{X}), g(\boldsymbol{X})\boldsymbol{X})\right] \,:\, g : \mathbb{R}^d \to [0,1] \;\text{ measurable} \Big\}\,. \tag{4}$$

The lift zonoid is a convex compact set in $\mathbb{R}^{d+1}$ that includes the expectations of all d-variate random vectors $(g(\boldsymbol{X}), g(\boldsymbol{X})\boldsymbol{X})$. With data

$\boldsymbol{X} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)'$, the definition (4) of the lift zonoid specializes to the previous one (3): By setting $\lambda_i = \frac{1}{n} g(\boldsymbol{x}_i)$ we see that

$$\begin{aligned}\widehat{Z}(\boldsymbol{X}) &= \left\{ \left(\sum_{i=1}^{n} \lambda_i, \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i \right) : 0 \le \lambda_i \le \frac{1}{n}, i = 1, \ldots, n \right\} \\ &= \frac{1}{n} \sum_{i=1}^{n} [(0, \mathbf{0}), (1, \boldsymbol{x}_i)] \ .\end{aligned}$$

The lift zonoid has many attractive properties, which make it useful for a broad range of applications. Here we mention only three of them. For proofs of the subsequent propositions and many more properties and applications the reader is referred to [21] and the comprehensive monograph [32].

The first important property is uniqueness. Like the generalized Lorenz curve in dimension one, the lift zonoid contains full information about the underlying distribution, that is, given the lift zonoid, the data can be completely regained.

Proposition 1 (Uniqueness). *The lift zonoid $\widehat{Z}(\boldsymbol{X})$ uniquely determines the distribution $F_{\boldsymbol{X}}$, for $\boldsymbol{X} \in \mathcal{X}^d$.*

The second property regards marginalization. If we restrict to one or several dimensions of the data, the lift zonoid of the marginal distribution is a simple projection of the lift zonoid of the joint distribution:

Proposition 2 (Marginalization). *For any $J \subset \{1, 2, \ldots d\}$ consider the projection $pr_J : \boldsymbol{z} \mapsto (z_0, \boldsymbol{z}_J)$, $\boldsymbol{z} \in \mathbb{R}^{d+1}$, where $\boldsymbol{z}_J$ is the subvector containing components $z_j, j \in J$. Then*

$$pr_J(\widehat{Z}(\boldsymbol{X})) = \widehat{Z}(\boldsymbol{X}_J) \,.$$

Thirdly, we state that the lift zonoid is additive in the distribution:

Proposition 3 (Additivity). *For two random vectors $\boldsymbol{X}$ and $\boldsymbol{Z}$ in $\mathcal{X}^d$ with distributions $F_{\boldsymbol{X}}$ and $F_{\boldsymbol{Z}}$ and $\alpha \in [0, 1]$,*

$$\widehat{Z}(\alpha F_{\boldsymbol{X}} + (1 - \alpha) F_{\boldsymbol{Z}}) = \alpha \widehat{Z}(F_{\boldsymbol{X}}) + (1 - \alpha) \widehat{Z}(F_{\boldsymbol{Z}}) \,.$$

For data $\boldsymbol{X}$ and $\boldsymbol{Y}$, the last equation reads

$$\widehat{Z}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m) = \frac{n}{n+m}\,\widehat{Z}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) + \frac{m}{n+m}\,\widehat{Z}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m)\,, \tag{5}$$

which is immediately seen from the definition (3). The additivity property is useful with indices based on the lift zonoid; it allows to decompose an index into subgroups of the population.

Now, the general definition of the *Lorenz zonoid* of a random vector and its properties are straightforward ([18]). The Lorenz zonoid of some $\boldsymbol{X} \in \mathcal{X}^{d+}$ is the lift zonoid of the relative vector, i.e. the vector componentwise divided by its expectation. With

$$\tilde{\boldsymbol{X}} = \left(\frac{X_1}{\mathrm{E}[X_1]}, \ldots, \frac{X_d}{\mathrm{E}[X_d]} \right)$$

define

$$LZ(\boldsymbol{X}) = \widehat{Z}(\tilde{\boldsymbol{X}}) = \left\{ \mathrm{E}\left[\left(g(\tilde{\boldsymbol{X}}), g(\tilde{\boldsymbol{X}})\tilde{\boldsymbol{X}} \right) \right] \;:\; g : \mathbb{R}^d \to [0,1] \;\text{ measurable} \right\} \tag{6}$$

The Lorenz zonoid of a random vector $\boldsymbol{X} \in \mathcal{X}^{d+}$ is a convex compact set in $\mathbb{R}^{d+1}$. Moreover, if $\boldsymbol{X} \in \mathcal{X}_+^{d+}$, i.e., has support in $\mathbb{R}_+^d$, the Lorenz zonoid is contained in the unit hypercube of $\mathbb{R}^{d+1}$.

Economic interpretation of the Lorenz zonoid: The function $g : \mathbb{R}_+^d \to [0,1]$ can be seen as a *selection* of some part of the population. Of all households which have relative endowment vector $\tilde{\boldsymbol{X}}$ the percentage $g(\tilde{\boldsymbol{X}})$ is selected. $\mathrm{E}[\tilde{\boldsymbol{X}} g(\tilde{\boldsymbol{X}})]$ amounts to the *total portion vector* held by this subpopulation and $\mathrm{E}[g(\tilde{\boldsymbol{X}})]$ is the size of the subpopulation selected by g.

The following three principal properties of the Lorenz zonoid correspond to those of the lift zonoid.
Uniqueness: The Lorenz zonoid of a random vector determines its distribution uniquely, up to d scale factors.
Marginalization: The Lorenz zonoid of a marginal $\boldsymbol{X}_J$ is equal to the projection of the Lorenz zonoid of $\boldsymbol{X}$, for any $J \subset \{1, 2, \ldots d\}$.

For the third property, additivity, the Lorenz zonoids have to be rescaled, which is done by multiplying them pointwise with proper diagonal matrices:

Proposition 4 (Additivity). *Given two random vectors* $\boldsymbol{X}$ *and* $\boldsymbol{Z}$ *in* $\mathcal{X}^{d+}$ *with distributions* $F_{\boldsymbol{X}}, F_{\boldsymbol{Z}}$ *and some* $\alpha \in [0,1]$ *consider a random vector* $\boldsymbol{Y}$ *that has distribution* $F_{\boldsymbol{Y}} = \alpha F_{\boldsymbol{X}} + (1-\alpha) F_{\boldsymbol{Z}}$*. Then*[2]

$$\begin{aligned} LZ(\boldsymbol{Y}) &= \alpha\ LZ(\boldsymbol{X})\ \mathrm{diag}\left(1, \frac{\mathrm{E}[X_1]}{\mathrm{E}[Y_1]}, \dots, \frac{\mathrm{E}[X_d]}{\mathrm{E}[Y_d]}\right) \\ &\quad + (1-\alpha)\ LZ(\boldsymbol{Z})\ \mathrm{diag}\left(1, \frac{\mathrm{E}[Z_1]}{\mathrm{E}[Y_1]}, \dots, \frac{\mathrm{E}[Z_d]}{\mathrm{E}[Y_d]}\right). \end{aligned}$$

For proof, use Proposition 2.24 in [32]. More properties of the Lorenz zonoid are found in [18] and [32].

4 Lorenz dominance

Here we define multivariate Lorenz dominance via set inclusion of zonoids. We do this for general random vectors, which includes the case of multivariate data.

For two random vectors in $\mathcal{X}^d$ define the *Lorenz dominance* $\preceq_L^d$ by

$$\boldsymbol{X} \preceq_L^d \boldsymbol{Y} \quad \text{if} \quad LZ(\boldsymbol{X}) \subset LZ(\boldsymbol{Y}). \tag{7}$$

As the Lorenz zonoid of a random vector depends on its distribution only, we also write $F_{\boldsymbol{X}} \preceq_L^d F_{\boldsymbol{Y}}$ in place of $\boldsymbol{X} \preceq_L^d \boldsymbol{Y}$.

First consequences of the definition are: The Lorenz dominance $\preceq_L^d$ is a preorder (reflexive and transitive) on $\mathcal{X}^{d+}$. Its smallest elements are the constant random vectors, which correspond to *egalitarian distributions*. The preorder is scale invariant, $(X_1, \dots, X_d) \preceq_L^d (Y_1, \dots, Y_d)$ implies $(\lambda_1 X_1, \dots, \lambda_d X_d) \preceq_L^d (\lambda_1 Y_1, \dots, \lambda_d Y_d)$ for any $\lambda_1, \dots, \lambda_d > 0$.

When $d = 1$, the Lorenz zonoid is the area between the Lorenz curve and the dual Lorenz curve; thus the preorder reduces to the usual Lorenz dominance.

The following Proposition 5 characterizes the multivariate Lorenz dominance by similar univariate dominances.

[2] $\mathrm{diag}(\lambda_1, \dots, \lambda_d)$ denotes a matrix having elements $\lambda_1, \dots, \lambda_d$ in the diagonal and 0 outside.

Proposition 5 (Characterization of Lorenz dominance). *The following statements ar equivalent:*
(i) $(X_1, \ldots, X_d) \preceq_L^d (Y_1, \ldots, Y_d)$

(ii) *For every* $\boldsymbol{p} = (p_1, \ldots, p_d) \in \mathbb{R}^d$*, the generalized Lorenz curve of*

$$\sum_{j=1}^{d} p_j \widetilde{Y}_j \text{ lies above that of } \sum_{j=1}^{d} p_j \widetilde{X}_j \,.$$

(iii) *For every convex function* $\psi : \mathbb{R} \to \mathbb{R}$ *and* $\boldsymbol{p} \in \mathbb{R}^d$*,*

$$\mathrm{E}\left[\psi\left(\sum_{j=1}^{d} p_j \widetilde{X}_j\right)\right] \leq \mathrm{E}\left[\psi\left(\sum_{j=1}^{d} p_j \widetilde{Y}_j\right)\right] . \tag{8}$$

(iv) *For every increasing*[3] *convex function* $\psi : \mathbb{R} \to \mathbb{R}$ *and* $\boldsymbol{p} \in \mathbb{R}^d$*,* (8) *is satified.*

(v) *For every* $\boldsymbol{p} \in \mathbb{R}^d$ *exists a random variable* $U_{\boldsymbol{p}}$ *such that*
$\mathrm{E}[U_{\boldsymbol{p}} | \sum_{j=1}^d p_j X_j] = 0$ *and*

$$\sum_{j=1}^{d} p_j \widetilde{Y}_j =_{st} \sum_{j=1}^{d} p_j \widetilde{X}_j + U_{\boldsymbol{p}} \,. \tag{9}$$

Here $=_{st}$ indicates equality in distribution. Part (ii) of the Proposition is also called *directional majorization* of $\tilde{Y}$ over $\tilde{X}$. See [5, 10]. It can be interpreted in terms of 'prices' and 'expenditures'. When, with properly chosen units, the mean endowment of each commodity amounts to one and $\boldsymbol{X}$ and $\boldsymbol{Y}$ signify alternative distributions of the commodity vector, then $\sum_{j=1}^d p_j \widetilde{X}_j$ and $\sum_{j=1}^d p_j \widetilde{Y}_j$ stand for the distributions of expenditures given the price vector $\boldsymbol{p}$. The proposition says that $\boldsymbol{X}$ has less multivariate disparity than $\boldsymbol{Y}$ if and only if the first distribution of expenditures is less unequal than the second – in the sense of usual Lorenz dominance – for every price vector in d-space. Note that this also includes negative prices. Later, in Section 6, we will discuss a related ordering that employs nonnegative prices only.

In terms of expected utility, Part (iv) of Proposition 5 means that every risk seeking person which has to choose between the random expenditures

[3] 'Increasing' is always meant in the weak sense.

$\sum_{j=1}^{d} p_j \widetilde{X}_j$ and $\sum_{j=1}^{d} p_j \widetilde{Y}_j$ will, for any prices, prefer the latter, and that every risk averse person will do the opposite. Part (iii) says the same, with not necessarily increasing utilities.

In Part (v), $U_{\boldsymbol{p}}$ may be interpreted as a perturbation of expenditures or 'noise'. The distribution of expenditures for $\tilde{\boldsymbol{Y}}$ is, for all prices, 'noisier' than the distribution of expenditures for $\tilde{\boldsymbol{X}}$.

From the projection property we conclude: $LZ(\boldsymbol{X}) \subset LZ(\boldsymbol{Y})$ implies $LZ(\boldsymbol{X}_J) \subset LZ(\boldsymbol{Y}_J)$ for all $J \subset \{1, \ldots d\}$. Hence Lorenz dominance of two random vectors implies Lorenz dominance of all subvectors. In particular, this holds for every single commodity:

Proposition 6 (Projection). *If* $(X_1, \ldots, X_d) \preceq_L^d (Y_1, \ldots, Y_d)$ *then* $X_j \preceq_L^1 Y_j$ *for* $j = 1, \ldots d$.

But the following example demonstrates that the reverse implication is not true.

Example 4: Let $d = 2, n = 3$, and

$$\tilde{\boldsymbol{X}} = \begin{pmatrix} .25 & .5 \\ .25 & .25 \\ .5 & .25 \end{pmatrix}, \qquad \tilde{\boldsymbol{Y}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\boldsymbol{Y}$ Lorenz dominates $\boldsymbol{X}$ in each single attribute. But the dimension of $LZ(\boldsymbol{X})$ is 3, whereas the dimension of $LZ(\boldsymbol{Y})$ is 2. Therefore $LZ(\boldsymbol{X})$ is no subset of $LZ(\boldsymbol{Y})$, and $\boldsymbol{Y}$ does not dominate $\boldsymbol{X}$.

If the attributes are stochastically independent, a reverse implication holds. Then the multivariate Lorenz dominance is equivalent to the usual Lorenz dominance of the marginals:

Proposition 7. *Assume that the components* $X_1, \ldots, X_d$ *as well the components* $Y_1, \ldots, Y_d$ *are independent. Then* $\boldsymbol{X} \preceq_L^d \boldsymbol{Y}$ *if and only if* $X_j \preceq_L^1 Y_j$ *for all* $j = 1, \ldots, d$.

Finally we mention a result about random vectors with given marginals: One cannot dominate the other unless they have the same distribution.

Proposition 8 (Equal marginals). *If the univariate marginal distributions of two random vectors* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *coincide, then*

$$\boldsymbol{X} \preceq_L^d \boldsymbol{Y} \quad \Rightarrow \quad F_{\boldsymbol{X}} = F_{\boldsymbol{Y}}.$$

5 Scaled convex order

There are many possibilities to extend the classical Lorenz dominance to the multivariate case. The Lorenz dominance introduced above is just one of them. Another one is the multivariate *scaled convex order*, which is now considered and contrasted with the Lorenz dominance.

Proposition 9 (Scaled convex order). *For $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathcal{X}^d$ the following statements are equivalent:*

(i) $\mathrm{E}[\psi(\tilde{\boldsymbol{X}})] \geq \mathrm{E}[\psi(\tilde{\boldsymbol{Y}})]$ *if $\phi : \mathbb{R}^d \to \mathbb{R}$ is concave and the expectations exist,*

(ii) $\mathrm{E}[\phi(\tilde{\boldsymbol{X}})] \leq \mathrm{E}[\phi(\tilde{\boldsymbol{Y}})]$ *if $\phi : \mathbb{R}^d \to \mathbb{R}$ is convex and the expectations exist,*

(iii) $\mathrm{E}[\phi(\tilde{\boldsymbol{X}})] \leq \mathrm{E}[\phi(\tilde{\boldsymbol{Y}})]$ *if $\phi : \mathbb{R}^d \to \mathbb{R}$ is increasing convex and the expectations exist,*

(iv) $\tilde{\boldsymbol{Y}} =_{st} \tilde{\boldsymbol{X}} + \boldsymbol{U}$ *for some $\boldsymbol{U}$ with* $\mathrm{E}[\boldsymbol{U}|\tilde{\boldsymbol{X}}] = 0$.

$\boldsymbol{X}$ is said to be not larger than $\boldsymbol{Y}$ in *scaled convex order*, $\boldsymbol{X} \preceq^d_{scx} \boldsymbol{Y}$, if one of these equivalent restrictions is satisfied. The scaled convex order is a preorder on $\mathcal{X}^d$.

In economic terms, $\boldsymbol{X} \preceq^d_{scx} \boldsymbol{Y}$ means that, if welfare is evaluated by a utilitarian social welfare function based on concave individual relative welfare functions, the distribution of X is always preferred over the distribution of Y.

Many results on the convex order are found in [38] and [35]. The random vector $\boldsymbol{U}$ in Proposition 9(iv) can be interpreted as 'noise', so that $\boldsymbol{Y}$ is distributed as $\boldsymbol{X}$ plus some noise.

In view of Propositions 5(iii) and 9(ii) and, as every convex-linear function is convex, the scaled convex order implies the Lorenz dominance,

$$\boldsymbol{X} \preceq^d_{scx} \boldsymbol{Y} \quad \Rightarrow \quad \boldsymbol{X} \preceq_L \boldsymbol{Y}\,. \tag{10}$$

In case $d = 1$ the scaled convex order is the same as the Lorenz dominance. But for $d \geq 2$ the two orderings coincide in special cases only.

Example 5 (Multivariate normal distributions): Among multivariate normal distributions the two dominance relations are equivalent and characterized as follows:

Assume $\boldsymbol{X} \sim N(\boldsymbol{a}, \boldsymbol{R})$ and $\boldsymbol{Y} \sim N(\boldsymbol{b}, \boldsymbol{S})$ with covariance matrices $\boldsymbol{R} = [r_{ij}]$, $\boldsymbol{S} = [s_{ij}]$ and positive (in each component) mean vectors $\boldsymbol{a} = (a_1, \ldots, a_d)$, $\boldsymbol{b} = (b_1, \ldots, b_d)$. Then

$$\begin{aligned}\boldsymbol{X} \preceq^d_{scx} \boldsymbol{Y} \quad &\Leftrightarrow \quad \boldsymbol{X} \preceq^d_L \boldsymbol{Y} \\ &\Leftrightarrow \quad \sum_{i=1}^{d}\sum_{j=1}^{d} p_i p_j \left(\frac{s_{ij}}{b_i b_j} - \frac{r_{ij}}{a_i a_j} \right) \geq 0 \quad \text{for all } \boldsymbol{p} \in \mathbb{R}^d ,\end{aligned}$$

that is, the covariance matrix of $\tilde{\boldsymbol{Y}}$ is larger than that of $\tilde{\boldsymbol{X}}$ in the usual covariance matrix ordering.

In view of Proposition 9(iv), checking data X and Y for scaled convex order is a relatively simple task. It amounts to solving a linear program:

Proposition 10 (Checking for scaled convex order). *Assume that all rows* $\widetilde{x}_i$ *of* $\tilde{X}$ *are different, and the same for all rows* $\widetilde{y}_i$ *of* $\tilde{Y}$*. Then* $\boldsymbol{X} \preceq^d_{scx} \boldsymbol{Y}$ *if and only there exists a bistochastic matrix* $\Pi = (\pi_{ij})$ *that satisfies*

$$x_i = \sum_{j=1}^{n} \pi_{ij} y_j \,, \quad i = 1, \ldots, n \,.$$

As scaled convex order implies Lorenz order, Proposition 10 provides a sufficient condition for $\boldsymbol{X} \preceq^d_L \boldsymbol{Y}$. A necessary condition has been given in Proposition 6: the Lorenz ordering of all marginals.

6 Extended lift zonoids and price Lorenz dominance

Here we consider a new object, the Minkowski sum of the lift zonoid and a certain orthant of $(d+1)$-space. We call it the extended lift zonoid. Similarly, an extended Lorenz zonoid is introduced, which is the extended lift zonoid of the relative distribution. The boundary of the extended Lorenz zonoid contains the Lorenz hypersurface.

The inclusion of these sets yields an ordering which has a similar characterization as the Lorenz dominance (Proposition 5). In this, the set of all prices (or directions) is replaced by the set of all nonnegative prices.

In the rest of the paper assume that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have support in $\mathbb{R}^d_+$, $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}^{d+}_+$. To construct the *extended lift zonoid* $e\widehat{Z}(\boldsymbol{X})$, the lift zonoid $\widehat{Z}(\boldsymbol{X})$ is augmented by all points that are below a point in $\widehat{Z}(\boldsymbol{X})$ regarding the first coordinate and above the point regarding the other d coordinates. More precisely,

$$\begin{aligned} e\widehat{Z}(\boldsymbol{X}) &= \{(v_0, v_1, \ldots, v_d) \,:\, v_0 \leq z_0, v_j \geq z_j, j = 1, \ldots, d\,, \\ &\qquad\qquad \text{for some } \ (z_0, z_1, \ldots, z_d) \in \widehat{Z}(\boldsymbol{X})\} \\ &= \widehat{Z}(\boldsymbol{X}) + (\mathbb{R}_- \times \mathbb{R}^d_+) \end{aligned}$$

When using the relative random vector $\tilde{\boldsymbol{X}}$ we obtain the *extended Lorenz zonoid*,

$$eLZ(\boldsymbol{X}) = e\widehat{Z}(\tilde{\boldsymbol{X}})\,. \tag{11}$$

Figure 3 exhibits the extended Lorenz zonoid of a two-attribute egalitarian distribution.

The set inclusion of extended Lorenz zonoids provides an ordering,

$$\boldsymbol{X} \preceq^d_{PL} \boldsymbol{Y} \quad \text{if} \quad eLZ(\boldsymbol{X}) \subset eLZ(\boldsymbol{Y})\,, \tag{12}$$

which can be characterized as follows.

Proposition 11. *The following statements ar equivalent:*
(i) $\boldsymbol{X} \preceq^d_{PL} \boldsymbol{Y}$

(ii) *For every* $\boldsymbol{p} = (p_1, \ldots, p_d) \in \mathbb{R}^d_+$,

$$\textit{the generalized Lorenz curve of } \sum_{j=1}^d p_j \widetilde{Y}_j \textit{ lies above that of } \sum_{j=1}^d p_j \widetilde{X}_j\,. \tag{13}$$

(iii) *For every convex function* $\psi : \mathbb{R} \to \mathbb{R}$ *and* $\boldsymbol{p} \in \mathbb{R}^d_+$,

$$\mathrm{E}\left[\psi\left(\sum_{j=1}^d p_j \widetilde{X}_j\right)\right] \leq \mathrm{E}\left[\psi\left(\sum_{j=1}^d p_j \widetilde{Y}_j\right)\right]\,. \tag{14}$$

(iv) *For every increasing convex function* $\psi : \mathbb{R} \to \mathbb{R}$ *and* $\boldsymbol{p} \in \mathbb{R}^d_+$, (14) *is satified.*

In view of Proposition 11, the ordering $\preceq_{PL}^d$ is named *price Lorenz dominance*. The interpretation is that, for all nonnegative price vectors p, expenditures are less dispersed with X than with Y.

When $d = 1$, the extended Lorenz zonoid covers the plane south and east of the Lorenz curve; it is included in another extended Lorenz zonoid if and only if its Lorenz curve lies below the other Lorenz curve.

Similarly, for set inclusion of extended Lorenz zonoids in higher dimensions, only that parts of their boundaries are relevant that lie in the unit hypercube of $\mathbb{R}^{d+1}$. This part of the boundary is named the *Lorenz hypersurface*, in symbols

$$LH(\boldsymbol{X}) = \partial eLZ(\boldsymbol{X}) \cap [0,1]^{d+1} ,$$

where ∂S denotes the boundary of a set S. $LH(\boldsymbol{X})$ is the graph of a function $l_{\boldsymbol{X}}$ which is called the *inverse Lorenz function (ILF)*,

$$l_{\boldsymbol{X}}(\boldsymbol{y}) = \max \mathrm{E}[g(\tilde{\boldsymbol{X}})] , \quad \boldsymbol{y} \in [0,1]^d , \tag{15}$$

where the maximum extends over all functions $g : \mathbb{R}_+^d \to [0,1]$ for which

$$\mathrm{E}[g(\tilde{\boldsymbol{X}})\tilde{\boldsymbol{X}}] \leq \boldsymbol{y}$$

holds in the usual componentwise order of $\mathbb{R}^{d+1}$.

In particular, the ILF of an empirical distribution is a piecewise linear function,

$$l_{\boldsymbol{X}}(\boldsymbol{y}) = \max \left\{ \sum_{i=1}^n \lambda_i \, : \, \sum_{i=1}^n \lambda_i \widetilde{\boldsymbol{x}}_i \leq \boldsymbol{y}, 0 \leq \lambda_i \leq \frac{1}{n} \right\} , \quad \boldsymbol{y} \in [0,1]^d , \tag{16}$$

and the Lorenz hypersurface is a piecewise linear hypersurface whose extreme points are solutions to a linear program.

In case $d = 1$ the ILF is the inverse function of the usual Lorenz function. The Lorenz dominance between univariate distributions consists in the pointwise ordering of their Lorenz functions or, equivalently, of their ILFs.

For general d, the ILF is defined on the unit cube of $\mathbb{R}^d$. Its argument $\boldsymbol{y}$ represents a vector of relative endowments with respect to mean endowments in the d attributes. $l_{\boldsymbol{X}}(\boldsymbol{y})$ equals the maximum percentage of households who hold a vector of shares less than or equal to $\boldsymbol{y} = (y_1, \dots, y_d)$.

Example 6 (Egalitarian distribution): With $\boldsymbol{X} = \boldsymbol{E_c}$ holds $\widetilde{\boldsymbol{x}}_i = (1, \dots, 1)$ for all i,

$$\sum_{i=1}^{n} \lambda_i \widetilde{\boldsymbol{x}}_i = \sum_{i=1}^{n} \lambda_i \, (1, \dots, 1) \leq \boldsymbol{y} \quad \Leftrightarrow \quad 0 \leq \sum_{i=1}^{n} \lambda_i \leq \min_j y_j \, .$$

Hence the ILF is

$$l_{\boldsymbol{E_c}}(\boldsymbol{y}) = \min_j y_j \quad \text{for } \; \boldsymbol{y} \in [0,1]^d \, ,$$

which does not depend on $\boldsymbol{c}$. See also Figure 3.

Example 7 (Antiegalitarian distribution): An antiegalitarian distribution $\boldsymbol{A}_{\boldsymbol{c},i^*}$ gives all to one household i^*. E.g., with $i^* = 1$,

$$\boldsymbol{A}_{\boldsymbol{c},1} = \begin{pmatrix} c_1 & \dots & c_d \\ 0 & \dots & 0 \\ & \dots & \\ 0 & \dots & 0 \end{pmatrix} .$$

Now let $\boldsymbol{X} = \boldsymbol{A}_{\boldsymbol{c},i^*}$ for some i^*. Then $\widetilde{x}_i = (n, \dots, n)$ if $i = i^*$ and $\widetilde{x}_i = (0, \dots, 0)$ otherwise,

$$\sum_{i=1}^{n} \lambda_i \widetilde{\boldsymbol{x}}_i = \lambda_{i^*} (n, \dots, n) \leq \boldsymbol{y}$$
$$\Leftrightarrow \quad 0 \leq \lambda_{i^*} \leq \frac{1}{n} \min_j y_j \quad \text{and} \;\; \lambda_i \leq \frac{1}{n} \;\; \text{for} \;\; i \neq i^* \, .$$

Hence

$$l_{\boldsymbol{A}_{\boldsymbol{c},i^*}}(\boldsymbol{y}) = \frac{n-1}{n} + \frac{1}{n} \min_j y_j \quad \text{for } \; \boldsymbol{y} \in [0,1]^d \, ,$$

independently of $\boldsymbol{c}$ and i^*.

Principal properties of the ILF are:

- $l_{\boldsymbol{X}}$ is monotone increasing, concave, and has values in $[0,1]$,
- $l_{\boldsymbol{X}}(\boldsymbol{1}) = 1 \, , \quad l_{\boldsymbol{X}}(\boldsymbol{0}) = Pr[\boldsymbol{X} \leq \boldsymbol{0}] \, ,$

- $l_{\boldsymbol{X}}(\boldsymbol{y}_J, \mathbf{1}_{-J}) = l_{\boldsymbol{X}_J}(\boldsymbol{y}_J)$.

Here, $(\boldsymbol{y}_J, \mathbf{1}_{-J})$ is the vector in $\mathbb{R}^d$ with components $y_j, j \in J$, and remaining components equal to 1.

The Lorenz hypersurface defines the underlying distribution uniquely, up to a vector of scale factors. It is additive in the distribution, and its proper projections form the Lorenz hypersurfaces of the marginals. For more properties and details, see [22] and [32, ch.9].

7 Decomposition of the volume-Gini index

In the multivariate case, there are two interesting extensions of the usual Gini index; see Koshevoy and Mosler [20]. One, called *volume-Gini index*, is based on the definition of the univariate Gini index as the area between the Lorenz curve and its symmetric curve, that is the area of the Lorenz zonoid. The *volume-Gini index* is essentially the volume of the Lorenz zonoid; see below. The second extension of the usual Gini index, called *distance-Gini index*, uses the fact that this area equals the mean difference over the mean. In dimensions $d \geq 2$ the two approaches lead to different indices.

An important question is, whether an index can be decomposed into indices of subgroups of the population. Here, we will provide a decomposition of the volume-Gini mean difference, which is essentially the volume of the lift zonoid. The volume-Gini index, then, is the volume-Gini mean difference of the relative data. For this, we exploit the additivity property of lift zonoids.

Given $\boldsymbol{X} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k)$ and $\boldsymbol{Y} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_l)$ consider the combined data set $\boldsymbol{X} \cup \boldsymbol{Y} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k, \boldsymbol{y}_1, \dots, \boldsymbol{y}_l)$. Then, by Proposition 3,

$$\widehat{Z}(\boldsymbol{X} \cup \boldsymbol{Y}) = \frac{k}{k+l}\,\widehat{Z}(\boldsymbol{X}) + \frac{l}{k+l}\,\widehat{Z}(\boldsymbol{Y}). \tag{17}$$

Various indices that are based on lift or Lorenz zonoids can be decomposed by combining the additivity property with some version of Minkowski's theorem. For the Minkowski theorem (the Steiner decomposition indeed) and its variants, see [36].

The *volume-Gini mean difference* M_V of a d-variate data set $\boldsymbol{X}$ is defined by (see [20])

$$M_V(\boldsymbol{X}) = \frac{1}{2^d - 1}\left(\mathrm{vol}\left(\widehat{Z}(\boldsymbol{X}) + C^d\right) - 1\right),$$

where $C^d = \{(0, z_1, \ldots, z_d) : 0 \le z_i \le 1, i = 1, \ldots, d\}$ is a d-dimensional unit cube in $\mathbb{R}^{d+1}$.

From Minkowski's theorem we obtain a formula for the volume of (17):

$$\begin{aligned}\mathrm{vol}(\widehat{Z}(\boldsymbol{X} \cup \boldsymbol{Y})) &= \left(\frac{k}{k+l}\right)^{d+1} \mathrm{vol}(\widehat{Z}(\boldsymbol{X})) + \left(\frac{l}{k+l}\right)^{d+1} \mathrm{vol}(\widehat{Z}(\boldsymbol{Y})) \\ &+ \sum_{m=1}^{d} \binom{d+1}{m} \frac{k^m l^{d+1-m}}{(k+l)^{d+1}} \mathrm{mixvol}_m(\widehat{Z}(\boldsymbol{X}), \widehat{Z}(\mathbf{Y})). \quad (18)\end{aligned}$$

Here $\mathrm{mixvol}_m(A, B)$ denotes the mixed volume of $(A, \ldots, A, B, \ldots, B)$ with m repetitions of A and $d+1-m$ repetitions of B. A formula similar to (18) holds true for M_V,

$$\begin{aligned}&(2^d + 1)M_V(\boldsymbol{X} \cup \boldsymbol{Y}) + 1 = \quad (19)\\ &\left(\frac{k}{k+l}\right)^{d+1} \mathrm{vol}(\widehat{Z}(\boldsymbol{X}) + C^d) + \left(\frac{l}{k+l}\right)^{d+1} \mathrm{vol}(\widehat{Z}(\boldsymbol{Y}) + C^d) \\ &+ \sum_{m=1}^{d} \binom{d+1}{m} \frac{k^m l^{d+1-m}}{(k+l)^{d+1}} \mathrm{mixvol}_m(\widehat{Z}(\boldsymbol{X}) + C^d, \widehat{Z}(\boldsymbol{Y}) + C^d).\end{aligned}$$

We conclude:

Proposition 12.

$$\begin{aligned}M_V(\boldsymbol{X} \cup \boldsymbol{Y}) &= \left(\frac{k}{k+l}\right)^{d+1} M_V(\boldsymbol{X}) + \left(\frac{l}{k+l}\right)^{d+1} M_V(\boldsymbol{Y}) \\ &+ \sum_{m=1}^{d} \binom{d+1}{m} \frac{k^m l^{d+1-m}}{(k+l)^{d+1}} \left[\mathrm{mixvol}_m(\widehat{Z}(\boldsymbol{X}) + C^d, \widehat{Z}(\boldsymbol{Y}) + C^d) - 1\right].\end{aligned}$$

For convex compacts $K_1, \ldots, K_n$ in $\mathbb{R}^n$, the mixed volume has a simple expression as an alternating sum of ordinary volumes:

$$\mathrm{mixvol}(K_1, \ldots, K_n) = \frac{1}{n!} \sum_{j=1}^{n} (-1)^{n-j} \sum_{i_1 < \cdots < i_j} \mathrm{vol}(K_{i_1} + \ldots + K_{i_j}) \quad (20)$$

From this representation of the mixed volume it is seen that $\mathrm{mixvol}(K_1,\dots,K_n)$ is a symmetric, multilinear and equivariant (under linear transformations) function of its variables. It holds $\mathrm{mixvol}(K,\dots,K) = \mathrm{vol}(K)$. Moreover, the mixed volumes are monotone (which is a nontrivial result):

$$\mathrm{mixvol}(K_1',\dots,K_n') \le \mathrm{mixvol}(K_1,\dots,K_n) \quad \text{if } K_i' \subset K_i \text{ for all } i\,.$$

If $K_1,\dots,K_{d+1}$ are lift zonoids of probability distributions $F_1,\dots,F_{d+1}$ in $\mathbb{R}^d$, a formula for their mixed volume is obtained from (20):

$$\mathrm{mixvol}(K_1,\dots,K_{d+1}) = \tag{21}$$
$$\frac{1}{(d+1)!}\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\left|\det\begin{pmatrix}1 & \cdots & 1\\ \boldsymbol{x}_1 & & \boldsymbol{x}_{d+1}\end{pmatrix}\right| dF_1(\boldsymbol{x}_1)\dots dF_{d+1}(\boldsymbol{x}_{d+1})\,.$$

Now, consider the case $d=1$. By (20) the mixed volume of two convex compact sets in $\mathbb{R}^2$ is given as

$$\mathrm{mixvol}(K,L) = \frac{1}{2}(\mathrm{vol}(K+L) - \mathrm{vol}(K) - \mathrm{vol}(L))\,. \tag{22}$$

Further, for univariate data $\boldsymbol{X} = (x_1,\dots,x_k), \boldsymbol{Y} = (y_1,\dots,y_l)$, it holds $\mathrm{vol}(\widehat{Z}(X) + C^1) = \mathrm{vol}(\widehat{Z}(X)) + 1$ and $\mathrm{vol}(\widehat{Z}(X) + C^1 + \widehat{Z}(Y) + C^1) = \mathrm{vol}(\widehat{Z}(X) + \widehat{Z}(Y)) + 4$, hence, by (22), $\mathrm{mixvol}(\widehat{Z}(\boldsymbol{X}) + C^1, \widehat{Z}(\boldsymbol{Y}) + C^1) = \mathrm{mixvol}(\widehat{Z}(X), \widehat{Z}(Y)) + 1$. From (21) we obtain

$$\mathrm{mixvol}(\widehat{Z}(X), \widehat{Z}(Y)) = \frac{1}{2}\frac{1}{kl}\sum_{i=1}^{k}\sum_{j=1}^{l}|y_j - x_i|\,,$$

and Proposition 12 yields the following three-term decomposition of the volume-Gini mean difference:

$$\begin{aligned} M_V(\boldsymbol{X}\cup\boldsymbol{Y}) &= \left(\frac{k}{k+l}\right)^2 M_V(\boldsymbol{X}) + \left(\frac{l}{k+l}\right)^2 M_V(\boldsymbol{Y}) \\ &\quad + \frac{1}{(k+l)^2}\sum_{i=1}^{k}\sum_{j=1}^{l}|y_j - x_i|\,. \end{aligned} \tag{23}$$

Another decomposition of M_V can be obtained from Theorem 5.2 ([20]) and (21). The details of this decomposition are left to the reader.

In dimension $d > 1$ more than three terms in the decomposition of the volume-Gini mean difference do appear. Note that the distance-Gini index ([20]) has, for any dimension d, a three-term decomposition similar to (23).

Let us note that Minkowski type theorems hold for other characteristics, say the surface area, of convex sets, too. Again, consider univariate data and observe (22). Let B denote the unit disc in $\mathbb{R}^2$. Then, $2 \cdot \mathrm{mixvol}(K, B)$ amounts to the circumference of K. For the unit segment S in direction $\boldsymbol{p}$, $2 \cdot \mathrm{mixvol}(K, S)$ is the width of K in the direction orthogonal to $\boldsymbol{p}$. By this, for univariate data, the additivity of the length of the Lorenz curve can be established.

8 Final remarks

Several multivariate generalizations of classical Lorenz curve and ordering have been surveyed in this paper. The key notions were the lift zonoid and the Lorenz zonoid of a d-variate distribution, and the graph of the inverse Lorenz function, which is part of the Lorenz zonoid's boundary. In particular, the pointwise ordering of inverse Lorenz functions is equivalent to the price Lorenz dominance. The Lorenz hypersurface is visual in the same way as the usual Lorenz curve is: increasing deviation from the egalitarian hypersurface indicates more disparity. The Lorenz hypersurface defines the underlying distribution uniquely, up to a vector of scaling constants.

As Lorenz did, the above analysis regards distributions of relative endowments. Similar dominance relations can be considered for distributions of absolute endowments; see [21, 32]. Like usual Lorenz dominance corresponds to vector majorization, these relations correspond to super- and submajorization.

The lift zonoid provides an embedding of d-variate distributions into the convex compacts of $(d+1)$-space. This embedding has attractive analytical properties and opens a special, visual way to the investigation of multivariate distributions. The distributions can be fully described by zonoid data depth and zonoid central regions, which arise from d-dimensional cuts of the lift zonoid ([19, 7]). The approach lends itself not only to measuring disparity, as we did it here in the tradition of Lorenz, but also to compare distributions with respect to location and dependency ([33]). Zonoid central regions may also be employed to exclude outlying data and to restrict Lorenz dominance to central parts of the distributions ([34]). Moreover, the lift zonoid is related in a visual way to other notions of data depth like the Oja depth and the Tukey depth ([17]). Also, statistical inference has been based on the zonoid data depth: testing for location and scale ([6]), testing on dependency ([25]), and estimating a scatter matrix ([24]).

Beyond probability distributions in $\mathbb{R}^d$, the lift zonoid of a capacity and the ordering induced by it has been discussed in [16].

For future research on measuring multidimensioned economic inequality four topics appear to be most relevant: (1) relations that are weaker (implied by) multivariate Lorenz dominance, (2) postulates for indices and dominance relations that model interaction effects between attributes, (3) subgroup decompositions of indices that allow an economic interpretation, (4) computational issues.

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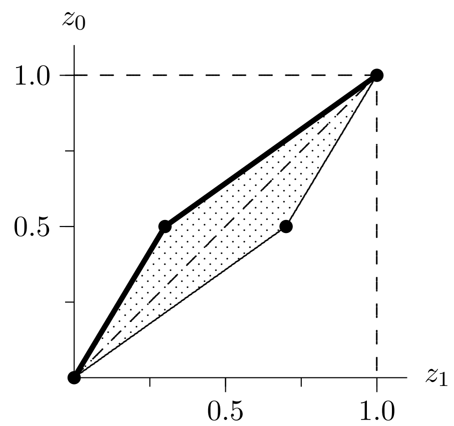
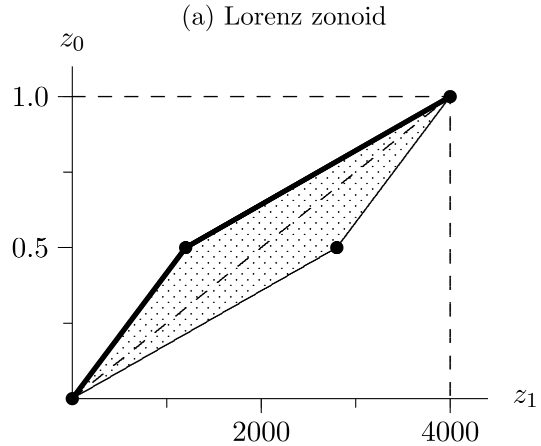


(a) Lorenz zonoid



(b) Lift zonoid

FIGURE 1: (a) LORENZ CURVE AND LORENZ ZONOID $LZ(2400, 5600)$, (b) GENERALIZED LORENZ CURVE AND LIFT ZONOID $\widehat{Z}(2400, 5600)$ OF THE UNIVARIATE TWO-POINT DISTRIBUTION IN EXAMPLE 1.

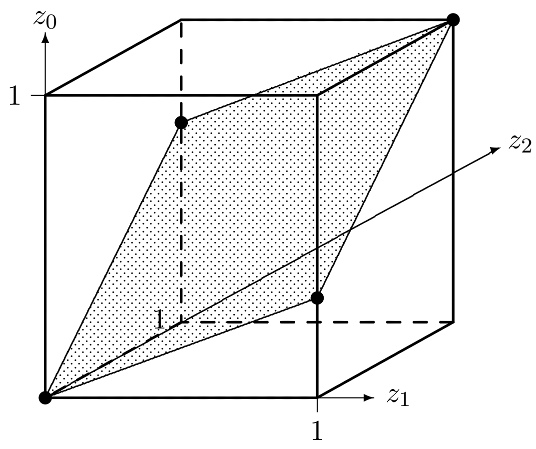


Figure 2: Lorenz zonoid of bivariate data. See Example 3.

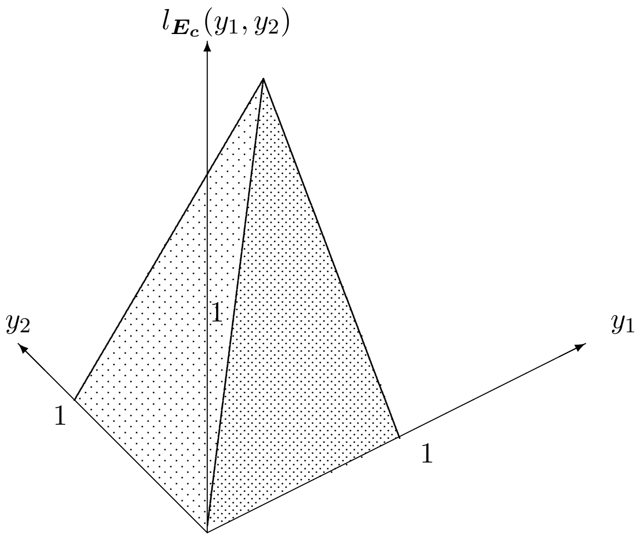


Figure 3: Lorenz hypersurface of a two-variate egalitarian distribution