Attenuation of relative poverty by taxes and subsidies

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Abstract[1]

The paper is the first systematic investigation into income redistribution functions that reduce relative poverty as measured by indices and orderings. A general class of poverty indices is introduced which includes the indices due to Sen, Thon, Kakwani, Chakravarty, FGT and many others. The functional form of a tax and/or subsidies function is derived which decreases the indices in this class. The same is done for poverty orderings. Further, the attenuation of poverty indices is investigated when the poverty line is a percentage of a quantile. Finally, some of the results are extended to the measurement of multivariate poverty.

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1 Introduction

Public policy often aims at improving the economic situation of a part of the population that is relatively poor. The focus is on the individuals or households that live at the

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bottom of the income pyramid, and the goal is to decrease income poverty by a proper redistribution scheme of taxes and subsidies. However, there is an almost complete lack of literature comparing poverty before and after redistribution. For a recent investigation into anti-poverty programs, see Ebert (2005). The time seems therefore right to begin such a study. In our investigation two principal questions arise: How shall we measure poverty, and which redistribution scheme reduces poverty with respect to the poverty measure chosen?

Indices of income poverty have been proposed by many authors. Each index provides a special view on the nature of poverty, satisfies certain postulates (and others not), and evaluates the income distribution in a particular way. Also, partial orderings of poverty have been introduced and shown to be consistent with certain classes of poverty indices. A comprehensive survey of poverty measurement has been recently provided by Chakravarty and Muliere (2004). Earlier surveys offering different aspects are Seidl (1988) and Zheng (1997).

An index of poverty is commonly constructed in two steps. Firstly, identify the poor and, secondly, quantify to which extent they are poor. The poor part of the population is determined by an upper bound on income, the poverty line, which may be exogenously fixed (according to some level of subsistence) or endogenously given as a function of a distribution quantile or similar. Given the poverty line, the poverty index is designed as a function of the incomes of the poor and the size of the population.

A redistribution scheme of subsidies and/or taxes transforms a given income distribution into another one. The income of every individual – rich or poor – may be changed, but the ranking of incomes shall remain unchanged. Often a redistribution policy is aimed at a certain part of the voting population, here, the poor. Thus we are specifically interested in measuring the change in poverty for this part of the population.

This paper deals with redistributions in terms of income, and not explicitly with those in terms of assets. While income is regarded as the principal source of welfare, nonmonetary or targeted redistributions certainly play some role in alleviating poverty. However, these may be included by their monetary equivalents in our analysis.

Our first question, how to measure poverty, is difficult, if not impossible, to answer. Many authors have proposed a great variety of indices and there appears to be no general argument to pick one of these approaches for the purpose of comparing pre-government and post-government poverty. Therefore, our approach is to consider a large class of poverty indices that virtually contains all important popular measures –at least in the limit–, and to search for conditions on the redistribution scheme that uniformly reduce all measures in this comprehensive class.

We demonstrate that this general class of indices satisfies the principal postulates of poverty measurement. In evaluating redistribution schemes we focus on a given percentage of people with smallest income. The poverty line, call it π, corresponds to the largest among these incomes. Hence after redistribution with a strictly increasing transformation function g the poverty line becomes $g(\pi)$. We derive a condition for the transformation function, which is necessary and sufficient to reduce poverty as measured by any index in the class.

We then consider related poverty orderings and obtain similar results on their attenuation by redistribution functions. When the poverty line is a percentage of a quantile (*e.g.* 60 per cent of the median), the number of poor can be different after redistribution; a similar reduction result will be derived for this case.

Several authors have argued that economic status and, in particular, poverty is a multiattribute phenomenon. See Atkinson and Bourguignon (1982) and, for recent developments, Bourguignon and Chakravarty (2003). Special multivariate poverty indices have been proposed by Tsui (2002), Bourguignon and Chakravarty (2003) and others. For these indices we also characterize the redistribution functions which diminish multivariate poverty.

To our knowledge, there exists no systematic treatment of the reduction of poverty measures in the literature. However, the reduction of economic inequality has been already investigated in the 1970s. Fellman (1976) and Jakobsson (1976) have characterized redistribution functions that reduce the Lorenz dominance and, by this, all Schur-convex inequality indices. See also Eichhorn et al. (1984), Arnold (1987, Chapter 4) and Arnold (1991). For absolute inequality see Moyes (1988).

Overview: Section 2 introduces the general class of poverty indices, discusses the postulates fulfilled and mentions many special established poverty indices that are included in this class. In Section 3 we characterize the redistributions by which these indices are lowered. Two cases are distinguished, pure taxation and redistribution allowing for taxes and subsidies. Section 4 investigates the same for poverty orderings. In Section 5 we generalize the result of Section 3, by defining the poverty line as a percentage of a quantile. Section 6 investigates the measurement of multivariate poverty. Section 7 concludes.

2 A general class of poverty indices

This Section introduces a general class of poverty indices which contains a large number of established special indices.

Consider a population of N individuals and let

$$\mathcal{D} = \{(x_1, \ldots, x_N) \in \mathbb{R}_+^N | x_1 \leq x_2 \leq \ldots \leq x_N\}$$

denote the set of all ordered income vectors, that is, income distributions. To measure the poverty of an income distribution we use a social evaluation function φ in connection with a poverty line π. The poverty line π is assumed to be a function of the income distribution,

$$\pi = \pi(x_1, \ldots, x_N)\,, \quad (x_1, \ldots, x_N) \in \mathcal{D}\,,$$

and $p = p(x_1, \ldots, x_N) = \#\{i | x_i < \pi, i = 1, \ldots, N\}$ denotes the number of poor persons. In the sequel we consider a general class $\mathcal{P}$ of poverty indices. Each index $\varphi \in \mathcal{P}$ is a social evaluation function of the form

$$\varphi(x_1, \ldots, x_N) = v\left(\sum_{i=1}^{N} w_{p,N}(i) u\left(\frac{x_i}{\pi}\right)\right)\,, \quad (x_1, \ldots, x_N) \in \mathcal{D}\,, \tag{1}$$

where

- $w_{p,N} : \{1, 2, \ldots, N\} \to \mathbb{R}_+$ is a decreasing[2] weight function, which may additionally depend on the number p of poor persons and the population size N,
- $u : \mathbb{R}_+ \to \mathbb{R}_+$ is an individual illfare function that strictly decreases on $[0, 1[$ and vanishes on $[1, \infty[$.
- $v : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing function.

Recall that we have defined our indices for ranked incomes. In case of non-ranked incomes the argument i in the weight function has to be replaced by the rank $R(x_i)$.

Many postulates have been formulated in the literature that may or should be satisfied by a poverty index. Some postulates are seemingly undebated requirements of poverty measurement, others appear in multiple forms and different degrees of strength. The following Theorem 1 says that most of the standard postulates are met by all indices either in the class $\mathcal{P}$ or in certain subclasses of $\mathcal{P}$. A description of the postulates and a proof of Theorem 1 are given in the Appendix.

Theorem 1 (Postulates). *Every poverty index $\varphi \in \mathcal{P}$ satisfies the postulates of*

1. *focus, symmetry, and weak monotonicity,*
2. *strong monotonicity and increasing poverty line if $w_{p,N}$ is increasing in p,*
3. *continuity if u and v are continuous,*
4. *minimal transfer and weak transfer if $w_{p,N}$ is constant and u is strictly convex on $[0, 1]$; strong transfer if, moreover, $w_{p,N}$ is decreasing in p,*
5. *population principle if $w_{p,N}$ is proportional to $\frac{1}{N}$ or $\frac{1}{p}$,*
6. *subgroup consistency if $w_{p,N}$ only depends on p and N,*
7. *subgroup decomposability if $w_{p,N}$ is proportional to $\frac{1}{N}$,*

[2] In this paper, unless otherwise stated, the words "decreasing" and "increasing" are meant in the weak sense.

8. non-poverty growth if $w_{p,N}$ is decreasing with N.

Remark 1. *Note that the poverty growth postulate is not generally satisfied in $\mathcal{P}$. E.g., if $w_{p,N}(\cdot) = \frac{1}{N}$ and the illfare of the new person is small enough, $u(\frac{x_{new}}{\pi}) < \frac{1}{N}\sum u(\frac{x_i}{\pi})$, then the index does not rise but fall.*

Many important poverty indices are included in the class $\mathcal{P}$, see Table 1. Among the indices in $\mathcal{P}$, positional and non-positional indices may be distinguished. The weights of the former depend on the ranks, while the latter do not. The upper part of Table 1 exhibits six examples of non-positional indices, the lower part three positional ones.

Further, Foster and Shorrocks (1991, Prop. 4) consider a class of poverty measures by $F\left(\frac{1}{N}\sum_{i=1}^{N}\phi\left(\frac{x_i}{\pi}\right)\right)$.If F is strictly increasing and ϕ is strictly decreasing this class of poverty measures is included in $\mathcal{P}$.

While $\mathcal{P}$ is a rather comprehensive class, there exist well-known poverty measures which are not in $\mathcal{P}$. The most obvious one is the *headcount ratio*, which is excluded by our assumption of a *strictly* decreasing illfare function. However, the *headcount ratio* is obtained as a limit of indices in $\mathcal{P}$. Also, the indices by Jenkins and Lambert (1997), Atkinson (1987) and Zheng (2000a) are not contained in $\mathcal{P}$, since they either employ no strictly decreasing illfare function or are not scale invariant.

3 Index reducing redistributions

In this Section the effect of income redistributions by taxes and/or subsidies is investigated. For this, the values of a poverty index before and after redistribution are compared. As usual in politics the focus is on a part of a population, here on the 'poor' part, that is, on the individuals whose income before redistribution does not exceed a poverty line. After redistribution the same individuals are considered as 'poor' and the index is evaluated with the properly transformed poverty line. For redistribution, any increasing function of incomes is considered. Following a proposal by Hagenaars and van Praag (1985) we

Table 1: Some well-known poverty indices which are in $\mathcal{P}$, non-positional (upper part of the table) and positional indices. Notation: $z_+ = \max\{z, 0\}$.

Index	$\varphi(x_1,\ldots,x_N), (x_1,\ldots,x_N) \in \mathcal{D}$	with	$w_{p,N}(i)$	$u\left(\frac{x_i}{\pi}\right)$	$v(y)$
income gap ratio	$\frac{1}{p}\sum_{i=1}^{p}\frac{\pi - x_i}{\pi}$		$\frac{1}{p}$	$\left(1-\frac{x_i}{\pi}\right)_+$	y
Chakravarty (1983)	$\frac{1}{N}\sum_{i=1}^{p}\left(1-\left(\frac{x_i}{\pi}\right)^e\right)$	$0 < e < 1$	$\frac{1}{N}$	$\left(1-\left(\frac{x_i}{\pi}\right)^e\right)_+$	y
Foster et al. (1984)	$\frac{1}{N}\sum_{i=1}^{p}\left(\frac{\pi - x_i}{\pi}\right)^\alpha$	$\alpha > 0$	$\frac{1}{N}$	$\left(1-\frac{x_i}{\pi}\right)_+^\alpha$	y
Watts (1968)	$\frac{1}{N}\sum_{i=1}^{p}(\ln\pi - \ln x_i)$		$\frac{1}{N}$	$\left(-\ln\left(\frac{x_i}{\pi}\right)\right)_+$	y
(Clark et al., 1981, p. 519)	$\left(\frac{p^{\alpha-1}}{N^\alpha}\sum_{i=1}^{N}\left(\frac{\pi - x_i}{\pi}\right)^\alpha\right)^{\frac{1}{\alpha}}$	$\alpha \geq 1$	$\frac{p^{\alpha-1}}{N^\alpha}$	$\left(1-\frac{x_i}{\pi}\right)_+^\alpha$	$y^{\frac{1}{\alpha}}$
(Clark et al., 1981, p. 522)	$1 - \left(\frac{1}{N}\sum_{i=1}^{N}\left(\min\left\{\frac{x_i}{\pi}, 1\right\}\right)^\beta\right)^{\frac{1}{\beta}}$	$0 < \beta < 1$	$\frac{1}{N}$	$\left(1-\left(\frac{x_i}{\pi}\right)^\beta\right)_+$	$1-(1-y)^{\frac{1}{\beta}}$
		$\beta < 0$		$\left(\left(\frac{x_i}{\pi}\right)^\beta - 1\right)_+$	$1-(1+y)^{\frac{1}{\beta}}$
Sen (1976)	$\frac{2}{(p+1)N}\sum_{i=1}^{p}(p+1-i)\frac{\pi - x_i}{\pi}$		$\frac{2(p+1-i)}{(p+1)N}$	$\left(1-\frac{x_i}{\pi}\right)_+$	y
Thon (1979)	$\frac{2}{(N+1)N}\sum_{i=1}^{p}(N+1-i)\frac{\pi - x_i}{\pi}$		$\frac{2(N+1-i)}{(N+1)N}$	$\left(1-\frac{x_i}{\pi}\right)_+$	y
Kakwani (1980)	$\frac{p}{N\sum_{i=1}^{p} i^k}\sum_{i=1}^{p}(p+1-i)^k\frac{\pi - x_i}{\pi}$	$k \geq 0$	$\frac{p(p+1-i)^k}{N\sum_{i=1}^{p} i^k}$	$\left(1-\frac{x_i}{\pi}\right)_+$	y

assume that the poverty line is a quantile of the income distribution. In Section 5 we return to the common definition of a poverty line as a percentage of a quantile.

We provide a necessary and sufficient condition on the form of the redistribution function to reduce poverty as measured by any index from $\mathcal{P}$. Two types of redistributions will be distinguished, pure taxation and a combination of taxes and subsidies. The latter comes also under the headings 'negative income tax' and 'citizen's tax'.

Since we investigate relative poverty measures, proportional taxation will not change the indices. The case of reducing poverty by pure taxation is therefore equal to the comparison of the proportional with a different kind of taxation.

Consider some $(x_1, \ldots, x_N) \in \mathcal{D}$ and a strictly increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$. Let $y_i = g(x_i), i = 1, \ldots, N$. Then $(y_1, \ldots, y_N) \in \mathcal{D}$. We interpret $(x_1, \ldots, x_N)$ as a given income distribution before taxes and subsidies and $(y_1, \ldots, y_N)$ as the respective distribution after redistribution. The redistribution function g allows for taxes and subsidies. The restriction $g(x) \leq x$, with $g(0) = 0$, corresponds to pure taxation.

Theorem 2 (Index reduction). (i) *Taxes and subsidies: Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing redistribution function. Then for $\varphi \in \mathcal{P}$*

$$\varphi(g(x_1), \ldots, g(x_N)) \leq \varphi(x_1, \ldots, x_N) \quad \textit{for all} \;\; (x_1, \ldots, x_N) \in \mathcal{D} \tag{2}$$

if and only if

$$\frac{g(x)}{g(\pi)} \geq \frac{x}{\pi} \quad \textit{for} \;\; 0 \leq x \leq \pi \,, \tag{3}$$

where π is the poverty line of distribution $(x_1, \ldots, x_N)$.

(ii) *Taxes only: The same statement holds if g satisfies the restriction*

$$g(x) \leq x \,. \tag{4}$$

Inequality (3) says that the relative incomes of the poor, relative to the poverty lines π and $g(\pi)$, have to increase. Equivalently, this can be put in the following way:

The post-redistribution income of every poor individual must exceed the pre-government income times a constant factor, which is the ratio of the transformed poverty line over the original one,

$$g(x) \geq \frac{g(\pi)}{\pi} x \quad \text{for all } \ 0 \leq x \leq \pi \,.$$

Adding inequality (4) yields a condition which is necessary and sufficient for reduction by pure taxation:

$$\frac{g(\pi)}{\pi} x \leq g(x) \leq x \quad \text{for } \ 0 \leq x \leq \pi \tag{5}$$

To illustrate inequality (3) consider the following example:

Let t be a flat tax, *e.g.* $t = 25\%$, with basic allowance a, where $a < \pi$. Assume that the incomes x_i before taxation are $x_1 \leq \ldots \leq x_n$. Then the incomes after taxation are $x_1 \leq \ldots \leq x_j \leq a < x_{j+1} - (x_{j+1} - a)t \leq \ldots \leq x_n - (x_n - a)t$. The incomes of the poorest do not change and poverty line is now at $\pi - (\pi - a)t$. It holds

$$\frac{x_i}{\pi - (\pi - a)t} \geq \frac{x_i}{\pi} \quad \text{for all } x_i \leq a$$

and

$$\frac{x_i - (x_i - a)t}{\pi - (\pi - a)t} \geq \frac{x_i}{\pi} \quad \text{for all } a < x_i \leq \pi \,,$$

since poverty line decreases more than the incomes of the poor above basic allowance a. Hence, condition (3) is fulfilled. Altogether the incomes of the poor increase relative to the poverty line and relative poverty decreases by theorem 2.

Figure 1 shows the restrictions on g in the two cases: The graph of g is restricted to the shaded areas.

Proof of theorem 2. Since the x_i are ordered, the poverty line is a quantile, and g is strictly increasing, we obtain

$$\pi(g(x_1), \ldots, g(x_N)) = g(\pi(x_1, \ldots, x_N)).$$

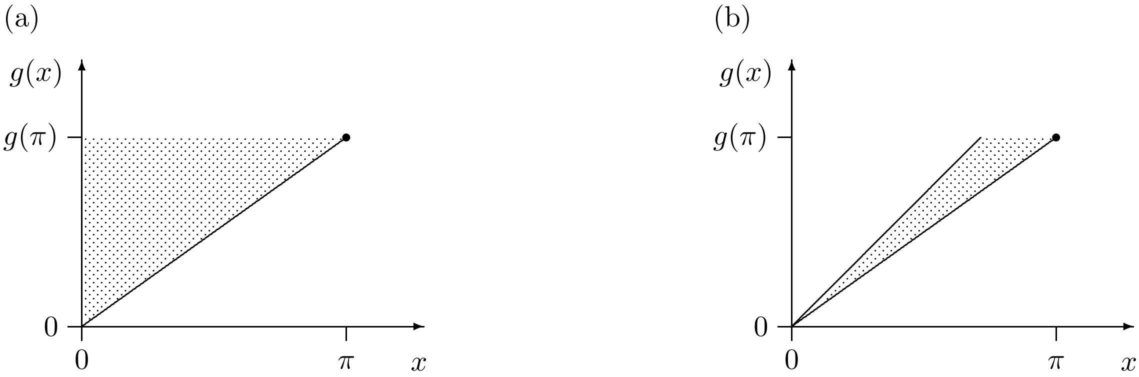

Figure 1: To reduce poverty, the redistribution function g has to be strictly increasing, with its graph being restricted to the shaded areas; (a) taxes and subsidies, (b) taxes only.

Now assume that the inequality (3) holds. Transforming both sides of it with a strictly decreasing u and summing up with weights $w_{p,N}(i)$, yields

$$\sum_{i=1}^{N} w_{p,N}(i) u\left(\frac{g(x_i)}{g(\pi)}\right) \leq \sum_{i=1}^{N} w_{p,N}(i) u\left(\frac{x_i}{\pi}\right) , \tag{6}$$

and, after transforming both sides with a strictly increasing v, (2) is obtained.

To prove the reverse, assume that there is some $x_0 < \pi$ with $\frac{g(x_0)}{g(\pi)} < \frac{x_0}{\pi}$. Using again the above arguments, we obtain

$$u\left(\frac{g(x_0)}{g(\pi)}\right) > u\left(\frac{x_0}{\pi}\right) ,$$

By choosing $x_i = x_0$ for all poor persons i, a contradiction is obtained. □

Note that condition (3) in theorem 2 characterizes those redistributions that reduce poverty as measured by any of the measures in class $\mathcal{P}$. That is, we may choose a specific poverty index (1) that satisfies certain restrictions and axioms, say, transfer principles, and the theorem tells us that this specific index is reduced if and only if the redistribution function satisfies inequality (3).

Similar results are obtained for absolute poverty measures *e.g.* the aggregate poverty gap: Consider a class $\mathcal{P}_{abs}$ of absolute poverty indices, with

$$\varphi(\mathbf{x}) = v\left(\sum_{i=1}^{n} w_{p,N}(i) u(x_i - \pi)\right) ,$$

where $u : [-\pi, \infty[\to \mathbb{R}_+$ is an individual illfare function that strictly decreases on $[-\pi, 0]$ and vanishes elsewhere; the other functions are defined as aforementioned. Theorem 2 holds for this class $\mathcal{P}_{abs.}$ with the condition

$$g(\pi) - g(x) \leq \pi - x \quad \text{for } \ 0 \leq x \leq \pi \, ,$$

in place of inequality (3).

4 Reduction of poverty orderings

The choices of a particular poverty line and a special poverty index are often difficult to justify. We therefore look for more general measurements that allow for some arbitrariness of poverty line and index, namely partial orderings of poverty. They are defined as uniform dominance of one income distribution over another with respect to a whole class of poverty lines and/or indices. See Zheng (2000b) for a survey of such orderings.

Definition 4.1 (Ordering uniform in index). *Consider a poverty line function π and a non-empty class of indices $\mathcal{P}_0 \subseteq \mathcal{P}$. For $(x_1, \ldots, x_N)$ and $(y_1, \ldots, y_N) \in \mathcal{D}$ define the poverty ordering*

$$(y_1, \ldots, y_N) \preceq_{\mathcal{P}_0, \pi} (x_1, \ldots, x_N)$$

if $\varphi(y_1, \ldots, y_N) \leq \varphi(x_1, \ldots, x_N)$ *holds for all* $\varphi \in \mathcal{P}_0$.

More generally, we may consider an interval of poverty lines, $]0, \pi_1]$, in definition 4.1, and generalize the ordering to an ordering that is also uniform in poverty lines.

Definition 4.2 (Ordering uniform in index and line). *Consider a poverty line π_1 and a non-empty class of indices $\mathcal{P}_0 \subseteq \mathcal{P}$. Define*

$$(y_1, \ldots, y_N) \preceq_{\mathcal{P}_0, \leq \pi_1} (x_1, \ldots, x_N)$$

if $\varphi(y_1, \ldots, y_N) \leq \varphi(x_1, \ldots, x_N)$ *holds for all* $\varphi \in \mathcal{P}_0$ *and all poverty lines* π, *with* $0 < \pi \leq \pi_1$.

Similar orderings have been defined in Section 6 of Foster and Shorrocks (1988). They involve a singleton $\mathcal{P}_0$ and an interval of poverty lines, and assume $g(\pi) = \pi$. Foster and Shorrocks demonstrate the equivalence of these orderings with welfare orderings for the corresponding income distributions, censored by the poverty line.

Theorem 2 immediately leads to

Corollary 1 (Reduction uniform in index). *Let g be a strictly increasing function and $\emptyset \neq \mathcal{P}_0 \subseteq \mathcal{P}$. Then*

$$(g(x_1), \ldots, g(x_N)) \preceq_{\mathcal{P}_0,\pi} (x_1, \ldots, x_N) \quad \textit{for all } (x_1, \ldots, x_N) \in \mathcal{D}$$

if and only if (3) holds.

Similarly, from Theorem 2 we obtain

Corollary 2 (Reduction uniform in index and line). *Let g be a strictly increasing function and $\emptyset \neq \mathcal{P}_0 \subseteq \mathcal{P}$. Then*

$$(g(x_1), \ldots, g(x_N)) \preceq_{\mathcal{P}_0,\leq\pi_1} (x_1, \ldots, x_N) \quad \textit{for all } (x_1, \ldots, x_N) \in \mathcal{D} \tag{7}$$

if and only if $-g(x)$ is star-shaped with respect to $(0,0)$ for $x \in]0, \pi_1]$, i.e. $\frac{g(x)}{x}$ is decreasing on $]0, \pi_1]$.

Proof: By Theorem 2, restriction (7) holds if and only if for every $\pi \in]0, \pi_1]$

$$\frac{g(x)}{g(\pi)} \geq \frac{x}{\pi} \quad \text{for all } x < \pi\,.$$

Equivalently, $\frac{g(x)}{x} \geq \frac{g(\pi)}{\pi}$ for all $x < \pi \leq \pi_1$. That is, $\frac{g(x)}{x}$ decreases on the interval $]0, \pi_1]$. □

Figure 2 shows two examples for g, one being consistent with (7), the other not.

The condition that $\frac{g(x)}{x}$ *is decreasing on* $]0, \pi_1]$ may be interpreted as follows: For a poor person, the ratio of post-redistribution income over pre-redistribution income should be greater than the same ratio of a relatively richer person. If only taxes are feasible, the condition means that

$$1 - \frac{g(x)}{x} = \frac{x - g(x)}{x} \quad \text{increases with } x\,,$$

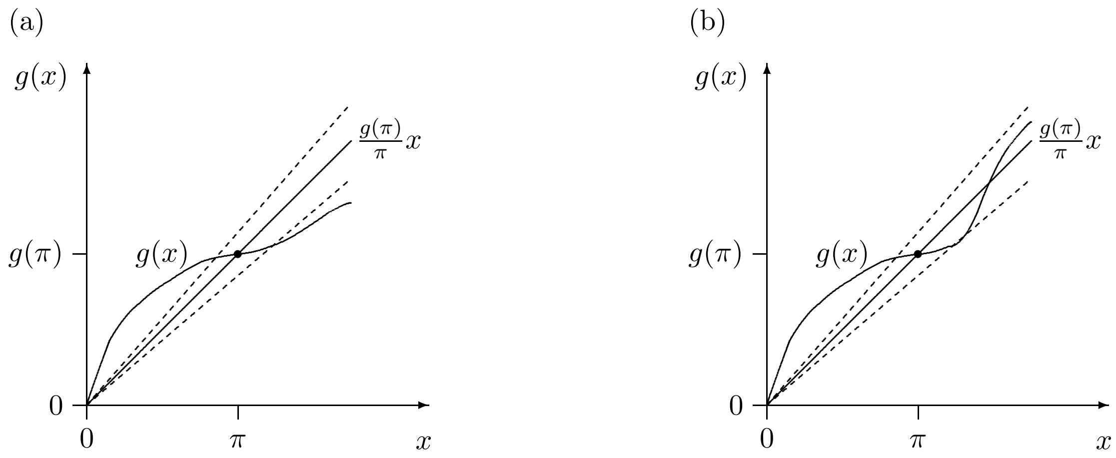

Figure 2: Two examples of g; (a) is consistent with (7), (b) is not consistent.

that is, the average tax rate increases. Note that this is one of the restrictions that characterize the reduction of the Lorenz order; see Fellman (1976), Jakobsson (1976), and many subsequent authors.

5 Index reduction when the poverty line is a quantile percentage

Up to now, reduction of poverty has been investigated when the number of poor people is fixed. In this Section, we assume that the poverty line is a given percentage α of some quantile x^* of the income distribution, $\pi = \alpha x^*$, $0 < \alpha \leq 1$. E.g., a widely used poverty line amounts to 60 (or 50) per cent of the median.

Consequently, if $\alpha < 1$, the number p of poor persons before and after redistribution may differ. Here, we restrict ourselves to the subset $\mathcal{P}' \subset \mathcal{P}$ of indices (1) that have weights $w'_{p,N}$ not depending on p. Note that,e.g., the indices of Thon, Chakravarty, FGT and Watts are still included in $\mathcal{P}'$.

As all indices (1) are scale invariant, instead of $(x_1, \ldots, x_N)$ with poverty line $\pi = \alpha x^*$ we may consider transformed incomes $\left(\frac{g(x^*)}{x^*}x_1, \ldots, \frac{g(x^*)}{x^*}x_N\right)$. The transformed incomes have poverty line $\frac{g(x^*)}{x^*}\pi = \frac{g(x^*)}{x^*}\alpha x^* = \alpha g(x^*)$, which is the poverty line of $(g(x_1), \ldots, g(x_N))$.

The number of poor people may increase or decrease, depending on g. Sufficient for

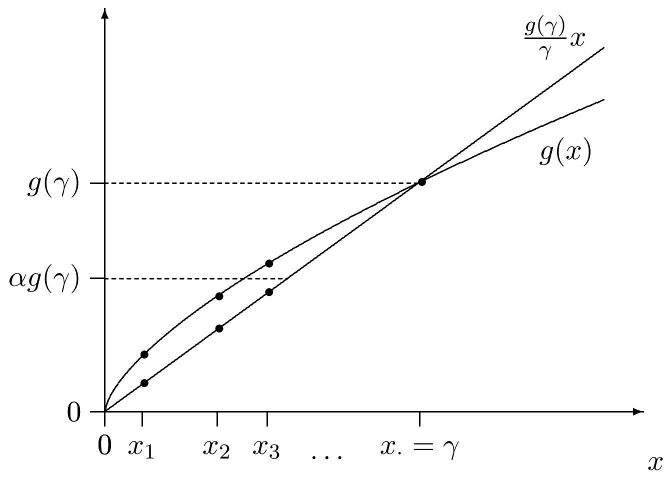

Figure 3: Sufficient condition for a decreasing the number of poor persons.

decreasing is the restriction

$$g(x) \geq \frac{g(x^*)}{x^*}x \quad \text{for all} \quad x \in]0, \alpha x^*[\,, \tag{8}$$

as it is illustrated in Figure 3.

The inequality (8) resembles condition (3), only the set of values on which the inequality holds differs slightly. By this, we are able to generalize Theorem 2, while restricting the indices to $\mathcal{P}'$:

Theorem 3 (Reduction with changing number of poor). (i) *Taxes and subsidies: Let x^* be a quantile and $0 < \alpha \leq 1$, and let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing redistribution function. Then*

$$\varphi(g(x_1), \ldots, g(x_N)) \leq \varphi(x_1, \ldots, x_N)$$

for all $(x_1, \ldots, x_N) \in \mathcal{D}$ and all $\varphi \in \mathcal{P}'$ if and only if

$$\frac{g(x)}{g(x^*)} \geq \frac{x}{x^*} \quad \textit{for } 0 \leq x \leq \alpha x^*\,. \tag{9}$$

(ii) *Taxes only: The same statement holds if g satisfies the restriction*

$$g(x) \leq x\,.$$

Proof. Let p_1 and p_2 be the number of poor persons before and after taxation and subsidies respectively. Assume first that condition (9) holds: $p_1 \geq p_2$. Similar to the first part of the proof of Theorem 2 we obtain:

$$\sum_{i=1}^{p_2} w'_{p,N}(i)u\left(\frac{g(x_i)}{\alpha g(x^*)}\right) \leq \sum_{i=1}^{p_2} w'_{p,N}(i)u\left(\frac{x_i}{\alpha x^*}\right).$$

Adding $\sum_{i=p_2}^{p_1} w'_{p,N}(i)u\left(\frac{x_i}{\alpha x^*}\right) \geq 0$ to the right side does not change the inequality.

To prove the reverse, assume there is $x_0 \in (0, \alpha x^*)$ with $\frac{g(x_0)}{g(x^*)} < \frac{x_0}{x^*}$. Under this condition anyone with income x_0 is poor before taxation and subsidies, because $x_0 \in (0, \alpha x^*)$ and also poor after redistribution, since $\frac{g(x_0)}{\alpha g(x^*)} < \frac{x_0}{\alpha x^*} \leq 1$. Now choose x belonging to $\mathcal{D}$ such that $x_i = x_0$ for at least one person and otherwise all incomes are above both poverty lines. As in the proof of Theorem 2, we again receive a contradiction immediately. $\square$

6 Reduction of multivariate poverty

Poverty may be (and is often) seen as a multivariate phenomenon. People are poor not only in terms of income but also in terms of wealth, education, health and other attributes of well-being. Some of these attributes are subject to redistribution policies, particularly the various forms of wealth (monetary wealth, pension schemes, *e.g.*) by wealth taxes and social welfare legislation. In this Section we consider multivariate indices of poverty and derive an attenuation result that corresponds to Theorem 2. Each attribute j is taxed by means of a redistribution function g_j, like income by income tax, wealth by wealth tax, etc.

Consider the class $\mathcal{P}^m$ of poverty indices defined by:

$$P(X, \pi) = \frac{1}{N}\sum_{i=1}^{N} f\left(\prod_{j=1}^{m} u_j\left(\frac{x_{ij}}{\pi_j}\right)\right),$$

where $X = (x_{ij})_{N,m}$ is a matrix representing a population of size N with m attributes and $\pi = (\pi_1, \ldots, \pi_m)$ is the vector of the corresponding poverty lines, $\pi_j = \pi_j(x_{1j}, \ldots, x_{Nj})$.

Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be a strictly increasing function and $u_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, \ldots, m$, be strictly decreasing functions on $[0, 1]$, which are constant elsewhere with $u_j(1) = 1$.

There are several poverty indices included, *e.g.* an index defined by Tsui (2002),

$$P(X, \pi) = \frac{1}{N} \sum_{i=1}^{N} \left(\prod_{j=1}^{m} \left(\frac{\pi_j}{x_{ij} \wedge \pi_j} \right)^{\alpha_j} - 1 \right),$$

where $\alpha_j > 0$ and $x_{ij} \wedge \pi_j = \min\{x_{ij}, \pi_j\}$. In this case $f(y) = y - 1$ and $u_j\left(\frac{x_{ij}}{\pi_j}\right) = \left(\frac{x_{ij}}{\pi_j}\right)^{-\alpha_j}$ for $x_{ij} \leq \pi_j$.

Another example is an index introduced by Bourguignon and Chakravarty (2003):

$$P(X, \pi) = \frac{1}{N} \sum_{j=1}^{m} \sum_{i \in S_j} h_j \left(\frac{x_{ij}}{\pi_j} \right),$$

where h_j is continuous, strictly decreasing on $[0, 1]$ and convex, $h_j(t) = 0$ for all $t \geq 1$, and S_j is the set of poor people with respect to attribute j. Here we have $f(y) = \ln y$ and $u_j(\frac{x_{ij}}{\pi_j}) = \exp(h_j(\frac{x_{ij}}{\pi_j}))$.

Tsui (2002) defines a special case of this index, with $h_j(\cdot) = -\delta_j \ln(\cdot)$, $\delta > 0$, *i.e.*

$$P(X, \pi) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{m} \delta_j \ln \left(\frac{\pi_j}{x_{ij} \wedge \pi_j} \right).$$

Consider now strictly increasing functions $g_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, \ldots, m$, as the redistribution functions with respect to attributes $j = 1, \ldots, m$. Thus $g(X)$ is the matrix with components $g_j(x_{ij})$, $i = 1, \ldots, N$, $j = 1, \ldots, m$.

Not surprisingly,

Theorem 4 (Reduction of multivariate index). *Let $g_j : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing redistribution function, $j = 1, \ldots, m$. Then*

$$\varphi(g(X)) \leq \varphi(X) \quad \textit{for all } X \in \mathbb{R}_+^{N \times m} \setminus \{\mathbf{0}\} \quad \textit{and all } \varphi \in \mathcal{P}^m \tag{10}$$

if and only if for all $j = 1, \ldots, m$

$$\frac{g_j(x)}{g_j(\pi_j)} \geq \frac{x}{\pi_j} \quad \textit{for all } x \in (0, \pi_j).$$

Proof. Assume first that $\frac{g_j(x)}{g_j(\pi_j)} \geq \frac{x}{\pi_j}$ and u_j is strictly decreasing. Thus,

$$u_j\left(\frac{g_j(x)}{g_j(\pi_j)}\right) \leq u_j\left(\frac{x}{\pi_j}\right) \qquad \text{for all } x \in (0, \pi_j),\ j = 1, \ldots, m\,.$$

After multiplying, transforming and summing as mentioned in the proof of Theorem 2, inequality (10) is shown.

To prove the reverse, assume that there is j_0, w.l.o.g. $j_0 = 1$, with $\frac{g_1(x_0)}{g_1(\pi_1)} < \frac{x_0}{\pi_1}$ for at least one $x_0 \in]0, \pi_j[$. By choosing *e.g.*

$$X = \begin{pmatrix} x_0 & \pi_2 & \ldots & \pi_m \\ \vdots & \vdots & \ddots & \vdots \\ x_0 & \pi_2 & \ldots & \pi_m \end{pmatrix},$$

again a contradiction is obtained. □

Remark 2. *Theorem 4 holds as well if we transform $P(X, \pi)$ by a strictly increasing function v, as done in Section 3.*

7 Final remarks

In this paper we have derived the functional form of redistribution schemes which reduce relative poverty. Simple and easily interpretable restrictions on the redistribution function have been given that are necessary and sufficient to attenuate poverty in terms of indices from a rather comprehensive class. By this class, many different views on poverty are included. Our results form a counterpart to the classical Fellman-Jakobsson Theorem on reduction of economic inequality (Fellman (1976); Jakobsson (1976)). A specific feature of our investigation is that, following common politics, we firstly have focussed on the poor part of the population and compared its situation before and after redistribution. Then we have extended the results in several respects: to poverty orderings, poverty lines which are percentages of a quantile, and to multivariate poverty measurement.

The above results are relevant for designing taxes on income and monetary wealth, for granting social security rights and for constructing public pension schemes. An interesting question is the design of a comprehensive redistribution function that affects these attributes in a joint way, reflecting the interdependencies of their use. Another important question is, whether existing systems of taxes and subsidies conform to the above restrictions. In countries like Germany income redistribution is effected by a highly complex and non-transparent system of progressive tax rates, deductions and tax exemptions, and rights for various social transfers. From microdata on pre- and post-government incomes that have been recently made available the effective redistribution function may be estimated and checked whether it really decreases poverty.

8 Appendix

In this Appendix we present a list of postulates to be imposed on a poverty index and provide the proof of Theorem 1.

Postulates (from Chakravarty and Muliere (2004)):

Postulate 1 (Focus). *The poverty index does not change if a rich person's income changes and the person remains rich.*

Postulate 2 (Weak monotonicity). *The poverty index rises if a poor person's income sinks.*

Postulate 3 (Strong monotonicity). *The poverty index sinks if a poor person's income rises.*

Postulate 4 (Minimal transfer principle). *The poverty index rises by a regressive transfer between two poor persons, provided the recipient remains poor.*

Postulate 5 (Weak transfer principle). *The poverty index rises by a regressive transfer form a poor person to another (not necessarily poor) person, provided the recipient does not change his or her status.*

Postulate 6 (Strong transfer principle). *The poverty index rises by a regressive transfer from a poor person to another (not necessarily poor) person.*

Postulate 7 (Symmetry). *The poverty index remains unchanged under a permutation of incomes.*

Postulate 8 (Increasing poverty line). *The poverty index is an increasing function of the poverty line.*

Postulate 9 (Population principle). *The poverty index remains unchanged if the population is n times replicated, $n \in \mathbb{N}$.*

Postulate 10 (Continuity). *The poverty index is continuous in the income vector.*

Postulate 11 (Subgroup consistency). *Consider two populations of size N where poverty is larger in the first, consider two other populations of size M that are equal in poverty, and merge them to two populations of size $N+M$. Then poverty is still larger in the first merged population than in the second one.*

Postulate 12 (Subgroup decomposability). *The poverty index is a weighted sum of subgroup poverty indices. The weights are the subgroup's proportion of population.*

Postulate 13 (Poverty growth). *The poverty index rises if a poor person enters the population.*

Postulate 14 (Non-poverty growth). *The poverty index declines if a non-poor person enters the population.*

Proof.(of Theorem 1)

1. As $u(x_i/\pi)$ vanishes if $x_i \geq \pi$, the *focus* postulate is satisfied. – *Symmetry* is obvious, since renaming the x_i does not change the index. – *Weak monotonicity* holds too: u is strictly decreasing and therefore the decrease of a poor person's income x_i will increase $u(\frac{x_i}{\pi})$. If the income decrease preserves ranks, φ will clearly increase. If it does not, we invoke the symmetry postulate and consider more than one decreasing income under rank preservation; again, φ increases.

2. Now assume that $w_{p,N}$ is increasing in the number p of poor. If a poor person's income x_i grows, then p and hence $w_{p,N}$ may decline. The *strong monotonicity* postulate is fulfilled by the same arguments as the weak monotonicity postulate. – If the poverty line π moves upwards, at least all poor persons stay poor and both $w_{p,N}$ and $u(\frac{x_i}{\pi})$ increase, hence the index. That is, the the *increasing poverty line* postulate holds.

3. Clearly, if u and v are continuous, the index depends continuously on the income vector.

4. Assume that $w_{p,N}$ is constant and u is strictly convex. After a regressive transfer among the poor, the illfare increase of the poorer person is larger than the illfare decrease of the less poor person, since u is strictly convex. Hence the index rises with such a transfer, that is, the *minimal transfer* postulate is satisfied. – Under the same assumptions, the *weak transfer* postulate holds, since either we have a minimal transfer, or the money is given to a rich person, and then we can use the weak monotonicity and focus postulates.

   The *strong transfer* postulate holds as well. A strong transfer is either a weak transfer or someone poor becomes rich. Strict convexity implies that the decrease in the poorer person's illfare is larger than that in the richer person's illfare and $w_{p,N}$ decreases in p.

5. The *population principle* is not generally satisfied. But, if $w_{p,N}$ is proportional $\frac{1}{N}$ or $\frac{1}{p}$, it is fulfilled, because the n-fold replication of the population is just compensated by the n-times smaller weights.

6. Now let $w_{p,N}(\cdot)$ only depend on N and p. Then the summands of the index are only affected by changes in N and/or p. Consequently, the *subgroup consistency* postulate holds.

7. If $w_{p,N}$ is proportional to $\frac{1}{N}$, the *subgroup decomposability* postulate is obvious.

8. If $w_{p,N}$ is decreasing in the size of the population N, adding a rich person does not change the value of the illfare function. Hence the *non-poverty growth* postulate is satisfied. $\square$

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