DISCUSSION PAPERS IN STATISTICS AND ECONOMETRICS

SEMINAR OF ECONOMIC AND SOCIAL STATISTICS
UNIVERSITY OF COLOGNE

No. 4/96

Inequality and negative income

by

Andreas Stich*

May 1996

Abstract

This paper deals with some problems in the measurement of inequality when negative incomes are allowed. A helpful axiom is defined, called the **G**reatest **G**ets **M**ore axiom. Using this axiom it can be shown that the properties of some inequality measures depends on whether there are negative incomes or not. In this paper for the intermediate measures of Eichhorn and the centrist inequality measures of Kolm a threshold value is given above which the Greatest Gets More axiom holds. Furthermore, a simple proof is given for the fact that there exists no function which fulfills the three axioms Pigou–Dalton, homogeneity and additive invariance when the data contain negative incomes.

JEL classification: D63

*Seminar für Wirtschafts– und Sozialstatistik, Universität zu Köln, Albertus–Magnus–Platz, 50923 Köln, Deutschland; Tel: +49/221/470 2809, Fax: +49/221/470 5074, e–mail: stich@wiso.uni-koeln.de

1 Introduction

Usually, negative incomes are excluded from the measurement of income inequality. This only makes sense for some definitions of income. However, for other, broader, definitions negative incomes are possible. For example, if income is defined as gross disposable income (i.e., real earnings plus real capital income plus total nonfamily transfers less income tax), gross factor income or real capital income (see Morgan et al. (1962), p. 494 and p. 498–500.).

Although there exists much literature about inequality measurement only a few authors have contributed to the measurement of inequality in the case of negative income, see, e.g., Chen et. al. (1982) and (1985), Berrebi and Silber (1985) and Arora et al. (1990). These articles give a transformation of the Gini coefficient. They notice that the regular Gini index is not normalized when some incomes are negative. Therefore, they transform this function to fulfill this axiom. Arora et al. (1990) show the asymptotic distribution of these transformations and proof them to be identical with that of the regular Gini index. But what about other inequality measures or other properties of the Gini index such as homogeneity? This axiom says that the inequality should not change when all incomes are multiplied by the same constant, i.e., there is no difference whether income is measured in US–dollar or DM. This paper shows some relations between the axioms of inequality measures. With the help of these it is easy to see that there exists no function fulfilling the axioms of Pigou–Dalton, homogeneity and additive invariance.

Section 2 gives some preliminaries and the axioms. The third section shows the interactions between the axioms and how they change when negative incomes are considered. Section 4 prove the non-existence of functions which fulfill the Pigou–Dalton principle, homogeneity and additive invariance. Some conclusions are drawn in the final section.

2 Axioms and Preliminaries

Let $x = (x_1, \ldots, x_n)$ be the vector of incomes. The elements of this vector are assumed to be ordered, i.e., $x_1 \leq \ldots \leq x_n$ because the order of the incomes does not change the degree of inequality. Let the vector $e = (1, \ldots, 1)$ be a n–dimensional vector of ones.

The following notation is used

$$\begin{aligned}
\mathbb{R}^n_+ &= \{x \in \mathbb{R}^n \;:\; x_i \geq 0\} \\
\mathcal{D}_+ &= \bigcup_{n=2}^{\infty} \mathbb{R}^n_+ \\
\mathcal{D} &= \bigcup_{n=2}^{\infty} \mathbb{R}^n \\
\mathcal{D}^* &= \bigcup_{n=2}^{\infty} \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i > 0 \right\}.
\end{aligned}$$

Although negative incomes should be considered the set of all vectors in $\mathbb{R}^n$ is restricted to those whose sum is positive. This is done because looking at vectors with mostly negative income is not suggestive.

Definition:
A function $I : \mathcal{D}^* \to \mathbb{R}$ is an inequality measure if it fulfills the Pigou–Dalton principle, i.e., if a richer person gives some of his or her income to a poorer person (so that the richer is not poorer than the poorer was before) the inequality should fall.

In the following a transfer from a poorer to a richer person will be called an inverse Pigou–Dalton transfer. From the above definition follows that an inverse Pigou–Dalton transfer increases inequality.

Now some useful axioms of inequality measures are defined. The first axiom is the Pigou–Dalton principle (PD) assigned in the above definition.

A further axiom is: If all elements of the vector x are multiplied by the same constant $\lambda > 0$ than the inequality should not change. This means the inequality index is independent of the unit in which the income is measured. This axiom is called homogeneity

$$(H) \quad I(x) = I(\lambda x) \quad \forall \lambda > 0.$$

Measures which fulfill (PD) and (H) are called rightist inequality measures according to Kolm. Their counterpart are leftist inequality measures. The argument which is used here is that the rich gets richer and the poor gets poorer if all vector elements are increased by the same proportion. So the inequality should change. It would be even fairer if all people get the same amount in absolute terms, i.e., to every element the same constant should be added. In this case the inequality should not change. Hence leftist inequality measures should fulfill the axiom of additive invariance

$$(AI) \quad I(x) = I(x + \delta e) \quad \forall \delta > 0.$$

A leftist inequality measure is given in Kolm (1976a, b)

$$K_\alpha(x) = \frac{1}{\alpha} \ln \left[\frac{1}{n} \sum_{i=1}^{n} e^{\alpha(\bar{x} - x_i)} \right].$$

A compromise between these extremes are the intermediate inequality measures. They are defined through the Pigou–Dalton principle and λ invariance (see Eichhorn (1988), Pfingsten (1988) and Bossert and Pfingsten (1990))

$$(\lambda I) \quad I(x) = I(x + \tau(\lambda x + (1 \Leftrightarrow \lambda)e))$$

with $0 \le \lambda \le 1$ fixed, τ is any scalar such that $x + \tau(\lambda x + (1 \Leftrightarrow \lambda)e) \in \mathbb{R}^n_+$. Obviously, this requirement reduces to (H) for $\lambda = 1$ and to (AI) for $\lambda = 0$. Furthermore it allows some intermediate value judgments, too.

Eichhorn (1988) shows that λ–invariant functions for $\lambda \in]0, 1]$ are of the form

$$I(x) = f\left(\frac{\lambda x + (1 \Leftrightarrow \lambda)e}{\lambda \bar{x} + (1 \Leftrightarrow \lambda)} \right), \tag{2.1}$$

where $f : \mathbb{R}_+^n \backslash \{0\} \to \mathbb{R}$ is an arbitrary function. Demanding the (PD) axiom reduces the set of functions to all schur–convex functions f. Note that the Lemma of Eichhorn is also true in the case of negative income because in the proof the positivity of the x–values is not used.

Kolm (1976a, b) proposed a centrist inequality concept. The corresponding axiom is

$$(CI) \quad I(\alpha(x \Leftrightarrow \beta e) + \beta e) = \alpha I(x)$$

with $\Leftrightarrow\infty < \beta \leq 0$, $\alpha \in \mathbb{R}$ such that $[\alpha(y \Leftrightarrow \beta e) + \beta e) \in \mathcal{D}_+$. For $\beta \to \Leftrightarrow\infty$ (CI) approaches (AI). The drawback of this class is that for $\beta = 0$ not (H) but $I(\alpha x) = \alpha I(x)$ is obtained (see Kolm (1976a)).

Another helpful axiom is the **G**reatest **G**ets **M**ore axiom. Obviously the inequality rises if the greatest income increases ceteris paribus. So the inequality index should rise if the greatest ceteris paribus gets more, i.e.,

$$(GGM) \quad I(x) < I(x_1, \ldots, x_{n-1}, x_n + \kappa) \quad \forall \kappa > 0.$$

Let

$$y_i = \frac{x_i}{\sum\limits_{j=1}^{n} x_j}$$

be the income share of the i–th unit. Then the Gini coefficient can be written as

$$G(x) = \frac{2}{n} \sum_{i=1}^{n} i y_i \Leftrightarrow 1 \Leftrightarrow \frac{1}{n}.$$

The first transformation proposed by Chen et al. (1982) is

$$G^*(x) = \frac{G(x)}{1 + \frac{2}{n} \sum\limits_{j=1}^{k} j y_j}$$

where k is defined in such a way that $\sum\limits_{i=1}^{k} y_i = 0$ and $\sum\limits_{i=1}^{n} y_i$ has to be greater than zero. In most cases there is no k satisfying the first restriction. So Chen et al. (1982,1985) and Berreby and Silber (1985) gave a generalization

$$G^{**}(x) = \frac{G(x)}{1 + \frac{2}{n} \sum\limits_{j=1}^{k} j y_j + \frac{1}{n} \sum\limits_{j=1}^{k} y_j \left[\frac{\sum\limits_{i=1}^{k} y_i}{y_{k+1}} \Leftrightarrow (1 + 2k) \right]}$$

with k defined so that $\sum\limits_{i=1}^{k} y_i \leq 0$ and $\sum\limits_{i=1}^{k+1} y_i > 0$. Notice that for $\sum\limits_{i=1}^{k} y_i = 0$ both indices $G^*(x)$ and $G^{**}(x)$ are the same. A graphical interpretation is given in Chen et al. (1982). All indices fulfill (H) because the income shares are invariant against multiplications of the income with some constant. It is well known that the Gini index fulfills the Pigou–Dalton principle.

3 Relations between the axioms

The first Theorem in this section shows that if only positive values are allowed (GGM) follows from (PD) and (λI). This includes that (GGM) follows from (PD) and (H) or from (PD) and (AI) . In the second Theorem also negative values are considered. It shows that the result of the first Theorem holds only for (PD) and (AI). For $\lambda \in]0,1]$ all elements of x have to be greater than a threshold value to implement (GGM). This value depends only on the parameter λ. This is also be done for (CI) and (PD). In this case the same conclusion can be drawn.

Theorem (3.1)

Let $I : \mathcal{D}_+ \to \mathbb{R}$.

(i) (PD) and (λI) $\Rightarrow$ (GGM).

(ii) (PD) and (CI) $\Rightarrow$ (GGM).

Proof

(i):
Let $\lambda \in [0,1]$ be fixed. Take $\frac{\tau\lambda}{1+\tau\lambda} \cdot 100\%$ with $\tau > 0$ of the $n \Leftrightarrow 1$ poorest and give it to the richest, i.e.,

$$y = \left(\left(1 \Leftrightarrow \frac{\tau\lambda}{1+\tau\lambda}\right) x_1, \ldots, \left(1 \Leftrightarrow \frac{\tau\lambda}{1+\tau\lambda}\right) x_{n-1}, x_n + \frac{\tau\lambda}{1+\tau\lambda} \sum_{i=1}^{n-1} x_i \right). \tag{3.2}$$

Because x can be created from y by $n \Leftrightarrow 1$ Pigou–Dalton transfers the inequality of y is higher than that of x, $I(y) > I(x)$. Now take $\frac{\tau(1-\lambda)}{1+\tau\lambda}$ of the $n \Leftrightarrow 1$ poorest and give it to the richest, i.e.,

$$\begin{aligned} y' \;=\; & \left(\frac{1}{1+\tau\lambda} x_1 \Leftrightarrow \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda}, \ldots, \frac{1}{1+\tau\lambda} x_{n-1} \Leftrightarrow \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda}, \right. \\ & \left. x_n + \frac{\tau\lambda}{1+\tau\lambda} \sum_{i=1}^{n-1} x_i + (n \Leftrightarrow 1) \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda} \right) \end{aligned} \tag{3.3}$$

with $I(y') > I(y)$ $[> I(x)]$ following the same argument as in the first step. Using (λI) the inequality of y' and

$$\begin{aligned} z \;=\; & \left((1+\tau\lambda) \left[\frac{1}{1+\tau\lambda} x_1 \Leftrightarrow \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda} \right] + \tau(1 \Leftrightarrow \lambda), \ldots, \right. \\ & (1+\tau\lambda) \left[\frac{1}{1+\tau\lambda} x_{n-1} \Leftrightarrow \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda} \right] + \tau(1 \Leftrightarrow \lambda), \qquad (3.4) \\ & \left. (1+\tau\lambda) \left[x_n + \frac{\tau\lambda}{1+\tau\lambda} \sum_{i=1}^{n-1} x_i + (n \Leftrightarrow 1) \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda} \right] + \tau(1 \Leftrightarrow \lambda) \right) \\ =\; & \left(x_1, \ldots, x_{n-1}, x_n + \tau\lambda \sum_{i=1}^{n} x_i + n\tau(1 \Leftrightarrow \lambda) \right) \qquad (3.5) \\ =\; & (x_1, \ldots, x_{n-1}, x_n + \kappa) \,. \end{aligned}$$

with

$$\kappa = \tau\lambda \sum_{i=1}^{n} x_i + n\tau(1 \Leftrightarrow \lambda) > 0 \tag{3.6}$$

is the same ($I(z) = I(y')$). Putting the three steps together gives

$$I(x) < I(z)$$

and every vector z can be transformed in this way. Because for every $\kappa > 0$ with λ, n and $\sum_{i=1}^{n} x_i$ given τ can be chosen so that κ can be written as in (3.6). τ has to be

$$\frac{\kappa}{\lambda \sum_{i=1}^{n} x_i + n(1 \Leftrightarrow \lambda)}.$$

This is (GGM).

(ii):
With $\alpha > 1$,

$$y' = \left(\frac{1}{\alpha}x_1 \Leftrightarrow \frac{\beta(1 \Leftrightarrow \alpha)}{\alpha}, \ldots, \frac{1}{\alpha}x_{n-1} \Leftrightarrow \frac{\beta(1 \Leftrightarrow \alpha)}{\alpha}, \frac{1}{\alpha}x_n + \frac{\alpha \Leftrightarrow 1}{\alpha}\sum_{i=1}^{n} x_i + (n \Leftrightarrow 1)\frac{\beta(1 \Leftrightarrow \alpha)}{\alpha}\right)$$

and

$$\begin{aligned} z &= \alpha y' + \beta(1 \Leftrightarrow \alpha)e \\ &= (x_1, \ldots, x_{n-1}, x_n + \kappa) \\ &\quad \text{with } \kappa = (\alpha \Leftrightarrow 1)\sum_{i=1}^{n} x_i + n\beta(1 \Leftrightarrow \alpha) > 0 \end{aligned} \tag{3.7}$$

follows $I(z) = \alpha I(y') > I(y') > I(x)$ by the arguments as in (i). And for every $\kappa > 0$ an $\alpha > 1$ can be found so that κ can be written as in (3.7), i.e.,

$$\alpha = \frac{\kappa}{\sum_{i=1}^{n} x_i \Leftrightarrow n\beta} + 1.$$

But this is (GGM) and the proof is complete. □

The result of Theorem (3.1) is that (GGM) holds for all intermediate and centrist measures, especially for rightist and leftist inequality measures due to their definition. Before Theorem (3.2) is shown, a short example. Let $x = (\Leftrightarrow 40,\ 10,\ 30,\ 60)$ and $z = (\Leftrightarrow 40,\ 10,\ 30,\ 70)$ The values of the inequality indices are

$$\begin{array}{rclrcl} G(x) &=& 1.3333 & G(z) &=& 1.2500 \\ G^{**}(x) &=& 0.8421 & G^{**}(z) &=& 0.8333 \\ K_{0.2}(x) &=& 48.0688 & K_{0.2}(z) &=& 50.5688 \end{array}$$

Notice that $G^*(x)$ cannot be calculated. It is obvious that $G(x)$ violates (GGM). The same holds to $G^{**}(x)$ the only difference to $G(x)$ being that the values are normalized. Only the measure of Kolm orders the two vectors in the right way and fulfills (GGM). This behaviour can be proved with Theorem (3.2).

Theorem (3.2)

Let $I : \mathcal{D}^* \to \mathbb{R}$ and $\lambda \in]0, 1]$. Assume (PD) and (λI)

(i) (GGM) holds if

$$x_1 > \Leftrightarrow \frac{1 \Leftrightarrow \lambda}{\lambda}$$

(ii) (GGM) does not hold if

$$x_{n-1} < \Leftrightarrow \frac{1 \Leftrightarrow \lambda}{\lambda}$$

(iii) (GGM) holds if $\lambda = 0$.

Let $I : \mathcal{D}^* \to \mathbb{R}$ and $\beta \in] \Leftrightarrow \infty, 0]$. Assume (PD) and (CI)

(iv) (GGM) holds if

$$x_1 > \beta$$

Proof

$n = 2$
Let $x_1 < 0$ and $\tau > 0$. Look at

$$y' = \left(x_1 \Leftrightarrow \frac{\tau\lambda}{1+\tau\lambda} x_1 \Leftrightarrow \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda}, x_2 + \frac{\tau\lambda}{1+\tau\lambda} x_2 + \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda} \right)$$

If

$$(i) \qquad \Leftrightarrow \frac{\tau\lambda}{1+\tau\lambda} x_1 < \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda}$$

This is an inverse Pigou–Dalton transfer from x_1 to $x_2 \Rightarrow I(y') > I(x)$.

$$(ii) \qquad \Leftrightarrow \frac{\tau\lambda}{1+\tau\lambda} x_1 > \frac{\tau(1 \Leftrightarrow \lambda)}{1+\tau\lambda}$$

This is a Pigou–Dalton transfer from x_1 to $x_2 \Rightarrow I(y') < I(x)$

With

$$\begin{aligned} z &= y' + \tau(\lambda y' + (1 \Leftrightarrow \lambda)e) \\ &= (x_1, x_2 + \tau\lambda(x_1 + x_2) + 2\tau(1 \Leftrightarrow \lambda)) \end{aligned}$$

(see (3.5)) and the λ-invariance follows:

(i) $I(x) < I(y') = I(z)$ this is (GGM).

(ii) $I(x) > I(y') = I(z)$ this contradicts (GGM).

This proofs the Lemma for $n = 2$ using that the following equivalence holds:

$$\begin{aligned} \Leftrightarrow\frac{\tau\lambda}{1+\tau\lambda}x_1 &> \frac{\tau(1\Leftrightarrow\lambda)}{1+\tau\lambda} \\ \Leftrightarrow \Leftrightarrow x_1 &> \frac{\tau(1\Leftrightarrow\lambda)}{\tau\lambda} \\ \Leftrightarrow x_1 &< \Leftrightarrow\frac{1\Leftrightarrow\lambda}{\lambda} \end{aligned}$$

$n > 2$
(i):
Let $m := \min_i\{i \in \mathbb{N} | x_{(i)} < 0 \text{ and } x_{(i+1)} > 0\}$.
Only the m first elements have to be considered because if $x_j > 0$ the transfer is an inverse Pigou–Dalton transfer and obviously there is no problem in this case. Because the transactions between the single elements of the vectors can performed successively (GGM) is always fulfilled if

$$\begin{aligned} \max_{i=1,\ldots,m}\left(\Leftrightarrow\frac{\tau\lambda}{1+\tau\lambda}x_i\right) &< \frac{\tau(1\Leftrightarrow\lambda)}{1+\tau\lambda} \\ \Leftrightarrow \Leftrightarrow\frac{\tau\lambda}{1+\tau\lambda}x_1 &< \frac{\tau\lambda}{1+\tau\lambda} \quad \text{because of } x_1 \le x_2 \le \ldots \\ \Leftrightarrow x_1 &> \Leftrightarrow\frac{1\Leftrightarrow\lambda}{\lambda} \end{aligned} \tag{3.8}$$

(3.8) means that all transfers are inverse Pigou–Dalton transfers. This proofs (i).

(ii):
Obviously (GGM) is not fulfilled if all transfers are Pigou–Dalton transfers. This means

$$\begin{aligned} \min_{i=1,\ldots,n-1}\left(\Leftrightarrow\frac{\tau\lambda}{1+\tau\lambda}x_i\right) &> \frac{\tau(1\Leftrightarrow\lambda)}{1+\tau\lambda} \\ \Leftrightarrow \Leftrightarrow\frac{\tau\lambda}{1+\tau\lambda}x_{n-1} &> \frac{\tau(1\Leftrightarrow\lambda)}{1+\tau\lambda} \\ \Leftrightarrow x_{n-1} &< \Leftrightarrow\frac{1\Leftrightarrow\lambda}{\lambda}. \end{aligned} \tag{3.9}$$

(iii):
The third part of the Lemma is very simple. Because of

$$\lim_{\lambda\to 0} \Leftrightarrow\frac{1\Leftrightarrow\lambda}{\lambda} = \Leftrightarrow\infty$$

follows from (i) that (GGM) is implemented if $x_1 > \Leftrightarrow\infty$ and this is true for all $x \in \mathcal{D}$ This completes the proof.

(iv):
Let $\alpha > 1$. Looking at

$$y' = \left(x_1 \Leftrightarrow \frac{\alpha \Leftrightarrow 1}{\alpha} \Leftrightarrow \frac{\beta(1 \Leftrightarrow \alpha)}{\alpha}, x_2 + \frac{\alpha \Leftrightarrow 1}{\alpha} + \frac{\beta(1 \Leftrightarrow \alpha)}{\alpha}\right)$$

and the equivalence

$$\begin{aligned} \Leftrightarrow \frac{\alpha \Leftrightarrow 1}{\alpha} x_1 &> \frac{\beta(1 \Leftrightarrow \alpha)}{\alpha} \\ \Leftrightarrow \Leftrightarrow x_1 &> \frac{\beta(1 \Leftrightarrow \alpha)}{\alpha} \cdot \frac{\alpha}{\alpha \Leftrightarrow 1} \\ \Leftrightarrow x_1 &< \beta \end{aligned}$$

(iv) can be proved by following the argumentation of (i). □

The first limiting case is $\lambda \to 1$, i.e., (H). Here Lemma (3.2) says that all values of x must be positive to fulfill (GGM). The second case is $\lambda \to 0$, i.e., (AI). The consequence of the Lemma is that (GGM) is always fulfilled when (PD) and (AI) are required. On the other hand for all $\lambda \in]0, 1]$ follows that from (PD) and (λI) follows $\overline{(GGM)}$ in the entire set of $\mathcal{D}$.

The result of Theorem (3.2) (i) is consistent with the statement in Bossert and Pfingsten (1990) that intermediate inequality is relative inequality in a space where the origin is shifted from zero to $\Leftrightarrow(1 \Leftrightarrow \lambda)/\lambda$.

Between the two bounds found in Theorem (3.2) no general conclusion can be drawn. In this area the ordering of $I(x)$ and $I(z)$, i.e., the existence or non–existence of (GGM), depends on the behaviour of the function f. So it changes with the choice of the intermediate inequality measure.

Following Theorem (3.2) the Gini measures defined above have a great disadvantage. Their behaviour is incorrect when negative incomes are considered. Because of the negative reaction of measures which fulfill (PD) and (H) one should reconsider the use of these measures with negative income. An alternative is the measure of Kolm which does not have this disadvantage and reacts correctly also with negative values. Alternatives can be created using the intermediate measures in (2.1). But λ has to be very close to 0 to get a useful measure, e.g. with $\lambda = 0.001$ the threshold value becomes only -999. Other alternatives are the centrist measures of Kolm (1976a, b).

4 Homogeneity, additive invariance and negative income

It would be very nice to find an inequality measure which fulfills the three axioms (PD), (H) and (AI) and so have a function which satisfies not only the independence from the unit in which the income is measured but also the demand of the leftist indices. Eichhorn (1980) has shown that only very specific functions fulfills (H) and (AI) for positive values.

He also shows that these functions do not fulfill (PD) (Eichhorn (1980), Satz 9). The same can be shown if negative incomes are considered. With Theorem (3.2) it is easily proved that there exists no function for negative incomes which fulfills the axioms (PD), (H) and (AI).

Theorem (4.1)

There exists no function $I : \mathcal{D}^* \to \mathbb{R}$ which fulfills (PD), (H) and (AI).

Proof

The statement of Theorem (3.2) with $\lambda = 1$ is equivalent to

$$\text{(PD) and (H)} \Rightarrow \overline{(GGM)}$$
$$\Leftrightarrow$$
$$\text{(GGM)} \Rightarrow \overline{(PD)} \text{ or } \overline{(H)}.$$

Let $I(x)$ fulfill (PD) and (AI). From these axioms it follows that (GGM) is also true (Theorem (3.2) (iii)). So (PD) and (GGM) hold. From the above statement then follows that (H) cannot hold. This proofs the Theorem. □

5 Conclusion

Taking negative incomes into account raises a problem in inequality measurement. Measures which are accepted as "good" inequality measures suddenly show untypical reactions to transformations of the income vector. So these indices are not as "optimal" as in the measurement of inequality with positive income. But should the axiom of homogeneity be given up to save the axiom (GGM) or should the axiom (GGM) be neglected? Until now the only alternative is the measure of Kolm (or other leftist inequality measures) to save the (GGM) axiom for all observations. If the support is truncated at the lower end (say at a point $a < 0$) an intermediate inequality measure can be used with λ lower than $\Leftrightarrow(a \Leftrightarrow 1)^{-1}$. Then (GGM) is also fulfilled on this area. This is true for the centrist measures with $\beta < a$ also.

In coincidence to the result from Eichhorn (1980) Theorem (4.1) shows that there exists no function which fulfills (PD), (H) and (AI). So it is not possible to create an inequality index which fulfills both, (H) and (AI).

Finally, one may conclude that intermediate, centrist and leftist inequality measures are good alternatives to measure inequality with negative income. The only problem is choosing the parameter to accomplish both, (GGM) and the personal idea of value judgment.

References

Arora, S; Julka, A.A.; Bagai, O.P. (1990): Testing the significance related to Gini ratio, non-parametric test statistics. *Journal of the Indian Society of Agriculture Statistics,* XLII, 118-130.

Berreby, Z.M.; Silber, J. (1985): The Gini coefficient and negative income: A comment. *Oxford Ecomomic Papers,* 37, 525-526.

Bossert, W.; Pfingsten, A. (1990): Intermediate inequality, Concepts, indices, and welfare implications. *Mathematical Social Sciences,* 19, 117-134.

Chen, C.-N.; Tsaur, T.-W.; Rhai, T.-S. (1982): The Gini coefficient and negative income. *Oxford Economic Papers,* 34, 473-478.

Chen, C.-N.; Tsaur, T.-W.; Rhai, T.-S. (1985): The Gini coefficient and negative income: A reply. *Oxford Economic Papers,* 37, 527-528.

Eichhorn, W. (1980): Wirtschaftliche Kennzahlen. *in: Henn, R.; Schips, B.; Stähly, P. (eds): Quantitative Wirtschafts- und Unternehmensforschung, Springer-Verlag, Berlin.*

Eichhorn, W. (1988): On a class of inequality measures. *Social Choice and Welfare,* 5, 171-177.

Kolm, S.-C. (1976a): Unequal inequalities I. *Journal of Economic Theory,* 12, 416-442.

Kolm, S.-C. (1976b): Unequal inequalities II. *Journal of Economic Theory,* 13, 82-111.

Morgan, J.N.; David, M.H.; Cohen, W.J.; Brazer, H.E. (1962): *Income and welfare in the United States.* McGraw-Hill, New York.

Pfingsten, A. (1988): New concepts of Lorenz domination and risk aversion. *Methods of Operations Research,* 59, 75-85.