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Statistical Inference for Tail Behaviour of Lorenz Curves*

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Abstract: The appeal of the Lorenz dominance criterion is undermined by the fact that many sample Lorenz curves intersect in the tails. Tests for Lorenz dominance which ignore tails (such as those considering only deciles) are therefore invalidated. Moreover, the usual inferential methods, based on central limit theorem arguments, do not apply to the tails of the Lorenz curve since the tails contain too few observations. By contrast, we have proposed a test procedure, based on a domain of attraction assumption, which fully takes into account the tail behaviour of Lorenz curves. Our experiments and empirical examples demonstrate the success and usefulness of the proposed test: in many cases we are able to infer that despite sample tail crossings the population Lorenz curves do, in fact, exhibit Lorenz dominance.

Keywords: Lorenz curves; tail behaviour; extreme value theory; regular variation

JEL classification: D31, D63, I32

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1 Introduction

The main tool for analysing economic inequality is the Lorenz curve. In order to compare inequality between two distributions one draws their Lorenz curves and concludes that inequality is unanimously higher in one distribution if its Lorenz curve is everywhere below the curve of the other distribution: any inequality measure which satisfies the principles of transfer and of anonymity and the Pigou-Dalton principle will rank the two distributions in the same way as the Lorenz curves (Atkinson 1970). The Lorenz curve provides, however, only a partial ordering of income distributions. If the curves cross, no statement is possible unless one is willing to make further assumptions about the social welfare function.

Population Lorenz curves are rarely known since one rarely has information about the entire population, and empirical Lorenz curves have to be estimated from sample data. The statistical theory for the main body of the Lorenz curve, which contains many observations, is well-developed (Beach and Davidson 1983). However, these methods do not apply to the tails of the Lorenz curve since the tails contain too few observations to invoke the usual central limit theorem arguments. However, the tail behaviour is of considerable interest, and it is precisely in the tails that crossings often occur in practice. For instance, our experiments with realistically calibrated parametric models, reported in detail below, suggest that about 45 percent of sample Lorenz curves intersect in the tails, although the population curves do not. We propose statistical methods which address such tail behaviour.

For most applied problems, the relevant testable null hypothesis is that the Lorenz curves to be compared cross at least once. The alternative hypothesis is that there is Lorenz dominance in either direction. However, if our analysis is built on the empirical Lorenz curves we will never be able to reject this null hypothesis if the empirical Lorenz curves actually cross. This is sensible if the crossing occurs in the middle of the distributions. However, if the curves only cross in the tails, which contain few observations, outright rejection of the null hypothesis is problematic since extreme observations exert a large influence.

To overcome this tail behaviour problem we develop a test which is based on extreme value theory and the theory of regular variation. For income distributions we have in mind, it is reasonable to assume that their tails lie in the domain of attraction of the Fréchet distribution, i.e. they decay like power functions. Examples of parametric models which exhibit this characteristic are the generalised beta distributions of the second kind (McDonald and Xu 1995), and therefore the special cases of the Singh-Maddala distribution and the Dagum distribution, all of which fit real world income data reasonably well (Brachmann, Stich, and Trede 1996). We do not examine middle heavy and thin tailed distributions, which decay like exponential functions, such as log-normal distributions, since their associated Lorenz dominance results are often trivial. Moreover, the fit of parametric models based on power functions to the tails of real world income data is far superior to the fit of lognormal models.

The domain-of-attraction assumption permits us to estimate extreme quantiles outside the data range without imposing strong assumptions on the parametric form of the income distribution. The test procedure based on extreme value theory closes a vexing gap in the conventional approach to statistical inference for Lorenz curves. Using our test we are able to infer in many cases that despite sample tail crossings, the population Lorenz curves do in fact exhibit Lorenz dominance.

This paper is organised as follows. Section 2 provides a review of the relevant concepts of extreme value theory. Although our test procedure is not parametric it is nevertheless useful to investigate the tail behaviour of common parametric income distributions. Section 3 describes the statistical test for Lorenz curve tails. Section 4 gives two illustrations: a Monte-Carlo simulation and an empirical example using data on disposable personal income from the Luxembourg Income Study (LIS). Section 5 concludes.

2 Preliminaries

Let $X_1, \ldots, X_n$ be an i.i.d. sample from an absolutely continuous (income) distribution function F_X with $F_X(0) = 0$. As the Lorenz curve is scale invariant we assume with loss of generality that the mean of X is normalized to $\mu_X = 1$. The upper tail of F_X is denoted by $\overline{F}_X(x) = 1 - F_X(x)$, and order statistics by $X_{(1)} \geq \ldots \geq X_{(n)}$.

The Lorenz curve of X is given by

$$\{(p, L_X(p)), 0 \leq p \leq 1\} \qquad \text{with} \qquad L_X(p) = \int_{x \geq 0} I\left(x \leq F_X^{-1}(p)\right) x dF_X(x)$$

where $I(.)$ is the indicator function. Let Y be a similarly defined random variable. X Lorenz dominates Y if $L_X(p) \geq L_Y(p)$ for all $p \in [0,1]$ and $L_X(p_0) > L_Y(p_0)$ for at least one $p_0 \in [0,1]$.

Lorenz dominance can equivalently be expressed as second order stochastic dominance; since $E(X) = E(Y) = 1$,

$$X \text{ Lorenz dominates } Y \Longleftrightarrow \int_x^\infty \overline{F}_X(t)\, dt \leq \int_x^\infty \overline{F}_Y(t)\, dt \text{ for all } x > 0 \qquad (1)$$

Recall that the lower extreme order statistics are asymptotically independent from the upper extreme order statistics, and both are asymptotically independent from the sample mean. A well known result concerning the distribution of the maximum is that if there exist norming constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$\frac{X_{(1)} - d_n}{c_n} \xrightarrow{D} Z,$$

then Z is distributed as either of the following three distributions: (1) the Gumbel distribution defined by its distribution function $\exp(-\exp(-x))$ for $x \in \mathbb{R}$, (2) the Weibull distribution given by $\exp\left(-(-x)^\alpha\right)$ for $x \leq 0$ and 1 otherwise, with $\alpha > 0$, and (3) the Fréchet distribution

$$\Phi_\alpha(x) = \begin{cases} 0 & x \leq 0 \\ \exp\left(-x^{-\alpha}\right) & x > 0 \end{cases} \qquad (2)$$

with $\alpha > 0$. We make the following

Assumption A1: *The distribution F_X lies in the domain of attraction of the Fréchet distribution Φ_α.*

For income models we have in mind, the Fréchet distribution is the only relevant limiting distribution for reasons that will become clearer once we are able to translate the assumption about the maximum into a condition on the tail of the distribution. To this end, we use the concept of regular variation. Recall that a function g is called regularly varying at x_0 of index ρ if

$$\lim_{x \to x_0} \frac{g(tx)}{g(x)} = t^\rho, \qquad t > 0.$$

The class of all distribution functions with regularly varying tails with parameter ρ is denoted by R_ρ . If $\rho = 0$, the function is said to be slowly varying.

If the distribution is in the domain of attraction of the Fréchet distribution Φ_α, the index of regular variation of the upper tail $\overline{F}_X(x)$ at infinity equals $\rho = -\alpha$, i.e.

$$\lim_{x \to \infty} \frac{\overline{F}_X(tx)}{\overline{F}_X(x)} = t^{-\alpha}, \qquad t > 0. \tag{3}$$

Hence, even though the approach is not parametric we can use the model $\overline{F}_X(x) = x^{-\alpha} L_0(x)$ with $L_0 \in R_0$ asymptotically for the upper tail of the income distribution.

Assumption A1': *F_X satisfies for some $\alpha > 0$*

$$\overline{F}_X(x) = x^{-\alpha} L_0(x) \tag{4}$$

for some slowly varying function $L_0 \in R_0$.

Thus, the tails are heavy in that they decay like power functions. We do not examine distributions with middle heavy tails which decay exponentially fast, such as the lognormal distribution.

Similar arguments apply to the lower tail of $F_X(x)$ which we assume to be regularly varying at 0 with index β,

$$\lim_{x \to 0} \frac{F_X(tx)}{F_X(x)} = t^\beta, \qquad t > 0. \tag{5}$$

If F_X is regularly varying at zero with β then $\overline{F}_{X^{-1}}$ is regularly varying at infinity with $-\beta$:

$$\lim_{x \to \infty} \frac{\overline{F}_{X^{-1}}(tx^{-1})}{\overline{F}_{X^{-1}}(x^{-1})} = \lim_{x \to 0} \frac{F_X(t^{-1}x)}{F_X(x)} = \left(t^{-1}\right)^\beta = t^{-\beta}$$

since $\overline{F}_{X^{-1}}(x^{-1}) = F_X(x)$. This relationship allows us to deal with the statistical inference of upper tails only as the same results hold for the lower tails if we consider the reciprocals.

The parameter α of regular variation at infinity in (4) can be estimated by Hill's estimator given by[1]

$$\hat{\alpha} = H_{k,n}^{-1} \tag{6}$$

$$H_{k,n} = \frac{1}{k}\sum_{i=1}^{k} \ln\left(X_{(i)}\right) - \ln\left(X_{(k)}\right) \tag{7}$$

where k is the number of extreme observations to be included. This estimator was originally proposed by Hill (1975) as the maximum likelihood estimator of the parameter α of the Pareto distribution model $\overline{F}(x;c,\alpha) = cx^{-\alpha}$ (i.e. the special case in which the slowly varying function in (4) is a constant). However, more general properties of Hill's estimator are well-known. For fixed k, the estimator $H_{k,n}$ converges in distribution to a gamma distribution as $n \to \infty$. It follows immediately from a diagonalisation argument that for any $\overline{F}$ satisfying (4), $\sqrt{k}\left(H_{k,n} - \alpha^{-1}\right)$ converges in law to a normal distribution with variance α^{-2} provided k tends to infinity sufficiently slowly. Various theorems exist in the literature which make the last statement more precise. We present one of them below.[2]

Theorem 1 *Under assumption A1':*

(a) (weak consistency) if $k \to \infty$, $k/n \to 0$, for $n \to \infty$

$$\hat{\alpha} \xrightarrow{p} \alpha$$

(b) (strong consistency) if $k \to \infty$, $k/\ln\ln n \to \infty$, as $n \to \infty$

$$\hat{\alpha} \xrightarrow{a.s.} \alpha$$

(c) (asymptotic normality) assume $\lim_{x\to\infty} \frac{\overline{F}(tx)/\overline{F}(x) - t^{-\alpha}}{\gamma(x)} = t^{-\alpha}\frac{t^{-\rho}-1}{-\rho}$, $t > 0$ exists where $\gamma(x)$ is a measurable function of constant sign. We refer to this as a "second order condition" with the second order parameter of regular variation $-\rho$. Let $U(t) = F^{-1}(1 - t^{-1})$, and $\Gamma(x) = \alpha^{-2}\gamma(U(x))$ and $k \to \infty$ but $k/n \to 0$. If

$$\lim_{n\to\infty} \sqrt{k}\Gamma\left(\frac{n}{k}\right) = \lambda \in \mathbb{R}$$

then, as $n \to \infty$, the estimator $\hat{\alpha}$ is consistent and asymptotically normal with

$$\sqrt{k}\left(\hat{\alpha} - \alpha\right) \xrightarrow{D} N\left(\frac{\alpha^3}{-\rho-\alpha}\lambda, \alpha^2\right). \tag{8}$$

[1]Following the preceding remarks, Hill's estimator is readily adaptable for an estimation of β for the lower tails, being now based on the k smallest observations $X_{(n-k+1)}, \ldots, X_{(n)}$:

$$\hat{\beta} = \left(\frac{1}{k}\sum_{i=1}^{k} \ln\left(X_{(n-i+1)}^{-1}\right) - \ln\left(X_{(n-k+1)}^{-1}\right)\right)^{-1}.$$

[2]See, for instance Embrechts, Klüppelberg, and Mikosch (1997, chap. 6.4). Theorem 1.c is due to de Haan and Peng (1999). Another version is given in Haeusler and Teugels (1985). See also Weissman (1978).

Asymptotic normality is also obtained by Hall (1982) using a different approach. He assumes that the true distribution satisfies

$$\overline{F}(x) = x^{-\alpha}c(1 + dx^{-\rho} + o(x^{-\rho})) \tag{9}$$

asymptotically, an assumption which more stringent than (4). The Fréchet distribution can be expanded into the above form, i.e. $\overline{F}(x) = cx^{-\alpha}\left(1 - 0.5cx^{-\alpha} + o\left(x^{-\alpha}\right)\right)$. If the distribution can be expanded to $m+1$ terms, so that $\overline{F}(x) = cx^{-\alpha}(1 + d_1x^{-\alpha} + .. + d_mx^{-m\alpha} + o(x^{-m\alpha}))$, he shows that if $k \to \infty$ such that $k = o\left(n^{2m/(2m+1)}\right)$ then $\sqrt{k}\left(\hat{\alpha} - \alpha\right) \to^D N\left(0, \alpha^2\sigma^2\right)$. In particular, if $\overline{F}(x) = cx^{-\alpha}(1 + O(x^{-\rho}))$ as $x \to \infty$, if $k \to \infty$ and if $k = o\left(n^{2\rho/(2\rho+\alpha)}\right)$ as $n \to \infty$, then $\sqrt{k}\left(\hat{\alpha} - \alpha\right) \to N\left(0, \alpha^2\right)$.

Hall's result and theorem 1.c can be linked by observing that

$$L\left(x\right) = c\left(1 + dx^{-\rho} + O(x^{-2\rho})\right) \tag{10}$$

is a slowly varying function, $L \in R_0$. Moreover,

$$\begin{aligned} \frac{L\left(tx\right)}{L\left(x\right)} - 1 &= \frac{(1 + t^{-\rho}dx^{-\rho} + O(x^{-2\rho}))}{(1 + dx^{-\rho} + O(x^{-2\rho}))} - 1 \\ &= \left(t^{-\rho} - 1\right)dx^{-\rho} + O\left(x^{-2\rho}\right) \end{aligned} \tag{11}$$

(after expanding $\left(1 + dx^{-\rho} + O(x^{-2\rho})\right)^{-1}$) so $-\rho$ in (9) is in fact the second order variation parameter of theorem 1.c, and the required function is

$$\gamma\left(x\right) = \left(-\rho\right)dx^{-\rho}. \tag{12}$$

In order to implement the Hill estimator, it remains to choose k appropriately. For a sample with given size, there is no universal optimal choice, and different methods have been proposed. One method is a Hill plot: plot the estimate $H_{k,n}^{-1}$ against k and select a value of k for which the plot is (roughly) constant. Embrechts, Klüppelberg, and Mikosch (1997, p. 194) observe that the Hill estimator can perform poorly if the slowly varying function in (4) is far from being a constant. This poor performance manifests itself in a volatile "Hill's horror plot". It is therefore informative in a parametric context to examine whether a given parametric model is close to the Pareto model asymptotically. We examine this point below. If the Hill plot is too volatile, using a logarithmic scale for k may increase the display space taken up by $H_{k,n}^{-1}$ around the true value α. This "alternative Hill plot", proposed in Drees, de Haan, and Resnick (2000) is thus given by $\{(\theta, H_{[n^\theta],n}^{-1}), 0 \le \theta \le 1\}$. We consider both methods below.

In order to illustrate theorem 1 we discuss some parametric models. The results will also be of use in the simulation study below in which k will be chosen by minimising the mean-squared error of the Hill estimator $\hat{\alpha} = H_{k,n}^{-1}$.

2.1 Pareto distribution

The Pareto model $\overline{F}(x; c, \alpha) = cx^{-\alpha}$ appears to capture empirically well the upper tails of actual income and wealth distributions (Pareto 1965). However, it obviously performs empirically inadequately for the lower tail and the main body of income distributions. It suffices to note that the Hill estimator is the maximum likelihood estimator of the parameter α of the Pareto model.

2.2 The generalised beta distribution of the second kind

This class of distributions, proposed in McDonald and Xu (1995), has density

$$f(x;a,b,c,d)=\frac{bx^{bd-1}}{a^{bd}B(d,c)\left[1+(x/a)^{b}\right]^{d+c}} \tag{13}$$

where $B(\cdot,\cdot)$ denotes the Beta function, and nests various distributions as special cases. For instance, if $d=1$ then (13) reduces to the Singh-Maddala distribution, which captures many actual income distributions, as regards both tails and the main body (Singh and Maddala 1976). Its tail is given explicitly by $\overline{F}(x;a,b,c)=(1+(x/a)^{b})^{-c}$.[3] Another example is the Dagum distribution (for $c=1$).

To obtain an approximation to the upper tail of the distribution function, expand $\left[(x/a)^{b}\right]^{d+c}/\left[1+(x/a)^{b}\right]^{d+c}$ to second order and integrate:

$$\overline{F}(x;a,b,c,d)=g_1x^{-bc}\left(1+g_2x^{-b}+O\left(x^{-2b}\right)\right) \tag{14}$$

for some constants g_i. Thus, the upper tail is regularly varying with parameter $-\alpha=-bc$, and using (10) and (11), the second order parameter is $-\rho=-b$.[4] Equation (14) is also in a form which permits direct application of Hall's result, so that k of the Hill estimator $\hat{\alpha}=H_{k,n}^{-1}$ must satisfy $o\left(n^{2/(2+c)}\right)$ to ensure unbiasedness. To apply theorem 1.c directly, it follows from (12) that $\gamma(x)=(-b)\,g_2x^{-b}$. In order to derive $u(.)$, just consider the first order term in (14) and invert to get $u(x)\propto x^{1/bc}$. Hence, $k^{0.5}\Gamma\left(\frac{n}{k}\right)\propto k^{0.5+1/c}n^{-1/c}$, so to obtain no bias we require $k=o\left(n^{2/(2+c)}\right)$.

As regards the lower tail, the usual expansion yields

$$F(x,a,b,c,d)=g_3x^{bd}\left(1+g_4x^{b}+O\left(x^{2b}\right)\right)$$

for some constants g_i. Hence the lower tail varies with parameter $\beta=bd$, and the second order parameter is b. Direct application of Hall's result shows that k of the Hill estimator $\widehat{\beta}=H_{k,n}^{-1}$ must satisfy $o\left(n^{2/(2+d)}\right)$ to ensure unbiasedness. The "second order condition" can be verified in a similar fashion.

[3] Note that this distribution is of the Pareto type for large x since $\bar{F}(x)=a^{bc}x^{-bc}+O(x^{-b(1+c)})$. Thus x needs to be large to avoid Hill's horror plots for the upper tail estimation. As regards the lower tail, we observe that $F(x)=x^{b}\left(ca^{-b}+O\left(x^{b}\right)\right)$. Hence a good result for the lower tail estimation is to be expected.

[4] Note that the first order result could also have been obtained directly from (13) using the lemmas in Embrechts, Klüppelberg, and Mikosch (1997, pp. 564) by observing that its numerator is regularly varying at infinity with parameter $bd-1$, the denominator with $bd+bc$, so the ratio regularly varies with $-bc-1$, and the tail of the distribution function with $-bc$.

2.3 The lognormal distribution

It is evident from the definition of the lognormal distribution that its tails decay exponentially fast, i.e. much faster than the class of distribution functions with regularly varying tails given in (4), which decrease like power functions. This distribution is rapidly varying. We do not examine such middle heavy tailed or thin tailed distributions in this paper. Note, however, that this is no restriction since the Lorenz ordering of Lorenz curves of lognormal distributions is trivial.[5]

3 Statistical inference for Lorenz curve tails

The income distribution functions of X and Y are given by F_X and F_Y, respectively. Let $-\alpha_X$ be the index of regular variation of $\overline{F}_X$ (at infinity) and β_X the index of regular variation of F_X at zero, and define $-\alpha_Y$ and β_Y for F_Y similarly.

Extreme upper and lower order statistics are asymptotically independent. Let $\overline{k}$ and $\underline{k}$ denote the number of upper and lower extreme observations, respectively, to be included in the estimators. Under conditions of Theorem 1.c with $\overline{k}$ and $\underline{k}$ growing sufficiently slowly so that the bias term λ equals 0, the joint asymptotic distribution of $\hat{\alpha}_X$ and $\hat{\beta}_X$ is

$$\left[\begin{array}{c} \sqrt{\overline{k}}\left(\hat{\alpha}_X - \alpha_X\right) \\ \sqrt{\underline{k}}\left(\hat{\beta}_X - \beta_X\right) \end{array}\right] \rightarrow N\left(\left[\begin{array}{c} 0 \\ 0 \end{array}\right], \left[\begin{array}{cc} \alpha_X^2 & 0 \\ 0 & \beta_X^2 \end{array}\right]\right).$$

Further, if the samples of X and Y are independent, so are $\hat{\alpha}_X$ and $\hat{\alpha}_Y$, and $\hat{\beta}_X$ and $\hat{\beta}_Y$.

Since this paper is concerned with inference for tail behaviour of Lorenz curves, and methods of inference for the main body of the Lorenz curve are well-known, we assume the following:

Assumption A2: *F_X Lorenz dominates F_Y in the middle of the distribution.*

Theorem 2: *Under assumptions A1 and A2,*

$$X \text{ Lorenz dominates } Y \iff \alpha_X \geq \alpha_Y \text{ and } \beta_X \geq \beta_Y. \tag{15}$$

A proof of the sufficiency statement of the theorem is given in Kleiber (1999), which we reproduce here for the upper tail only. From (1), Lorenz dominance of X over Y is equivalent to $g(x) = \int_x^\infty \overline{F}_Y(t)\,dt / \int_x^\infty \overline{F}_X(t)\,dt \geq 1$ for all $x > 0$. By assumption the tail of $\overline{F}_Y$ is regularly varying with parameter $-\alpha_Y$, and its integral with $-\alpha_Y + 1$. Hence g regularly varies with $\alpha_X - \alpha_Y$, but $\lim_{x\to\infty} g(x) \geq 1$ iff $\alpha_X \geq \alpha_Y$. The result for the lower tail is established similarly. The necessity statement is obvious considering assumption A2.

[5] For instance, if $X \sim N\left(\mu_1, \sigma_1^2\right)$ and $Y \sim N\left(\mu_2, \sigma_2^2\right)$, the Lorenz curves for the lognormal distributions of $\exp(X)$ and $\exp(Y)$ cannot cross. They are either identical ($\sigma_1 = \sigma_2$) or there is Lorenz dominance.

To test whether F_X Lorenz dominates F_Y throughout, the null and alternative hypotheses concerning the tails are

$$\begin{aligned} H_0 &: \text{ the population Lorenz curves cross at the bottom or the top} \\ &= (\alpha_X < \alpha_Y \text{ or } \beta_X < \beta_Y) \\ H_1 &: \text{ not } H_0 \\ &= (\alpha_X \geq \alpha_Y \text{ and } \beta_X \geq \beta_Y) \end{aligned}$$

Because of theorem 2, rejecting the null hypothesis firmly establishes Lorenz dominance of X over Y. Let $\overline{k}_X, \underline{k}_X$ and $\overline{k}_Y, \underline{k}_Y$ denote the number of extreme observations for the estimation of $\hat{\alpha}_X, \hat{\beta}_X$ and $\hat{\alpha}_Y, \hat{\beta}_Y$, respectively. A suitable test is based on the two statistics

$$T_1 = \frac{\hat{\alpha}_X - \hat{\alpha}_Y}{\sqrt{\frac{\hat{\alpha}_X^2}{\overline{k}_X} + \frac{\hat{\alpha}_Y^2}{\overline{k}_Y}}}, \tag{16}$$

$$T_2 = \frac{\hat{\beta}_X - \hat{\beta}_Y}{\sqrt{\frac{\hat{\beta}_X^2}{\underline{k}_X} + \frac{\hat{\beta}_Y^2}{\underline{k}_Y}}}. \tag{17}$$

These test statistics are asymptotically normal, despite the dependence between numerator and denominator, as can be seen from an application of Slutsky's theorem[6].

As T_1 and T_2 are asymptotically independent and standard normally distributed, the null hypothesis is rejected when both T_1 and T_2 are too large. The critical value δ for significance level γ is chosen such that $P(T_1 > \delta \text{ and } T_2 > \delta) \leq \gamma$ under the null hypothesis. Because of

$$\begin{aligned} P(T_1 > \delta \text{ and } T_2 > \delta) &= P(T_1 > \delta) \times P(T_2 > \delta) \\ &= (1 - \Phi(\delta))^2 \end{aligned}$$

the critical value is given by $\delta = \Phi^{-1}\left(1 - \sqrt{\gamma}\right)$ where Φ^{-1} is the quantile function of $N(0,1)$.

If the parameters are on the boundary of H_0 the (true) null hypothesis is rejected (asymptotically) with a probability of γ. If the parameters are inside H_0 the error probability of the first kind is less than γ. The power of the test depends, of course, on the true parameter values, and we have approximately

$$\begin{aligned} & P(H_0 \text{ rejected} | \alpha_X \geq \alpha_Y \text{ and } \beta_X \geq \beta_Y) \\ = \; & P(T_1 > \delta \text{ and } T_2 > \delta) \\ = \; & \left(1 - \Phi\left(\delta - \frac{\alpha_X - \alpha_Y}{\sqrt{\frac{\alpha_X^2}{k_X} + \frac{\alpha_Y^2}{k_X}}}\right)\right) \times \left(1 - \Phi\left(\delta - \frac{\beta_X - \beta_Y}{\sqrt{\frac{\beta_X^2}{k_X} + \frac{\beta_Y^2}{k_X}}}\right)\right). \end{aligned}$$

[6]Consider $\tau = (\hat{\alpha}_X - \hat{\alpha}_Y)\left(\alpha_X^2/\overline{k}_X + \alpha_Y^2/\overline{k}_Y\right)^{-0.5}$. It is easily seen that τ has a limiting Gaussian distribution, but it cannot be implemented since α_X and α_Y are, of course, unknown. Define the random sequence $c_{\overline{k}_X,\overline{k}_Y} = \left(\alpha_X^2/\overline{k}_X + \alpha_Y^2/\overline{k}_Y\right)^{0.5} / \left(\widehat{\alpha}_X^2/\overline{k}_X + \widehat{\alpha}_Y^2/\overline{k}_Y\right)^{0.5}$, which has the property that $c_{\overline{k}_X,\overline{k}_Y} \xrightarrow{p} 1$. By Slutsky's theorem, $T_1 = \tau c_{\overline{k}_X,\overline{k}_Y}$ converges to the same (Gaussian) distribution as τ. Similarly for T_2.

4 Illustrations

We first present some evidence which reveals that sample Lorenz curves may intersect in the tails, although the population Lorenz curves do not cross: in our experiments 45% of sample Lorenz curves intersect in the tails. This is precisely the situation about which we would like to make statistical inference. Using our test we are able to infer in many cases that despite sample tail crossings, the population Lorenz curves exhibit statistically significant Lorenz dominance.

4.1 The experiments

We let X and Y have Singh-Maddala distributions defined in (13) with $d = 1$ such that X Lorenz dominates Y: X is distributed with densities $f_X(.; 5, 2.8, 1.7, 1)$ and Y with $f_Y(.; 5, 2.4, 1.8, 1)$. The parameters are chosen such that (a) the Lorenz curves look similar to curves encountered in empirical applications and (b) the Lorenz curves are far apart in the middle of the distribution, in order to make assumption A2 sensible. Comparing the parameters of regular variation at the upper and lower tails, it follows that the parameter choice is consistent with Lorenz dominance[7]. The analytical form of the Lorenz curves, given for the Singh-Maddala distribution by

$$p \mapsto IB_{1-(1-p)^{1/c}}\left(\frac{1}{b}+1, c-\frac{1}{b}\right)$$

where $IB(\cdot,\cdot)$ is the incomplete Beta function, establishes Lorenz dominance (Schader and Schmid 1988). The population Gini coefficients are $Gini_X = 0.2887$ and $Gini_Y = 0.3275$. Figure 1 displays the theoretical Lorenz curves. X Lorenz dominates Y, but the corresponding empirical Lorenz curves $\hat{L}_X$ and $\hat{L}_Y$ may, of course, intersect. A Monte Carlo simulation with $N = 10\,000$ replications of empirical Lorenz curves estimated from samples of size $n = 5\,000$ reveals that the proportion of non-intersecting $\hat{L}_X$ and $\hat{L}_Y$ is as low as 0.549, see table 1. In about 45 percent of the cases the empirical curves cross even though the theoretical curves do not. The reason for this unsatisfactory performance is the large number of intersections in the tails as shown in the table (where the lower and upper tails are defined by the 0.05- and 0.95-quantiles, respectively).

intersecting curves	45.1 %
intersections in lower tail	21.8 %
intersections in upper tail	29.3 %
rejections of H_0	62.3%

Table 1: Results of the Monte Carlo simulations: intersections of sample Lorenz curves when the population Lorenz curve of X dominates that of Y. The lower and upper tail regions are defined by the 0.05- and 0.95-quantiles.

[7] Using (15) since, as regards the upper tail, $-2.8 \times 1.7 < -2.4 \times 1.8$ and, for the lower tail, $-2.8 < -2.4$.

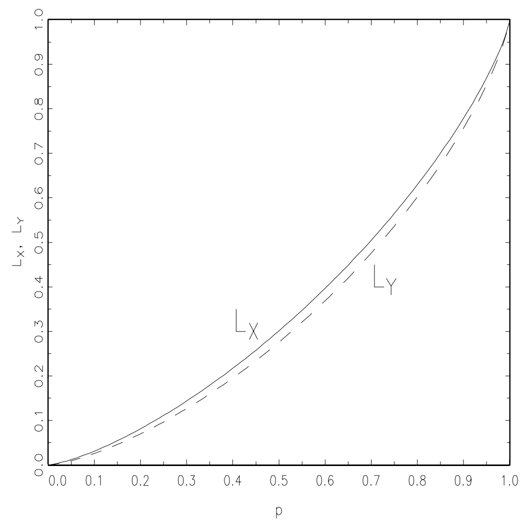

Figure 1: Theoretical Lorenz curves for the Monte Carlo experiment

We proceed to examine the performance of our test. For each of the N replications we estimate the parameters α_X, α_Y, β_X and β_Y using the respective Hills's estimators. In order to do so, a choice about the number of extreme observations to be used must be made. Since it is impractical to evaluate a Hill plot for each iteration of the simulation, we let $\overline{k}$ and $\underline{k}$ minimise the mean-squared error of the Hill estimator in the parametric model on which the simulation is based. Clearly, this simplification is not possible in empirical applications where the population values are unknown. From theorem 1.c it is immediate that the mean-squared error of the upper tail parameter estimate is

$$MSE_{\hat{\alpha}} = \frac{1}{k}\left(\alpha^2 + \frac{\alpha^6 \lambda_{k,n}^2}{(\rho-\alpha)^2}\right) \quad \text{and } \overline{k} = \arg\min_k MSE_{\hat{\alpha}}$$

where ρ is the second order regular variation parameter and $\lambda = \lambda_{k,n}$ is defined in the theorem.

For the Singh-Maddala distribution it follows that, as regards the upper tail, $\alpha = bc$, $\rho = -b$, and $\lambda_{k,n} = (bc)^{-1} k^{1/c+1/2} n^{-1/c}$. For a sample size $n = 5,000$ and the Singh-Maddala distribution $F_X(.; 100, 2.8, 1.7)$ we obtain $\overline{k}_X = 142$, and for $F_Y(.; 100, 2.4, 1.8)$, $\overline{k}_Y = 128$.

For the lower tail, let $\underline{k}$ minimise $MSE_{\widehat{\beta}}$ which is defined as above. From section 2.2 we know that $\alpha = b$, $\rho = -b$, and $\lambda_{k,n} = b^{-1}k^{3/2}n^{-1}$. Irrespective of the parameters of the Singh-Maddala distribution, $\underline{k} = 2^{1/3}n^{2/3}$, so for $n = 5000$ we have $\underline{k}_X = \underline{k}_Y = 369$.

From the estimates $\hat{\alpha}_X$, $\hat{\alpha}_Y$, $\hat{\beta}_X$, $\hat{\beta}_Y$ and from $\overline{k}_X$, $\underline{k}_X$, $\overline{k}_Y$ and $\underline{k}_Y$ we compute the statistics (16) and (17). The last row of table 1 shows the proportion of rejections of the null hypotheses that

H_0 : the population Lorenz curves cross at the bottom or the top at a significance level of $\gamma = 0.1$. We conclude that rejecting the null hypothesis implies strong statistical evidence in favour of the alternative hypothesis, i.e. that the population Lorenz curves exhibit dominance. The formal test procedure therefore improves considerably on merely basing one's judgement on whether or not the sample Lorenz curves intersect.

4.2 Empirical examples

The LIS database provides comprehensive and comparable information about household composition and income for many countries, and has been used for many inequality analyses (see e.g. Atkinson, Rainwater, and Smeeding (1994)). We illustrate the merits of our test procedure by investigating Lorenz dominance relations of four major economies in 1994: the United States, Canada, Italy and, at the other end of the inequality spectrum, Germany. The LIS definition of disposable income[8] includes earnings, other factor income, means and non means tested social insurance transfers and public and private pension transfers; mandatory social insurance contributions and income tax are subtracted. The left hand side of table 2 reports summary statistics of the income distributions (number of observations, coefficient of variation and the Gini coefficient).

Country	No. obs.	CV	Gini	$\overline{k}$	$\underline{k}$	$\hat{\alpha}$	$\hat{\beta}$
Canada (CN)	100207	0.5562	0.2835	101	101	4.83	1.27
Germany (GE)	15084	0.5379	0.2460	123	93	4.16	1.60
Italy (IT)	23725	0.7807	0.3431	255	155	2.83	0.82
USA (US)	162380	0.7286	0.3629	403	403	4.26	0.58

Table 2: Summary statistics of income distributions, number of extreme values, and Hill estimates

Table 3, reporting the pairwise Lorenz orderings, makes clear that the tails cannot be ignored: apart from the pair Canada-Italy all other pairs have intersecting sample Lorenz curves, in most cases the intersection occurs in the tails. Even despite the large difference between the Gini coefficients, Germany does not appear to Lorenz dominate the USA. Considering just the Lorenz curve ordinates at deciles, a common practice, is an improper procedure because the entire sample Lorenz curve needs to be taken into account. The sample tail crossings invalidate conclusions about Lorenz dominance based on the limited consideration of deciles. However, this "decile approach" does yield a positive insight: recognising that this approach focuses only on the main body of the Lorenz curve, we can use it to test our assumption A2. Our test becomes appropriate if assumption A2 can be inferred to be met, and sample tail crossings occur. Table 3 suggests a crossing of the Lorenz curves of Italy and the USA in the main body of the distribution (more precisely at about $p = 0.85$). Our test is then appropriate for all other pairs.

[8] See http://www.lis.ceps.lu/summary.htm for a detailed description.

	at deciles			entire curve		
	Canada	Germany	Italy	Canada	Germany	Italy
Germany	$>$			x		
Italy	$<$	$<$		$<$	x	
USA	$<$	$<$	x	x	x	x

Table 3: Lorenz dominance. Note: "$<$" means that the row country is dominated by the column country, "$>$" the reverse, "x" indicates crossing

In order to test whether the tail crossings are statistically significant we apply our test procedure. The number of extreme observations to be included into the estimators are determined by investigating the alternative Hill plots (see appendix). The right part of table 2 gives the numbers of extremes ($\overline{k}$ and $\underline{k}$) as well as the resulting Hill estimates of the index of regular variation for the upper tail ($\hat{\alpha}$) and for the lower tail ($\hat{\beta}$).

Table 4 states the null hypotheses that the Lorenz curves cross, given that there is Lorenz dominance in the main body of the distribution (hence Italy-US is disregarded). Further, we provide the values of the test statistics T_1 and T_2, the test results at 10% significance level are reported in the right-most column.

Pair	Null hypothesis (Lorenz curves cross)	T_1	T_2	Test result
CN-GE	$\alpha_{CN} > \alpha_{GE}$ or $\beta_{CN} > \beta_{GE}$	1.1082	-1.5777	do not reject
CN-IT	$\alpha_{CN} < \alpha_{IT}$ or $\beta_{CN} < \beta_{IT}$	3.9195	3.1354	reject
GE-IT	$\alpha_{GE} < \alpha_{IT}$ or $\beta_{GE} < \beta_{IT}$	3.2151	4.3479	reject
CN-US	$\alpha_{CN} < \alpha_{US}$ or $\beta_{CN} < \beta_{US}$	1.0904	5.2965	reject
GE-US	$\alpha_{GE} < \alpha_{US}$ or $\beta_{GE} < \beta_{US}$	-0.2381	6.0340	do not reject

Table 4: Null hypotheses, test statistics, and test results

We conclude that there is strong statistical evidence that Canada Lorenz dominates Italy, Germany Lorenz dominates Italy, and that Canada Lorenz dominates the USA, even if the tails are taken into account. This demonstrates the success and usefulness of the proposed test: in many cases we are able to infer that despite sample tail crossings the population Lorenz curves do, in fact, exhibit Lorenz dominance.

5 Conclusion

The appeal of the Lorenz dominance criterion is undermined by the fact that many sample Lorenz curves intersect in the tails. Tests for Lorenz dominance which ignore tails (such as those considering only deciles) are therefore invalidated. Moreover, the usual inferential methods, based on central limit theorem arguments, do not apply to the tails of the Lorenz curve since the tails contain too few observations. By contrast, we have proposed a test procedure, based on a domain of attraction assumption, which fully takes into account the tail behaviour of Lorenz curves. Our experiments and empirical examples demonstrate the success and usefulness of the proposed test:

in many cases we are able to infer that despite sample tail crossings the population Lorenz curves do, in fact, exhibit Lorenz dominance.

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A Alternative Hill plots

The alternative Hill plots for Canada (CN), Germany (GE), Italy (IT), and the USA (US) are shown in figures 2 (parameter α for the upper tail) and 3 (parameter β for the lower tail).

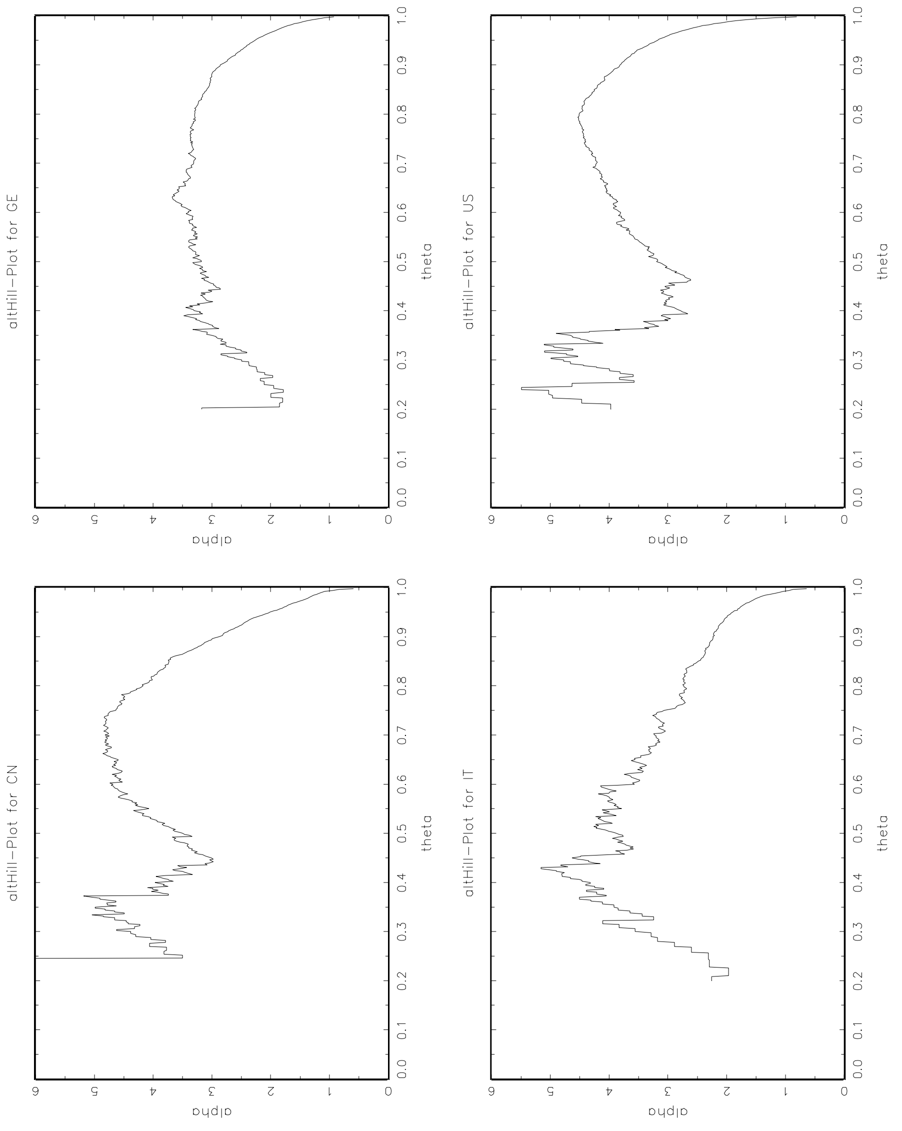

Figure 2: Alternative Hill plots for the upper tail parameter (α)

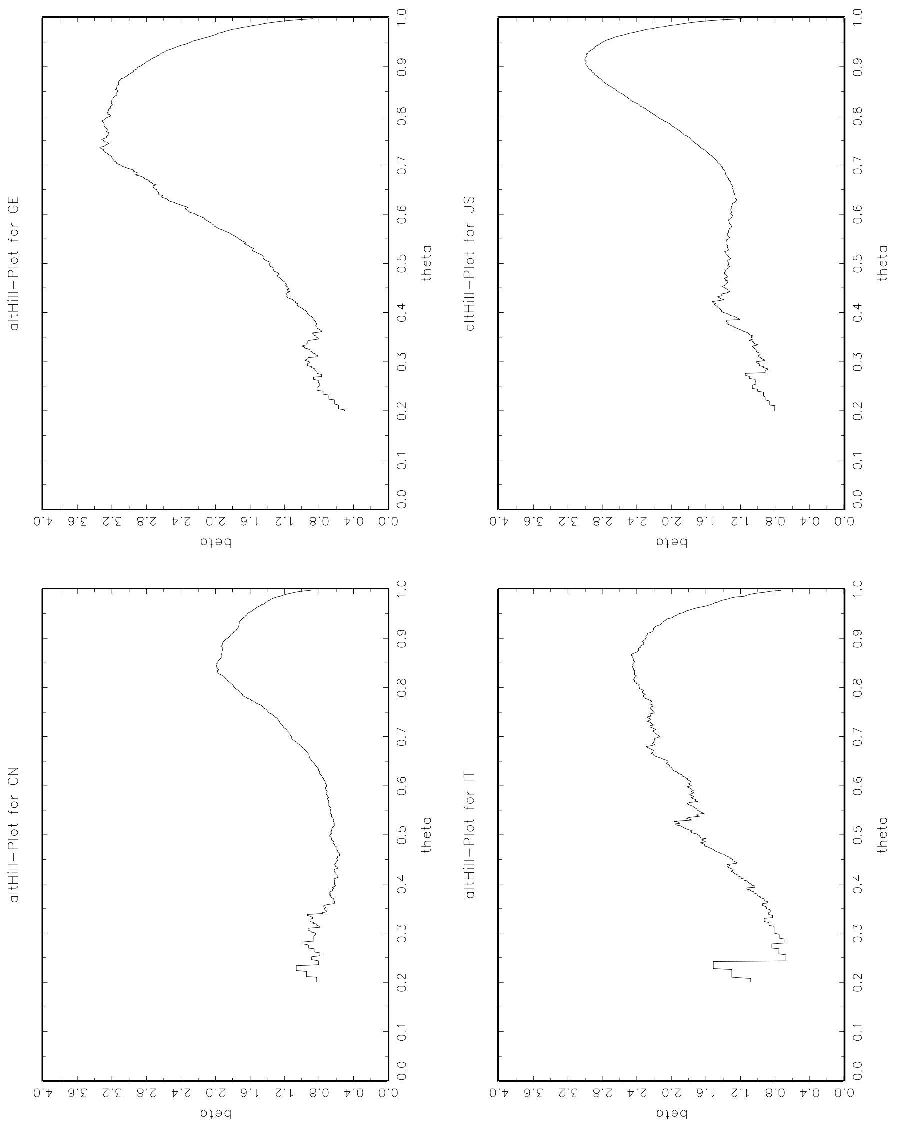

Figure 3: Alternative Hill plots for the lower tail parameter (β)

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