The declining middle class in one and many attributes*

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Version August 14, 2007

Abstract

A new multivariate approach is introduced to identify the middle class of a society and to measure its possible decline. The middle class is a properly defined central region in attribute space that contains a fixed portion of the population. The decline of the middle class is measured by comparing volumes of the middle class. The use of different notions of central regions is discussed in one and several dimensions. An empirical application is given to German data on income and wealth.

Keywords: Middle class, central region, zonoid region, halfspace region, German Sample Survey of Income and Expenditure, EVS.

JEL: D63, D31, I30.

1 Introduction

The stabilizing role of the "middle class" is a commonly accepted principle in social science. By this class a group of people is meant who are close enough in their socio-economic status to be able to cooperate and form a common

*We are indebted to Christoph Scheicher for many useful remarks and to Rainer Dyckerhoff for his support in computing zonoid central regions.

political will. A large middle class has a beneficial influence on the society, as it provides a buffer between the extreme tendencies of the lower and upper social classes.

In this context, the decline of the middle class in a developed country signifies a threat for economic growth and socio-political stability. Several political scientists and economists have addressed such issue; in particular, Esteban and Ray (1999) develop a behavioral model that shows higher societal conflict in presence of a weaker middle class.

The problem of the weakening middle class has been studied already by several authors in the 1980s, mainly concerning the distribution of income and labor earning in the United States; see, for example, Blackburn and Bloom (1985), Horrigan and Haugen (1988), Rosenthal (1985), Beach (1989), Beach et al. (1997), Ilg and Haugen (2000). These authors measure changes in the middle class in basically two ways:

One approach, followed, for example, by Blackburn and Bloom (1985), Horrigan and Haugen (1988) and Beach et al. (1997), is to define the middle class as an interval of income values, with lower and upper bounds given by some fractions of the median, and measure the number of people inside such central group. The decline of the middle class in this sense means that less and less people have income endowments close enough to each other that they are able and willing to politically cooperate and possibly majorize the remaining population.

A second approach is proposed, in particular, by Beach (1989) and Beach et al. (1997). The middle class is defined as a central population group, i.e. the central 20%, 30% or 60% of the population, and the decline is measured in terms of income share or income range possessed by such population class. It means that, chosen the proportion of individuals, the middle class disappears, if either its economic power or its internal cohesion weaken. Building on but different from this approach, Wolfson (1997) defines the middle class as those individuals having median income and then monitors the weakening of the class in terms of how far is the rest of the population from median income.

The problem of the declining middle class is also related to the measurement of societal polarization; see, in particular, Esteban and Ray (1994), who provide a class of indices through an axiomatic treatment of splitting the society into possibly conflicting groups, and Wolfson (1994), who constructs an index that captures the hollowing of the middle class.

Whether an individual belongs to the middle class or not, is determined by his or her socio-economic attributes. Income is an important attribute but, as we shall argue in the sequel, other attributes like wealth and education appear to be class building as well. In this paper we shall discuss and investigate two problems: (1) How should the middle class and other central parts of the society be defined in terms of socio-economic status, particularly when more than one attributes are relevant? (2) How should the decline of the middle class be measured?

We follow the authors of the 1980s, in extending their second approach: The middle class is defined as a properly fixed proportion of individuals whose endowments are contained in a most central part of the distribution.

Our multivariate analysis of the disappearing middle class considers, therefore, the "middle h per cent" of the population, say $h = 50\%$, and compares different distributions with respect to the dispersion of the corresponding middle class. Decline in this sense implies that the central majority of the population differs more and more in status so that, in a system of proportional representation, people become less and less able to form a political coalition against the rest.

Different from inequality measurement, we do not focus on interpersonal differences over the entire population, but rather we monitor the dispersion of a well-defined subgroup, the middle class.

Key concepts of our analysis will be two particular notions of geometric bodies, central regions and minimum volume ellipsoids. The analysis will be, therefore, divided into two approaches, according to the geometric bodies used.

Concerning the central regions, for a given distribution in d-space ($d \geq 1$), regions of different centrality α are defined. The regions are nested and convex, and the most central region – often a singleton – is regarded as the center of the distribution. As an example, consider balls with varying diameter around the mean of the distribution. In order to measure the decline of the middle class, we determine, for each distribution, that central region which contains h (e.g. fifty) percent of the population and we measure its dispersion.

With the minimum volume ellipsoid, the middle class is defined as the ellipsoid that covers at least $h\%$ of population and has minimum volume among all such ellipsoids. Two distributions are compared by evaluating the volumes (or other dispersion measures) of their minimum volume ellipsoids.

In both the cases, the choice of the middle class is more or less arbitrary, as remarked also in Beach (1989), since it depends on the choice of the parameter h, that divides the middle class from the rest of the population. In order to reduce such arbitrariness, we consider an interval of values for h, as it is done in the theory of poverty measurement; see, e.g., Zheng (2000). Therefore, comparisons of distributions may be done that are uniform in a range of values of the parameter.

Finally, we apply the two multivariate approaches to the analysis of the decline of the middle class in Germany, during the years 1998 and 2003, using data from the German Sample Survey of Income and Expenditure[1].

The paper is organized as follows: Section 2 discusses the idea of defining the middle class as a central region, introducing several special notions of central regions and demonstrating their suitability in identifying a proper middle class and in measuring polarization. Section 3 presents the middle class in terms of minimum volume ellipsoid and proposes a measure for its decline. Section 4 discusses the main relations between the two approaches and the properties of the proposed indices. Section 5 presents the results of the empirical analysis and Section 6 concludes.

Some notation: We consider a population of N individuals having d attributes. A *distribution*[2] F is an $N \times d$ matrix $[x_{ij}]$, where each row $\mathbf{x}_i = [x_{i1}, \ldots, x_{id}]$ corresponds to an *individual,* and each column to an *attribute.* The set of all $N \times d$ matrices is denoted by $\mathcal{P}$. The mean vector and the covariance matrix of F are, respectively, μ and Σ. $\mathbb{R}^d$ is the d-dimensional Euclidean space, the transpose of $\mathbf{x}$ is indicated with $\mathbf{x}^T$, while $||\cdot||$ and $\langle \cdot, \cdot \rangle$ are, respectively, the Euclidean norm and the inner product in $\mathbb{R}^d$. $|S|$ signifies the number of elements of a finite set S.

[1] German title: *Einkommens- und Verbrauchsstichprobe (EVS)*

[2] F may be regarded as an empirical distribution in $\mathbb{R}^d$, that is, a probability distribution giving mass $\frac{1}{N}$ to each row of F. Note that the analysis of this paper can be straightforwardly extended to infinite populations and general probability distributions.

2 Middle class as a central region

In the univariate analysis of the income distribution, a middle class may be defined as an interval that contains a fixed h percent of the distribution. The interval's boundaries are either given as functions of quantiles or defined by the restriction that the interval should have minimum length; see, e.g., Beach (1989) and Beach et al. (1997). Our first multivariate counterpart of such intervals is based on so called the central regions, which are particular sets able to isolate the most central points from the rest of the data cloud and whose contours are based on sort of quantiles, like the boundaries of the univariate middle class.

We firstly give a general definition of central regions, following Mosler (2002), and then consider special notions of them.

Definition 2.1 (Family of central regions, median)**.** *Let $\mathcal{P}$ be the set of all $N \times d$ matrices.*

- *A* family of central regions $\mathcal{D} = \{D_\alpha\}_{\alpha \in]\alpha_{min}, \alpha_{max}]}$ *consists of functions D_α that map $F \in \mathcal{P}$ to a convex compact set $D_\alpha(F) \subset \mathbb{R}^d$ such that*

$$D_\alpha(F) \subset D_{\alpha'}(F) \quad \text{if } \alpha_{min} < \alpha' \leq \alpha \leq \alpha_{max} \,.$$

- *The* $\mathcal{D}$-median set *of F is given by $median_{\mathcal{D}}(F) = D_{\alpha_{max}}(F)$, and the* $\mathcal{D}$-median *is its gravity center.*

- *$D_{\alpha_{min}}$ is defined as the convex hull of the support of F.*

Given a d-variate distribution F, its central regions, therefore, form a family of nested, convex, compact sets $\{D_\alpha(F)\}_\alpha$ in $\mathbb{R}^d$. The smallest and most central region is given by the median set, and the parameter α expresses the degree of centrality of a region, maximum for $\alpha = \alpha_{max}$. Thus, the family of central regions describes the distribution F, summarizing information on its location, spread, and shape. For a given univariate distribution F, the *interquantile intervals* $[Q(\alpha), Q(1-\alpha)], \alpha \in]0, 1/2]$, with $Q(t) = \inf\{x : F(x) \geq t\}, t \in [0, 1]$, form a family of central regions.

Important properties require central regions to be equivariant to certain transformations of d-space, e.g., affine transformations[3]. Instead of convexity,

[3] D_α is affine equivariant if $D_\alpha(F_{\mathbf{XA}+\mathbf{b}}) = D_\alpha(F_{\mathbf{X}})\mathbf{A}+\mathbf{b}$, where $\mathbf{A}$ is a $d \times d$ non-singular matrix and $\mathbf{b} \in \mathbb{R}^d$.

a weaker property can be considered, according to which D_α is starshaped[4] about every point $\mathbf{x} \in D_{\alpha_{max}}$.

In correspondence to each level α of centrality, a central region divides the distribution's support into those points in $\mathbb{R}^d$ that are closer to the median set and those points that are distant from the center and from each other. For an appropriate value of the parameter α, it seems quite natural to think of the individuals in region D_α as the middle class, i.e. as the set of individuals having similar endowments, and being isolated from the remaining population.

To identify the middle class of a distribution F, we consider the smallest central region $D_{\alpha(h)}(F)$ that contains $h\%$ of the individuals. The set of these individuals forms the middle class, which is denoted by $M_h(F)$. The weakening of the middle class is then measured in terms of the volume of the central region $D_{\alpha(h)}(F)$. The smaller the region is, that is, the closer to the median set the middle class is situated, the more homogeneous and the more able to cooperate are the middle class individuals. The formal definitions are:

Definition 2.2 (Central regions middle class)**.** *Let $\mathcal{D} = \{D_\alpha\}$ be a family of central regions. For $F \in \mathcal{P}$ and $0 < h < 100$, define*

$$\alpha(h) = \inf\left\{\alpha : \frac{1}{N}\left|\{i : x_i \in D_\alpha(F)\}\right| \geq \frac{h}{100}\right\}, \quad (1)$$

$$M_h^{\mathcal{D}}(F) = \left\{i : x_i \in D_{\alpha(h)}(F)\right\}. \quad (2)$$

$M_h^{\mathcal{D}}$ is named the $\mathcal{D}$-middle class of level *h.*

Note that, in identifying the middle class, the parameter to be exogenously chosen is not the centrality parameter α, but rather the percentage h of the population to be included in the middle class, on which the value of α depends. With Definition 2.2 we determine the middle class by a specific central region and measure the middle class decline by the volume of this central region in d-space. The volume

$$S_h^{\mathcal{D}}(F) = \mathrm{vol}_d(D_{\alpha(h)}(F)\,, \quad (3)$$

[4]A set D is starshaped about $\mathbf{x} \in D$ if, for every $\mathbf{y} \in D$, the straight line connecting $\mathbf{x}$ and $\mathbf{y}$ lies in D.

is a *measure of absolute weakening* and not invariant to changes of scale in the attributes. To obtain a relative measure we relate this volume to the volume of either the whole population or a slightly trimmed version of it. That is, we divide the volume of the middle class by the volume $\mathrm{vol}_d(D_{\alpha(h^*)}(F))$, where h^* is either equal to 1 or close to 1. A *measure of relative weakening* of the h-middle class is then defined as

$$S_h^{\mathcal{D}*}(F) = \frac{\mathrm{vol}_d(D_{\alpha(h)}(F))}{\mathrm{vol}_d(D_{\alpha(h^*)}(F))}\,. \tag{4}$$

Its values are bounded between 0 and 1. Comparing the dispersion of the middle class with that of the entire population, such normalized index monitors the relative decline of the middle class, while measure $S_h^{\mathcal{D}}$ focusses on the absolute one.

Our approach can be seen as a direct multivariate extension of an index considered by Wolfson (1997), who measures the declining middle class in terms of income range of some central population group, normalized by the median[5].

In the presence of potential outliers, instead of considering the whole population in the denominator in (4), it is more appropriate to use a trimmed population, such as the central region which covers the 95 percent of the most central individuals. This implies, of course, that an additional parameter has to be chosen, *viz.* the portion of outlying observations to be excluded.

The particular value of h and, consequently, of $\alpha(h)$ has to be chosen exogenously and different values can order distributions, in terms of middle class decline, in different ways. For that reason, in their univariate study of the disappearing of the middle class, Beach et al. (1997) define more than one middle class, inducing a sensitivity analysis, and propose a semi-ordering of distributions that unanimously respects the various definitions of middle class decline.

Here, we follow the same idea, proposing at first, in Definition 2.2, a measure of weakening of the middle class, which depends on the particular value of h. We then consider not only a single middle class, but rather a set of reasonable middle classes, in order to reduce the subjectivity due to the choice of h. We can, therefore, order different distributions, according to the

[5]Observe that such index is not the popular measure proposed in the same paper by Wolfson (1997).

dominance criterion in Definition 2.3, which is uniform for all the values of h in a properly chosen interval[6] $[h_1, h_2]$.

Definition 2.3 (Uniformly weaker)**.** *Let* $0 < h_1 \leq h_2 < 100$*, and* F *and* G *be two distributions in* $\mathcal{P}$*;* F *is uniformly weaker in middle class than* G*,* $F \succeq_v G$*, if*

$$S_h^{\mathcal{D}*}(F) \geq S_h^{\mathcal{D}*}(G) \quad \textit{for all } h \in [h_1, h_2] .$$

The order in Definition 2.3 is scale invariant, reflexive and transitive. It is partial unless $h_1 = h_2$.

2.1 Special notions of central regions

Several proposals of central regions are present in the literature; see, e.g., Mosler (2002). In the following, we review four families of central regions and explore their potential to provide a meaningful and operationable notion of the middle class.

1. **Balls around the spatial median**

   The simplest case is to consider the *family of balls* around a proper central point,

   $$D_\alpha^B(F) = \left\{ \mathbf{y} : (||\mathbf{y} - \mathbf{c}|| + 1)^{-1} \geq \alpha \right\} .$$

   For $\mathbf{c}$ we insert the spatial median med_F of $F = [x_{ij}]$, which is defined as the point $\mathbf{c}$ that minimizes $\frac{1}{N} \sum_{i=1}^{N} ||\mathbf{x}_i - \mathbf{c}||$. Such central regions are bounded, closed and convex, but not affine equivariant.

   If based on balls around the spatial median, the middle class consists of those h% individuals whose Euclidean distance from the spatial median is less than or equal to $\frac{1-\alpha(h)}{\alpha(h)}$. Of course, this notion is invariant to constant shifts of the distribution, but sensitive to changes in the scales of the attributes. The next notion of middle class will be invariant to shift and scale and, moreover, to affine transformations of the data.

[6] In such a way, the arbitrariness is shifted from the choice of the specific h to the choice of the interval's boundaries h_1 and h_2.

2. **Mahalanobis central regions**

   The *Mahalanobis family* of central regions is given by

$$D_\alpha^M(F) = \left\{ \mathbf{y} : \left((\mathbf{y}-\mu)\Sigma^{-1}(\mathbf{y}-\mu)^T + 1 \right)^{-1} \geq \alpha \right\}, \ 0 < \alpha \leq 1.$$

   They are ellipsoids centered at the mean vector μ. Therefore, the Mahalanobis median set is a singleton, $\{\mu\}$, obtained for $\alpha = 1$. The Mahalanobis central regions are affine equivariant, bounded, closed and convex.

   Here, the middle class is built of individuals in a central ellipsoid, *viz.* those $h\%$ individuals whose Mahalanobis distance from the mean μ is less or equal $\frac{1-\alpha(h)}{\alpha(h)}$. It centers at the mean μ. One of their main advantages of this notion is the ease of computation. The main drawbacks of a Mahalanobis based middle class are its symmetry, whether the distribution is symmetric or not, and its lack of robustness in the presence of outliers. A more robust notion of middle class, which is also based on an ellipsoid, will be given below in Section 3.

3. **Halfspace central regions**

   The halfspace central regions are given as the upper level sets of the following function

$$f(\mathbf{y}|F) = \min \left\{ \beta : \beta = \frac{1}{N} |\{i : \mathbf{x}_i \in H\}|, H \text{ closed halfspace}, \mathbf{y} \in H \right\}.$$

   That is, the *halfspace central regions* are given by

$$D_\alpha^H(F) = \{\mathbf{y} : f(\mathbf{y}|F) \geq \alpha\}, \ F \in \mathcal{P}, \ \mathbf{y} \in \mathbb{R}^d.$$

   A *halfspace* is any of the two parts into which a hyperplane divides the space $\mathbb{R}^d$. A closed halfspace has form $H = \left\{\mathbf{x} \in \mathbb{R}^d : p_0 + \langle \mathbf{p}, \mathbf{x} \rangle \geq 0 \right\}$, with $p_0 \in \mathbb{R}, \mathbf{p} \in \mathbb{R}^d$. The function $f(\mathbf{y}|F)$ gives the smallest portion of data points contained in a closed halfspace whose boundary passes through $\mathbf{y} \in \mathbb{R}^d$. The gravity center of its median set is called the *Tukey median*. Each halfspace central region is affine equivariant, bounded, closed and convex. Its shape follows the shape of the distribution; in general, it shows no symmetry.

   The middle class based on halfspace central regions consists of those $h\%$ individuals for whom $f(\mathbf{y}|F) \geq \alpha(h)$ holds.

An important advantage of halfspace central regions is their robustness against outliers, while the drawback is their difficulty in computation; see Rousseeuw and Ruts (1996) and Rousseeuw and Struyf (1998).

4. **Zonoid central regions**

   The *family of zonoid central regions* of a distribution F is defined as

$$D_\alpha^Z(F) \quad = \quad \left\{ \sum_{i=1}^{N} \mathbf{x}_i \lambda_i : 0 \leq \alpha\lambda_i \leq \frac{1}{N}, \sum_{i=1}^{N} \lambda_i = 1 \right\} ,$$

   with $0 \leq \alpha \leq 1$. The zonoid central regions are affine equivariant, bounded, closed and convex. Like with the Mahalanobis regions, the median set of the zonoid regions is the singleton $\{\mu\}$. For that reason, such regions are not robust against outliers.

   The middle class based on zonoid central regions collects those $h\%$ individuals who are in $D_\alpha^Z(F)$. Such a middle class can be calculated in acceptable time if the dimension is low and the data set not excessively large; see Dyckerhoff (2000).

3 Middle class as a minimum volume ellipsoid

A second approach to identify the middle class and measure its decline employs another kind of geometric bodies: the minimum volume ellipsoid (MVE), i.e. the smallest regular ellipsoid covering at least the percentage h of the elements of the data cloud. It is a multivariate extension of the intervals with minimal length, given the percentage of population, that have been introduced by Beach (1989) and Beach et al. (1997) to define the univariate middle class. This approach does not constitute a family of nested regions; it rather selects the population, eliminating the furthest and "least typical" individuals and keeping the h percent closest to the center.

Let $\mathcal{S}$ denote the set of symmetric positive definite $d \times d$ matrices. Given some $\Sigma \in \mathcal{S}$ and $\mu \in \mathbb{R}^d$, consider the (μ, Σ)-distances of the data points from μ,

$$d_{\mu,\Sigma}^2(\mathbf{x}_i) = (\mathbf{x}_i - \mu)\Sigma^{-1}(\mathbf{x}_i - \mu)^T , \quad i = 1, \ldots, N ,$$

and let $\mathbf{x}_{(1)}, \ldots, \mathbf{x}_{(N)}$ denote the ordered data points in this distance, $\mathbf{x}_{(1)}$ being the point closest to μ. Now, for a given percentage h, let $\ell = \lceil \frac{h}{100} N \rceil$, which is the smallest integer greater or equal to $\frac{h}{100} N$, and solve the following minimization problem,

$$(\hat{\mu}, \hat{\Sigma}) := \operatorname{argmin}_{\mu \in \mathbb{R}^d, \Sigma \in \mathcal{S}} (\mathbf{x}_{(\ell)} - \mu) \Sigma^{-1} (\mathbf{x}_{(\ell)} - \mu)^T .$$

Then, following Croux et al. (2002), the h percent minimum volume ellipsoid E_h is defined[7] as the ellipsoid with center $\hat{\mu}$, shape matrix $\hat{\Sigma}$ and $\mathbf{x}_{(\ell)}$ located on its boarder,

$$E_h(F) = \left\{ \mathbf{x} \in \mathbb{R}^d : (\mathbf{x} - \hat{\mu}) \hat{\Sigma}^{-1} (\mathbf{x} - \hat{\mu})^T \leq (\mathbf{x}_{(\ell)} - \hat{\mu}) \hat{\Sigma}^{-1} (\mathbf{x}_{(\ell)} - \hat{\mu})^T \right\} . \quad (5)$$

Thus the $h\%$-MVE is the smallest $(\hat{\mu}, \hat{\Sigma})$-ellipsoid that isolates at least $h\%$ (that is, the first ℓ) individuals whose cumulative $(\hat{\mu}, \hat{\Sigma})$-distance from $\hat{\mu}$ is minimum.

As in our first approach, we think of the middle class as the set of $h\%$ individuals more similar to each other and closer to the center:

Definition 3.1 (MVE middle class)**.** *For $F \in \mathcal{P}$ and $0 < h < 100$, the set*

$$M_h^{\mathcal{E}} = \{ i : \mathbf{x}_i \in E_h(F) \} = \{ i : \mathbf{x}_i = \mathbf{x}_{(i)}, i = 1, \ldots, \ell \}$$

is defined as the MVE middle class of level *h.*

In order to measure the decline of the middle class, in terms of dispersion of the attributes inside such group, we use the volume of the MVE,

$$S_h^{\mathcal{E}}(F) = \operatorname{vol}_d(E_h(F)) , \quad (6)$$

which is proportional to

$$\left(d^2_{\hat{\mu}, \hat{\Sigma}}(\mathbf{x}_{(\ell)}) \right)^{d/2} \cdot \left(det(\hat{\Sigma}) \right)^{1/2} .$$

The more internally homogeneous is the central $h\%$ of the population, i.e. the stronger and powerful is the middle class, the smaller is the volume of the ellipsoid.

[7] A necessary and sufficient condition for the existence of the MVE with positive volume requires that the points $\mathbf{x}_i \in \mathbb{R}^d$ are in general position, i.e. that no d elements of the population are lying on a common $(d-1)$-dimensional hyperplane.

A normalized index of middle class weakening is, analogously to Section 2, defined as

$$S_h^{\mathcal{E}*}(F) = \frac{\mathrm{vol}_d(E_h(F))}{\mathrm{vol}_d(E_{h^*}(F))}\,, \tag{7}$$

Given F and $G \in \mathcal{P}$, F has a *weaker middle class* than G if

$$S_h^{\mathcal{E}*}(F) \geq S_h^{\mathcal{E}*}(G) \tag{8}$$

Here, the volume of the MVE middle class is related to the volume of the society, that is, the volume of the MVE that covers either the total data cloud ($h^* = 100\%$) or a trimmed version of it ($h^* < 100\%$).

Moreover, as already discussed in the previous sections, instead of a single value of h, a reasonable range of values $[h_1, h_2]$ may be chosen. The partial order of distributions, in terms of relative volume of middle class, parallels that in Definition 2.3,

$$S_h^{\mathcal{E}*}(F) \geq S_h^{\mathcal{E}*}(G) \quad \text{for all } \ h \in [h_1, h_2]. \tag{9}$$

4 Properties

We now turn to investigate the properties satisfied by the notions of the middle class and measures of its decline that we have introduced above in Sections 2 and 3. In particular, the advantages and disadvantages of the alternative notions shall be discussed.

We establish a set of desiderata for a general notion of the middle class and for a general index of middle class decline, and we check whether the proposed notions satisfy them. Such postulates are expressed in terms of invariance or equivariance regarding transformations of the distribution, of continuity, of monotonicity and of minima and maxima attained.

The first group of postulates (Properties 1 to 4) refers to the notion of the middle class M as well as to that of the index S of decline. The second group (Properties 5 to 10) concerns the index only.

1. **Anonymity:** The middle class and the index are invariant to the individual identities: $M(\Pi F) = M(F)$ and $S(\Pi F) = S(F)$ for any $N \times N$ permutation matrix Π.

2. **Translation Invariance:** The middle class and the index are invariant to shifts in the attributes: $M(G) = M(F)$ and $S(G) = S(F)$ if $G = F + [\mathbf{c}, \ldots, \mathbf{c}]^T$, for any $\mathbf{c} \in \mathbb{R}^d$.

3. **Vector Scale Invariance:** The middle class and the index are invariant to changes of scales in the attributes: $M(G) = M(F)$ and $S(G) = S(F)$ if $G = F\Lambda$, for any positive $d \times d$ diagonal matrix Λ.

4. **Nonnegative Affine Invariance:** The middle class and the index are invariant to nonnegative affine-linear transformations of the distribution: $M(G) = M(F)$ and $S(G) = S(F)$ if $G = FA + [\mathbf{c}, \ldots, \mathbf{c}]^T$, for any $\mathbf{c} \in \mathbb{R}^d$ and any nonnegative $d \times d$ matrix A that has full rank. This property includes the translation and vector scale invariances.

5. **Replication Invariance:** S depends on the frequency distribution of attributes only; replicating the population, without changing the distribution of the variables, does not influence the measure S: $S(G) = S(F)$ if G is the $mN \times d$ matrix $[FF \ldots F]^T$, $m \in \mathbb{N}$.

6. **Continuity:** S is a continuous function of F.

7. **Focus:** Given the middle class, the index S depends only on attribute vectors of members of the middle class.

8. **Maximum and minimum value:** The index S reaches its maximum value when all the individuals belonging to the middle class (and therefore the entire population) are located on the contour of the distribution's convex hull; the minimum value is attained if $h\%$ of the individuals are concentrated at the properly defined center of the distribution.

9. **Minimal Transfer Principle:** A progressive transfer between two individuals in the middle class reduces the index S: $S(G) \leq S(F)$ if G arises from F by substituting two rows, $\mathbf{x}_{i'}$ and $\mathbf{x}_{i''}$, with $\beta\mathbf{x}_{i'} + (1-\beta)\mathbf{x}_{i''}$ and $(1-\beta)\mathbf{x}_{i'} + \beta\mathbf{x}_{i''}$, respectively[8].

[8]Note that the Weak Transfer Principle and Strong Transfer Principle known from poverty measurement (e.g., Chakravarty and Muliere, 2004) are not appropriate as the complement of the middle class includes lower and upper parts of the distribution.

10. **Correlation Increasing Majorization:** A transfer that increases the correlation among attributes induces an increase[9] in S, if the attributes are considered as substitutes. The same transfer causes a decrease in S, if the attributes are complements. (For an analogous postulate regarding multivariate inequality indices, see Weymark (2006).)

The notions M of central region middle class and MVE middle class (Definitions 2.2 and 3.1) obviously satisfy Property 1. Property 4, hence Properties 2 and 3, are satisfied by the middle class notions obtained from affine equivariant central regions (Mahalanobis, zonoid, halfspace, but not the balls) and by a middle class based on MVE. For a middle class based on balls, Property 2 is fulfilled.

Equations (4) and (7) provide indices S of the relative middle class decline,

$$S_h^{\mathcal{D}*}(F) = \frac{\mathrm{vol}_d(D_{\alpha(h)}(F))}{\mathrm{vol}_d(D_{\alpha(h^*)}(F))} \quad \text{and} \quad S_h^{\mathcal{E}*}(F) = \frac{\mathrm{vol}_d(E_h(F))}{\mathrm{vol}_d(E_{h^*}(F))}\,.$$

They obviously satisfy Properties 1, 5, and also Property 8. The above four indices based on central regions and that based on MVE are continuous in the data, hence fulfill Property 6. As central regions and MVEs are convex, the principle of minimal transfers (Property 9) applies always to these indices. $S_h^{\mathcal{E}*}$ is affine invariant (Property 4); the same holds for $S_h^{\mathcal{D}*}$ if the regions are of Mahalanobis, halfspace or zonoid type. $S_h^{\mathcal{D}}$ based on Euclidean balls is only translation invariant. Concerning Property 10, a correlation increasing transfer typically reduces the volume of the central regions and MVE; therefore, the indices satisfy the second part of Property 10, since they consider the attributes only as complement, excluding the case of substitution. Property 7 is satisfied by the absolute indices, $S_h^{\mathcal{D}}$ and $S_h^{\mathcal{E}}$, but not by the relative ones.

An important property of the MVE, common also to the halfspace central regions, is their robustness against outliers, so that the presence of outliers does not substantially modify the analysis. Studying the disappearing of the middle class with the MVE approach has the important advantage, compared to the halfspace regions, to be easily computed, thanks to algorithms by Rousseeuw and van Zomeren (1990) and Rousseeuw and Leroy (1987).

[9] Throughout the paper, increase and decrease are meant in the weak sense.

Although such algorithms can not calculate the exact values of the MVE estimators, they provide good approximations.

One of the main disadvantages of the MVE is, instead, the impossibility to capture an eventual asymmetry of the data. In such a case, central regions like the halfspace and the zonoid regions are more suitable. In order to avoid multiple solutions, with MVE we have to assume $h > 0.5$; such restriction is not required in case of central regions.

Advantages and drawbacks of central regions vary according to the special notions presented in Subsection 2.1 and depend on the characteristics of the distribution F.

If F is centrally symmetric with respect to some center[10], then all central regions discussed in Subsection 2.1 reflect this symmetry.

In case of an asymmetric distributions, balls and Mahalanobis ellipsoids are not appropriate to describe the distribution, while the zonoid and the halfspace central regions are more flexible.

In order to avoid multiple solutions, with MVE we have to assume $h > 0.5$; such restriction is not required in case of central regions.

Also in the univariate case, the five approaches introduced in Sections 2.1 and 3 lead to different definitions of the middle class, that is, in general, to different income intervals. In particular, the univariate middle class for balls and Mahalanobis regions are symmetric intervals of type $I = \left[\mu - \left(\frac{1}{\alpha} - 1\right), \mu + \left(\frac{1}{\alpha} - 1\right)\right]$, with $\alpha = \alpha(h)$. The univariate middle class for MVE is, instead, the interval with minimum length, that covers $h\%$ of population. In both cases, univariate polarization is measured through the (normalized) length of the interval that defines the middle class, i.e. in terms of its relative range.

5 Application to German data

We now apply the two approaches of Sections 2 and 3, in order to analyze the possible disappearing of the middle class in Germany, from the year 1998 to the year 2003. The analysis is based on the German Sample Survey of Income and Expenditure, which is a cross-sectional survey provided by the

[10] F is centrally symmetric with center $\mathbf{c}$ if there exists a permutation π such that $\mathbf{x}_i - \mathbf{c} = \mathbf{c} - \mathbf{x}_{\pi(i)}$ for all i.

German Federal Statistical Office. It collects, every five years, information on income, wealth and consumption of the German households. In the two years of interest, the sample consists of 49,720 households in 1998, and of 42,744 households in 2003[11].

The unit of analysis is the household, defined as the income sharing unit, i.e. married couples, singles and cohabitants, with or without children. The variables of interest are income and wealth, considered as good proxies for the socio-economic conditions of families.

Income has always been regarded as a good indicator of the socio-economic status of persons; however, recently, has been underlined the necessity to introduce other attributes, monetary and non-monetary, in order to better understand the real conditions of the individuals.

Here, we consider wealth, as, following Wolff (1998), it provides an additional dimension of well-being. Wealth, in fact, constitutes, in the form of housing, a direct service to the owners and, in the form of financial assets, a liquidity in case of economic problems; moreover, wealth is a source of consumption independent from income, as the assets can be converted into cash and satisfy consumption needs. The definition of wealth considered here is the marketable annual household net wealth, constituted by the sum of gross property wealth (houses and real estates) and the gross monetary wealth (mainly, building society savings accounts, quoted shares, bonds, savings, life insurance and investment fonds), minus the liabilities, due to mortgage and consumer debts; excluded are consumer durables, cash money, jewelery and objects d'art. The income used is the quarterly household net income, given by the sum of the household income from employed and self-employed labor, from public and private transfers and from wealth, minus taxes and social contributions.

In order to take into account the differences in size and composition of the families, the household net income is transformed according to the old OECD equivalent scale[12], while wealth is made equivalent, dividing the household net wealth by the number of members in the household[13].

[11]Not covered from the survey are the homeless, the institutionalized persons and the households with very high monthly net income.

[12]Such scale gives weight equal to 1 for the head of household, equal to 0.7 for any other adult, i.e. older than 14 years old, and equal to 0.5 for each child.

[13]Same results are obtained, using other equivalent scales for wealth, such as the old

Equivalent income and wealth are, finally, deflated, in reference to the prices of the year 2000.

Selecting the appropriate variables that reflect the socio-economic conditions of the middle class constitutes a task that has to precede the statistical measurement. Other attributes of welfare, besides income and wealth, could have been chosen to define the belonging to the middle class (for instance, education, health, benefits in kind). However, we do not claim to derive, through this empirical analysis, exhaustive conclusions about the extent and potential decline of the middle class in Germany.

At first a marginal analysis is conducted, calculating measures of the middle class for the univariate distributions of income and wealth. For this we employ the measures of Beach et al. (1997) and Wolfson (1997). They amount to the range of income possessed by the central $h\%$ of the population, divided by the median. Table 1 shows that from the year 1998 to 2003, the middle class declines, in case of income distribution, but gets stronger, in terms of wealth distribution, according to all the measures considered.

Table 1: Univariate analysis.

	INCOME		WEALTH	
$h\%$	1998	2003	1998	2003
40%	0.5186	0.5365	3.1935	2.8924
45%	0.5945	0.6006	3.8457	3.2418
50%	0.6792	0.7008	4.2902	3.7677
median[(1)]	3,946	4,129	17,228	19,400

(1): amount in Euro

Now, it is interesting to study how the results change, when the joint distribution of the variables is considered. Moving to the bivariate analysis, we study the vanishing of the middle class, from the year 1998 to 2003, on a subsample of 1 000 households, randomly extracted from the entire survey and weighted with proper household weights to represent the entire population.

Both approaches require to choose a proportion of population which constitutes the middle class. According to the discussion in Section 2, we use

OECD equivalent scale and a mixed scale, which divides the property wealth by the old OECD equivalent scale and the financial assets by the number of household's components.

$h\% = 40\%, 45\%, 50\%$. Tables 2 and 3 show the values of the absolute indices $S_h^{\mathcal{D}}$ (respectively, $S_h^{\mathcal{E}}$) and the relative indices $S_h^{\mathcal{D}*}$ (respectively, $S_h^{\mathcal{E}*}$) for Mahalanobis ellipsoids, zonoid central regions and MVE.

Table 2: Central regions volume of the middle class: Absolute indices $S_h^{\mathcal{D}}$ and $S_h^{\mathcal{E}}$ (in millions of €^{2}).

	MAHALANOBIS		ZONOID		MVE	
$h\%$	1998	2003	1998	2003	1998	2003
40%	356.72	471.22	515.92	627.09	143.29	159.70
45%	390.35	567.05	612.13	762.19	181.84	207.78
50%	435.64	642.78	746.84	854.45	213.50	282.42

Table 3: Central regions volume of the middle class: Relative indices $S_h^{\mathcal{D}*}$ and $S_h^{\mathcal{E}*}$, with $h^* = 100\%$.

	MAHALANOBIS		ZONOID		MVE	
$h\%$	1998	2003	1998	2003	1998	2003
40%	0.00486	0.00307	0.03219	0.01797	0.00808	0.00122
45%	0.00532	0.00370	0.03819	0.02184	0.01025	0.00159
50%	0.00594	0.00419	0.04659	0.02448	0.01204	0.00215

A first obvious observation concerns the increase of the volume for both the measures, as the value of the parameter h becomes larger: All the three geometric bodies expand their area, as the number of points they cover increases.

Looking at Table 2, the non-normalized measure gives an unanimous result, for all the different middle classes, i.e. Mahalanobis ellipsoids, zonoid central regions and MVE: From 1998 to 2003, the dispersion of the middle class increases. Figures 2, 3 and 4 illustrate, clearly, the increase in volume of all the middle classes.

However, the normalized measures $S_h^{\mathcal{D}*}$ and $S_h^{\mathcal{E}*}$ in Table 3 shows an opposite tendency: For each definition of middle class, the volume of the middle class, as a portion of the volume of the whole society, show a sharp decrease from 1998 to 2003.

While the middle class becomes more dispersed over the years, the entire society grows apart at a much faster rate. It means that centrifugal forces, that involve the society, are much stronger outside than inside the middle class. Therefore, even if in 2003 the central group is less homogeneous than in 1998, its ability to cooperate cannot be considered damaged, if compared with the greater dispersion and consequent weakening of the rest of the population.

The volume of the whole society is determined by its extreme points. To reduce the influence of single outlying points, also trimmed versions of the relative indices, that is, with $h^* < 100\%$, have been determined. The values are reported in Tables 4 to 6. In case of zonoid regions and MVE they have been calculated as the mean values from 1000 runs (= subsamples drawn)[14].

Table 4: Central regions volume of the middle class: Relative indices $S_h^{\mathcal{D}*}$ and $S_h^{\mathcal{E}*}$, with $h^* = 98\%$.

	MAHALANOBIS		ZONOID		MVE	
$h\%$	1998	2003	1998	2003	1998	2003
40%	0.03330	0.03390	0.04786	0.03861	0.01205	0.00405
45%	0.03644	0.04080	0.05678	0.04693	0.01531	0.00541
50%	0.04067	0.04624	0.06928	0.05261	0.01843	0.00718

Table 5: Central regions volume of the middle class: Relative indices $S_h^{\mathcal{D}*}$ and $S_h^{\mathcal{E}*}$, with $h^* = 95\%$.

	MAHALANOBIS		ZONOID		MVE	
$h\%$	1998	2003	1998	2003	1998	2003
40%	0.08273	0.08316	0.09208	0.09963	0.02150	0.01893
45%	0.09053	0.10008	0.10925	0.12109	0.02733	0.02530
50%	0.10104	0.11344	0.13330	0.13575	0.03289	0.03360

We see from these tables that for each geometrical body considered and for each value of h, the measures $S_h^{\mathcal{D}*}$ and $S_h^{\mathcal{E}*}$ obviously increase, as the trimming percentage increases, i.e. as h^* reduces.

[14] We thank Karen Safarov for calculating these three tables and providing Figure 1.

Table 6: Central regions volume of the middle class: Relative indices $S_h^{\mathcal{D}*}$ and $S_h^{\mathcal{E}*}$, with $h^* = 90\%$.

	MAHALANOBIS		ZONOID		MVE	
$h\%$	1998	2003	1998	2003	1998	2003
40%	0.15898	0.17528	0.15806	0.16657	0.05336	0.04801
45%	0.17396	0.21092	0.18754	0.20246	0.06782	0.06416
50%	0.19415	0.23909	0.22881	0.22697	0.08160	0.08521

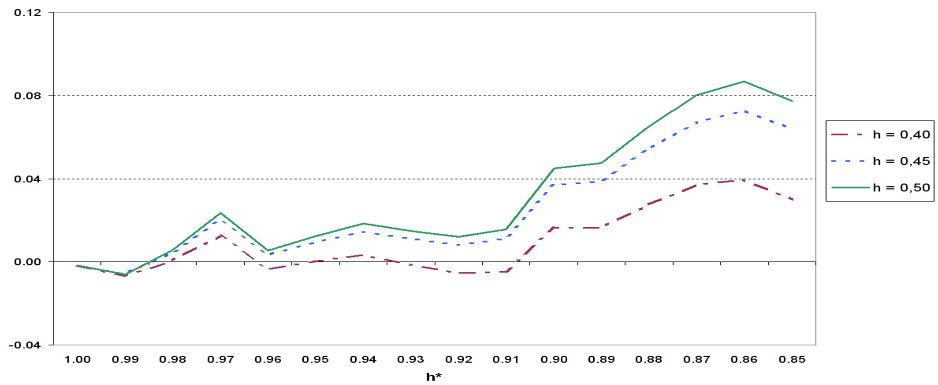
When introducing the trimming procedure, no overall unanimous results can be obtained from the different definitions of middle class. In particular, for each percentage of trimming, the relative measure $S_h^{\mathcal{D}*}$ identified by the Mahalanobis ellipsoids increases from the year 1998 to 2003. It means that once the whole population is purified from outliers, its dispersion increases at a rate that is no longer higher than the rate of the middle class. An analogous result holds for middle classes defined by zonoid central regions, when removing at least the extreme 5% observations. Differently, the middle class defined by MVE increases in volume at a lower rate than the population does; only in case of stronger trimming ($h^* = 95\%, 90\%$) and larger middle class ($h\% = 50\%$), such trend reverses.

Figure 1 exhibits the difference $(S^*_{2003} - S^*_{1998})$ depending on the trimming parameter h^* and for different values of the middle class proportion h.

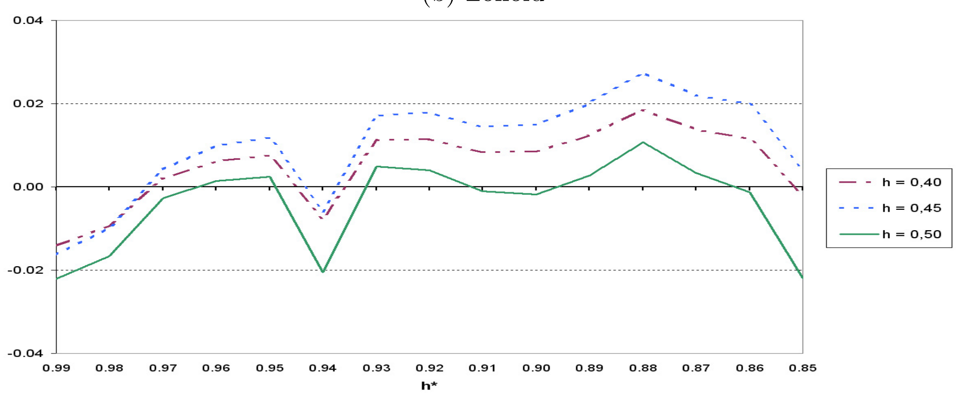
We see that such difference has an increasing trend with respect to h^*, mainly because the volumes of the two populations (in 1998 and in 2003) get closer to each other as the proportion of trimming increases. Moreover, for each kind of geometrical body, the difference $(S^*_{2003} - S^*_{1998})$ is characterized by a quite similar trend as h changes; it assumes higher values when employing Mahalanobis ellipsoids than when employing the other two geometrical bodies.

Figure 1: Difference $(S^*_{2003} - S^*_{1998})$ depending on the trimming parameter h^* and for different values of the middle class proportion h

(a) Mahalanobis



(b) Zonoid



(c) MVE

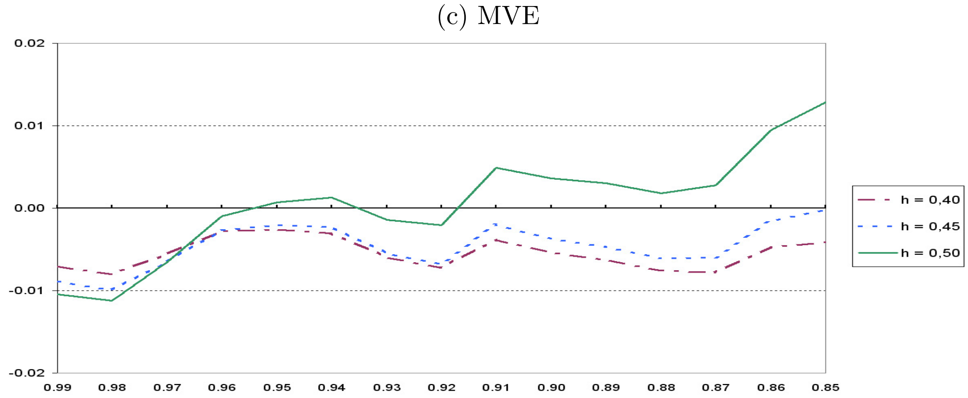


Figure 2: Mahalanobis for $h\% = 40\%, 45\%, 50\%$.

(a) Year 1998 (b) Year 2003

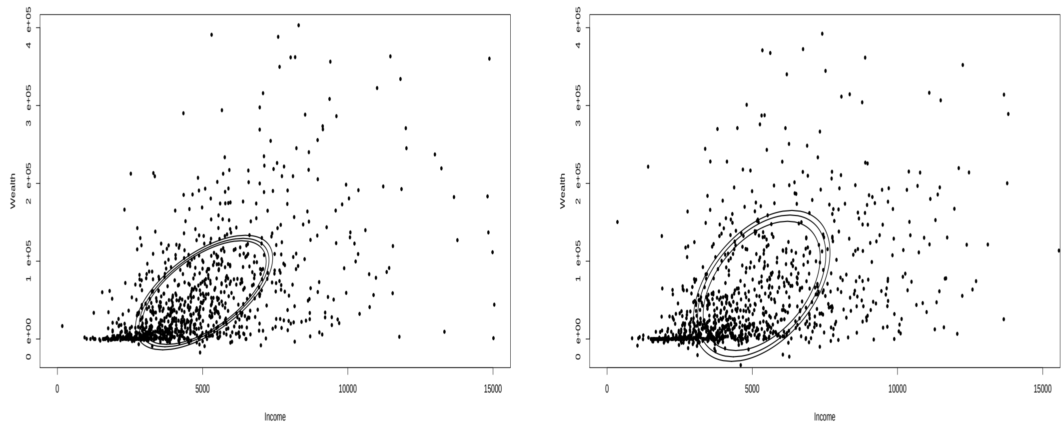


Figure 3: Zonoid for $h\% = 40\%, 45\%, 50\%$.

(a) Year 1998 (b) Year 2003

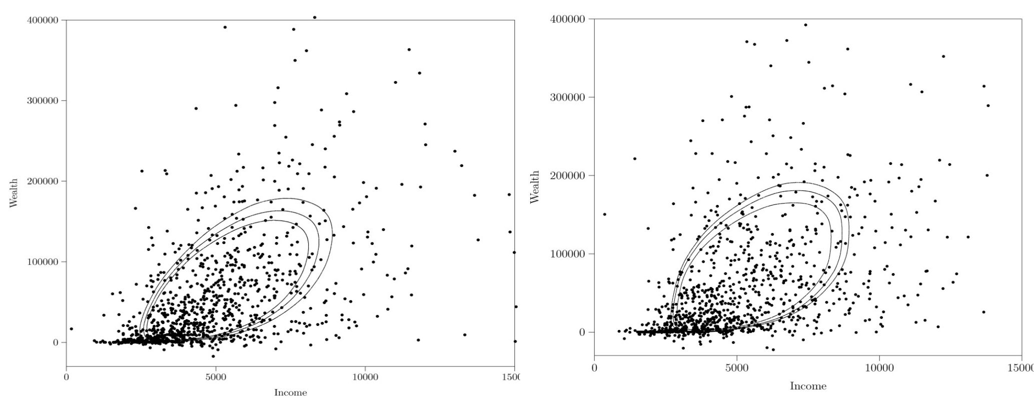
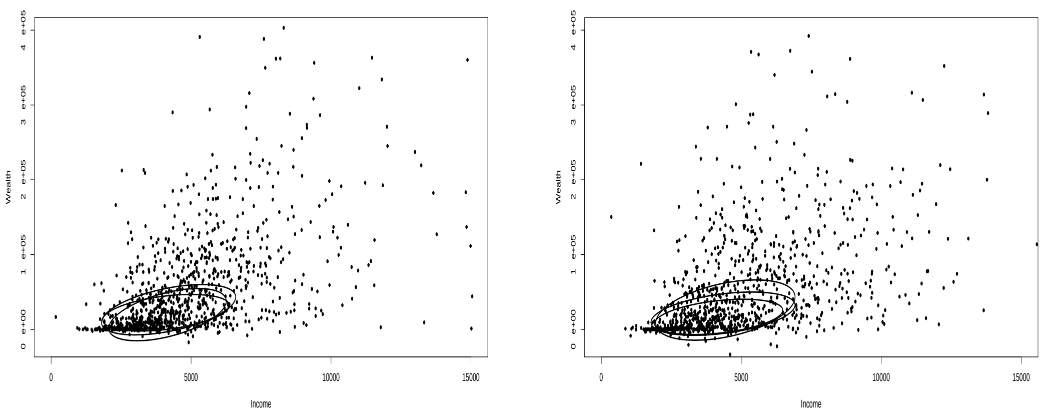


Figure 4: MVE for $h\% = 40\%, 45\%, 50\%$.

(a) Year 1998 (b) Year 2003



An interesting observation on the different definitions of middle class emerges from Tables 2 to 6: For each level of h, the volume, both normalized and not, of the zonoid regions is greater than with the other methods, i.e. the h percent individuals closest to the center are considered more dispersed with the zonoid central regions. It means that zonoid method is more sensible to polarization than the other methods. The same holds for the trimmed relative indices, as long as the proportion of trimming is less than 10%.

Another aspect about zonoid central regions has to be underlined: The graphics reveal how the zonoid central regions are more able to reflect to the strong asymmetry of the bivariate distribution of income and wealth than the other methods, as discussed in Section 4.

A natural question, after having defined the middle class and having studied its size, is the following: Which are the characteristics of the households that constitute such social group? Tables 7 and 8 compare the main aspects of the households of the middle class with the ones of the entire population (always referred to the subsample of 1.000 units), in case of middle class defined by the Mahalanobis ellipsoids.

A first obvious aspect can be observed both from all the graphics and from Table 7: The range of values for income and wealth (and also for the two components of property and financial assets) is smaller in the middle class than in the entire population.

Table 7: Main characteristics of the households.

	MIDDLE CLASS [1]		POPULATION	
	1998	2003	1998	2003
INCOME[2][3]	4,146	4,573	4,526	5,005
WEALTH[2][3] [4]	33,876	41,242	62,599	64,662
PROPERTY WEALTH[2][3]	22,576	27,919	46,243	46,139
FINANCIAL WEALTH[2][3]	11,300	13,323	16,356	18,523
NUMBER OF COMPONENTS[2]	2.20	2.36	2.08	2.15
NUMBER OF WORKERS[2]	0.93	1.02	0.82	0.83
NUMBER OF CHILDREN[2]	0.38	0.45	0.33	0.36
% IN FORMER WEST GERMAN	0.78	0.78	0.79	0.80

[1]The middle class is defined for Mahalanobis ellipsoids, with $h = 0.5$.
[2]Mean value.
[3]Amounts in Euro.
[4] The equivalence scale adopted is the mixed scale discussed in footnote 13.

Table 7 reveals that the families in the middle class have, on average, a higher number of components, of workers and of children under 14 years than in the population; moreover, in the middle class the percentage of households living in the regions of the former West Germany is slightly smaller than in the entire population.

Table 8 shows, moreover, that in the middle class the household's heads are in higher percentage male (even if the trend is declining), on average younger and with a more technical education (with a certificate in a vocational school or with a degree in a university of applied science) than in the entire population. In the middle class the percentage of employee householders (in particular white and blue collars) is greater, while the proportion of self-employed, unemployed and retired household's heads is smaller than in the entire society. Finally, in the middle class, the percentage of householders single, divorced or widowed is smaller and the proportion of married heads is greater than in the total population.

The characteristics of the middle class, shown in Tables 7 and 8, are not surprising; the two approaches proposed in the paper are, therefore, able to isolate a central class which corresponds to the common idea of middle class.

Table 8: Main characteristics of the heads of the household.

	MIDDLE CLASS (1)		POPULATION	
	1998	2003	1998	2003
AVERAGE AGE	50.80	50.29	51.81	50.74
MALE VS FEMALE (2)	64.20	62.37	59.34	61.74
EDUCATION (2)				
university	14.10	14.69	15.07	15.25
university of applied science	13.89	11.76	12.51	11.26
training in specialized academy	12.53	18.73	14.45	17.53
job training	51.25	51.09	47.49	46.87
other vocational education	1.85	2.08	4.19	2.86
student/ still in training	0.84	0.49	1.69	2.98
no training	5.54	1.15	4.60	3.25
SOCIAL CONDITION (2)				
self-employed farmer	0.53	0.21	0.54	0.45
self-employed craftsman	3.18	4.15	5.09	4.42
functionary	4.45	5.41	4.22	4.46
white collar	34.51	33.39	29.53	28.58
blue collar	18.62	24.28	15.03	17.74
unemployed	1.74	2.42	6.17	6.20
retired person/pensioner	32.15	29.63	33.72	33.08
student	2.51	0.00	1.76	2.09
others	2.30	0.50	3.93	2.98
MARITAL STATUS (2)				
single	22.93	22.91	23.44	25.78
married	56.28	57.10	51.44	51.64
widowed	9.82	8.24	10.52	7.63
divorced	9.84	9.76	12.29	12.50
separated	1.13	1.98	2.31	2.45

(1) The middle class is defined for Mahalanobis ellipsoids, with $h = 0.5$.
(2) Percentage of households.

6 Concluding remarks

At the beginning of this paper, two questions have been posed: How to define the middle class of a society, in case of one or more attributes of interest, and how to measure its possible disappearing.

In order to answer the first question, the use of particular geometric bodies has been introduced, leading to two approaches, one based on central regions (in particular, balls, Mahalanobis ellipsoids, zonoid and halfspace central regions) and the other one based on minimum volume ellipsoids.

The second question has been handled monitoring the level of dispersion inside the middle class. The middle class disappears, every time its members become more different to each other and less able to cooperate. In this paper we employed the volume and the relative volume of the relevant central set (or MVE) to measure the dispersion of the middle class. Note that in place of these volumes other multivariate indices of dispersion may be used, among them the Gini indices of Koshevoy and Mosler (1997).

We have applied the proposed measures to a German data set and analyzed the phenomenon both in absolute and in relative terms, by comparing the changes in dispersion between the middle class and the whole population, eventually trimmed. The application revealed different results, in the relative analysis, for the central regions (Mahalanobis and zonoid) and for the MVE. Mahalanobis ellipsoids and zonoid central regions, indeed, show an increase in the weakening of the middle class, from the year 1998 to 2003, both in absolute terms and in relative terms, for different degrees of population's trimming; the minimum volume ellipsoids, instead, are able to isolate a middle class that becomes, throughout the years, more dispersed in absolute terms but less dispersed when compared to the trimmed population.

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