Weighted-mean regions of a probability distribution

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Abstract

In this paper we investigate a new class of central regions for probability distributions on $\mathbb{R}^d$, called weighted-mean regions. Their restrictions to an empirical distribution are the weighted-mean trimmed regions investigated by Dyckerhoff and Mosler (2011) for d-variate data. Furthermore a new class of stochastic orderings of variability, the weighted-mean orderings, is introduced.

Keywords: Central regions, Continuous trimming, Lift zonoid regions, Expected convex hull, Variability order

1. Introduction

Let F be a probability distribution on the Borel sets of $\mathbb{R}^d$, and X a random vector that is distributed as F. A *family of central regions* is a family $(R_\alpha(X))_{\alpha\in I}$ of nested convex compacts $R_\alpha(X) \subset \mathbb{R}^d$ which are distribution invariant (i.e. depend only on F) and affine equivariant,

$$R_\alpha(AX+b) = A\,R_\alpha(X) + b \quad \text{for regular } A \in \mathbb{R}^{d\times d} \text{ and } b \in \mathbb{R}^d.$$

Here I is an interval in $\mathbb{R}$, and the regions $R_\alpha(X)$ decrease with $\alpha \in I$. Central regions describe a distribution regarding its location and dispersion (due to their affine equivariance), as well as its shape. They are also mentioned as *trimmed regions* and can be seen as level sets of a *data depth*. For a general discussion, see e.g. Zuo and Serfling (2000), Dyckerhoff (2004).

Dyckerhoff and Mosler (2011) have introduced the notion of weighted-mean (WM) trimming for multivariate data, i.e. for empirical distributions. This general class of trimmings includes the zoid trimming (Koshevoy and Mosler, 1997), the ECH (expected convex hull) trimming (Cascos, 2007), and other known trimmings as special cases. The WM trimmed regions of an empirical distribution are convex polytopes around the mean that have many attractive properties, including subadditivity and continuity in the data as well as in the trimming parameter. In the sequel we present a population version of the WM trimmed regions and derive their similar properties. These WM regions are defined for any probability distribution on $\mathbb{R}^d$ that has a finite first moment. Their restriction to an empirical distribution comes out as the weighted-mean trimmed regions for d-variate data. In fact, a law of large numbers holds: The WM trimmed regions of an independent sample converge, almost surely, to the trimmed regions of the underlying probability distribution; see Dyckerhoff and Mosler (2011).

2. Weighted-mean regions of a probability

As central regions are convex compacts in $\mathbb{R}^d$, we will define them through their support functions. Recall that a closed convex set $K \subset \mathbb{R}^d$ is uniquely characterized by its support function $h_K : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, $h_K(p) = \sup\{p'x \mid x \in K\}$. Further, the support function is finite for all $p \in \mathbb{R}^d$ if and only if K is compact, i.e., closed and bounded. A function $h : \mathbb{R}^d \to \mathbb{R}$ is the support function of a convex compact iff it is positive homogeneous ($h(\lambda p) = \lambda h(p)$ for $\lambda > 0$) and subadditive ($h(p+q) \leq h(p) + h(q)$). See, e.g., Rockafellar (1970).

Consider the class $\mathcal{F}_{wg}$ of *weight-generating functions* $f : [0,1] \to [0,1]$ that satisfy

(i) $f(0) = 0$, $f(1) = 1$,
(ii) f is increasing, convex, and continuous,
(iii) f' is bounded.

Note that every $f \in \mathcal{F}_{wg}$ is differentiable except at (at most) countable many points. As the derivative is increasing, we extend f' to a function on $[0,1]$ that is right continuous and bounded, in other words, to the distribution function of a finite measure.

Proposition 1 (Support function). *Let $I \subset \mathbb{R}$ be an interval and $\{r_\alpha\}_{\alpha \in I}$ a family of functions from $\mathcal{F}_{wg}$. Then, for any random vector X in $\mathbb{R}^d$ having probability distribution F and finite first moment, the function h,*

$$h(p) = \int_0^1 Q_{p'X}(t)\, dr_\alpha(t)\,, \quad p \in \mathbb{R}^d\,, \tag{1}$$

is the support function of a convex compact. Here $Q_{p'X}$ signifies the quantile function of $p'X$, i.e., $Q_{p'X}(t) = \inf\{x \in \mathbb{R} \mid P(p'X \leq x) \geq t\}$.

Proof: We have to show that h is positive homogeneous, subadditive and finite for every p. The positive homogeneity follows immediately from the fact that $Q_{(\lambda p)'X} = Q_{\lambda(p'X)} = \lambda Q_{p'X}$ for $\lambda > 0$.

For a (univariate) random variable X, functionals of the form $H_r : X \mapsto \int_0^1 Q_X(t)\,dr(t)$, where $r \in \mathcal{F}_{wg}$, have been studied in risk theory. It is well known that these functionals are subadditive, i.e.,

$$H_r(X+Y) \le H_r(X) + H_r(Y)\,.$$

For a proof of this result see, e.g., Wang and Dhaene (1998). In this article the functional H_r is written in the form $H_r(X) = \int_0^1 Q_X(1-t)\,dr^*(t)$ where $r^*(t) = 1 - r(1-t)$ is increasing and concave.

Therefore, for d-variate random vectors X and Y and $u, v \in \mathbb{R}^d$, it holds

$$\begin{aligned} h(u+v) = \int_0^1 Q_{(u+v)'X}(t)\,dr_\alpha(t) &= \int_0^1 Q_{u'X+v'X}(t)\,dr_\alpha(t) \\ &\le \int_0^1 \left[Q_{u'X}(t) + Q_{v'X}(t)\right]\,dr_\alpha(t) = h(u) + h(v) \end{aligned}$$

which was to be proved.

For finiteness of $h(p)$ notice that $E\big[\|X\|\big] < \infty$ implies $E[\,|p'X|\,] = \int_0^1 |Q_{p'X}(t)|\,dt < \infty$ for every $p \in \mathbb{R}^d$. Since r'_α is bounded, it holds that $|r'_\alpha(t)| < M$ for some $M > 0$. It follows that

$$\begin{aligned} \left|\int_0^1 Q_{p'X}(t)\,dr_\alpha(t)\right| &= \left|\int_0^1 Q_{p'X}(t) r'_\alpha(t)\,dt\right| \\ &\le \int_0^1 |Q_{p'X}(t)| \cdot |r'_\alpha(t)|\,dt \le M \int_0^1 |Q_{p'X}(t)|\,dt < \infty\,. \end{aligned}$$

Therefore, $h(p)$ is finite for every $p \in \mathbb{R}^d$. □

Based on the preceding proposition we define:

Definition 1. *The unique convex bodies whose support functions are given by* (1) *are called the* weighted-mean regions, *in short WM regions, of* X *and denoted by* $\mathrm{D}_\alpha(X)$, $\alpha \in I$.

Observe that the support function h and hence the region $D_\alpha(X)$ depends only on the distribution F of X. To show that the weighted-mean regions form indeed a family of central regions, we still have to establish their affine equivariance and nestedness.

Proposition 2 (Affine equivariance)**.** *For every matrix* $A \in \mathbb{R}^{m\times d}$ *and every* $b \in \mathbb{R}^m$ *it holds*

$$\mathrm{D}_\alpha(AX+b) = A\mathrm{D}_\alpha(X) + b\,.$$

Proof:

$$h_{\mathrm{D}_\alpha(AX+b)}(p) = \int_0^1 Q_{p'(AX+b)}(t)\, dr_\alpha(t) = \int_0^1 \left[Q_{(A'p)'X}(t) + p'b\right]\, dr_\alpha(t)$$
$$= h_{\mathrm{D}_\alpha(X)}(A'p) + p'b = h_{A\mathrm{D}_\alpha(X)+b}(p)$$

□

Note that Proposition 2 states much more than affine equivariance, as A can be *any* matrix.

Proposition 3 (Nestedness)**.** *Let the family of functions* $\{r_\alpha\}_{\alpha\in I}$ *satisfy*
(iv) $\alpha \mapsto r_\alpha(t)$ *is increasing for every* t*.*
Then, the WM regions are nested,

$$\alpha < \beta \quad \Longrightarrow \quad \mathrm{D}_\beta(X) \subset \mathrm{D}_\alpha(X)\,.$$

Proof: From $\alpha < \beta$ it follows $r_\alpha(t) \le r_\beta(t)$ for every $t \in [0,1]$. Thus, the probability distribution generated by r_α dominates that generated by r_β in the sense of first degree stochastic dominance. Since quantile functions are increasing it follows that

$$h_{\mathrm{D}_\alpha(X)}(p) = \int_0^1 Q_{p'X}(t)\, dr_\alpha(t) \ge \int_0^1 Q_{p'X}(t)\, dr_\beta(t) = h_{\mathrm{D}_\beta(X)}(p)\,,$$

and therefore $\mathrm{D}_\beta(X) \subset \mathrm{D}_\alpha(X)$. □

When we speak of weighted-mean regions we will henceforth assume that condition *(iv)* of Proposition 3 is satisfied.

Interesting is the special case when $d = 1$. Then it holds $h_{D_\alpha(X)}(1) = \int_0^1 Q_X(t)dr_\alpha(t)$ and $h_{D_\alpha(X)}(-1) = \int_0^1 Q_{-X}(t)dr_\alpha(t)$, hence

$$D_\alpha(X) = \left[-\int_0^1 Q_{-X}(t)dr_\alpha(t)\,,\ \int_0^1 Q_X(t)dr_\alpha(t)\right]. \tag{2}$$

For general $d \ge 1$ and the choice

$$r_\alpha(t) = \begin{cases} 0\,, & \text{if } t < 1-\alpha\,, \\ \frac{t-(1-\alpha)}{\alpha} & \text{if } t \ge 1-\alpha\,. \end{cases} \tag{3}$$

one gets the so-called *zonoid regions*, that have been extensively studied in the literature, see, e.g., Koshevoy and Mosler (1997), Mosler (2002). The univariate zonoid region is a closed interval whose lower bound is the negative of a popular univariate risk measure, the α-*expected shortfall.*

Another special case of weighted-mean regions is given by the *continuous* ECH* *regions* (shortly CECH* regions) that were introduced in Cascos (2007) and Dyckerhoff and Mosler (2011). These regions are defined by the weight-generating functions

$$r_\alpha(t) = t^{1/\alpha}\,, \quad \alpha \in (0,1]\,.$$

In the univariate case, if $\alpha = 1/n$, the lower bound of the CECH* region can be represented as

$$\min \mathrm{D}_{1/n}(X) = -h_{\mathrm{D}_{1/n}(X)}(-1) = E\left[\min\{X_1,\dots,X_n\}\right]\,,$$

where $X_1,\dots,X_n$ are independent copies of X. This again is a coherent risk measure, the *expected minimum*, also called *Alpha V@R* by Cherny and Madan (2006).

3. Properties of weighted-mean regions

In this section we will establish additional properties of the weighted-mean regions, such as subadditivity, monotonicity, and continuity.

Proposition 4 (Subadditivity)**.** *The WM regions are subadditive,*

$$\mathrm{D}_\alpha(X+Y) \subset \mathrm{D}_\alpha(X) \oplus \mathrm{D}_\alpha(Y)$$

Proof: Recall that support functions are additive w.r.t. the Minkowski addition $\oplus$ of sets, $h_K(p) + h_L(p) = h_{K\oplus L}(p)$. Thus, we have to show that $h_{\mathrm{D}_\alpha(X+Y)} \le h_{\mathrm{D}_\alpha(X)} + h_{\mathrm{D}_\alpha(Y)}$.

As in the proof of Proposition 1 we obtain

$$\begin{aligned} h_{\mathrm{D}_\alpha(X+Y)}(p) &= \int_0^1 Q_{p'(X+Y)}(t)\,dr_\alpha(t) \;=\; \int_0^1 Q_{p'X+p'Y}(t)\,dr_\alpha(t) \\ &\le \int_0^1 \left[Q_{p'X}(t) + Q_{p'Y}(t)\right]\,dr_\alpha(t) \;=\; h_{\mathrm{D}_\alpha(X)}(p) + h_{\mathrm{D}_\alpha(Y)}(p)\,. \end{aligned}$$

□

Let X, Y be d-variate random vectors. A set $U \subset \mathbb{R}^n$ is called *upper* if $x \in U$, $x \le y$ implies $y \in U$. The *strong first degree stochastic order* $\le_1$ on the space of d-variate random vectors is defined by

$$X \le_1 Y \quad \Longleftrightarrow \quad P(X\in U) \le P(Y \in U) \quad \text{for all upper sets } U \subset \mathbb{R}^d.$$

Proposition 5 (Monotonicity)**.** *Suppose $X \le_1 Y$, where $\le_1$ denotes strong first degree multivariate stochastic dominance. Then,*

$$\mathrm{D}_\alpha(Y) \subset \mathrm{D}_\alpha(X) \oplus \mathbb{R}^d_+ \quad \textit{and} \quad \mathrm{D}_\alpha(X) \subset \mathrm{D}_\alpha(Y) \oplus \mathbb{R}^d_- \,.$$

Proof: Since $h_{\mathbb{R}^d_+}(p) = \infty$ if $p \notin \mathbb{R}^d_-$, the first set inclusion is equivalent to $h_{\mathrm{D}_\alpha(Y)}(p) \le h_{\mathrm{D}_\alpha(X)}(p)$ for all $p \in \mathbb{R}^d_-$, which has to be checked.

If $X \le_1 Y$ then there exist random vectors $\tilde{X}$ and $\tilde{Y}$ such that $X \overset{d}{=} \tilde{X}$ and $Y \overset{d}{=} \tilde{Y}$ and $\tilde{X} \le \tilde{Y}$ with probability one (see, e.g., Kamae et al., 1977). Thus, $\tilde{Y} - \tilde{X} \ge 0$ almost surely, and $p'\tilde{Y} - p'\tilde{X} = p'(\tilde{Y} - \tilde{X}) \le 0$ for all $p \in \mathbb{R}^d_-$, hence, with probability one, $p'\tilde{Y} \le p'\tilde{X}$. We conclude

$$Q_{p'Y} = Q_{p'\tilde{Y}} \le Q_{p'\tilde{X}} = Q_{p'X} \,,$$

$$h_{\mathrm{D}_\alpha(Y)}(p) = \int_0^1 Q_{p'Y}(t)\, dr_\alpha(t) \le \int_0^1 Q_{p'X}(t)\, dr_\alpha(t) = h_{\mathrm{D}_\alpha(X)}(p)\,.$$

This proves $\mathrm{D}_\alpha(Y) \subset \mathrm{D}_\alpha(X) \oplus \mathbb{R}^d_+$. The second statement is analogously proven. □

Proposition 6 (Continuity in α)**.** *Assume that X is a random vector for which the expectation exists and is finite. Assume further that (r_{α_n}) converges pointwise to r_α whenever (α_n) converges to α. Then the map $\alpha \mapsto \mathrm{D}_\alpha(X)$ is continuous w.r.t. the Hausdorff metric.*

Proof: Since r_{α_n} is convex, pointwise convergence of (r_{α_n}) to r_α implies that (r'_{α_n}) converges to r'_α except possibly at countable many points in $[0,1]$. Since $\lim_{n\to\infty} r'_{\alpha_n}(1) = r'_\alpha(1) < M < \infty$ there exists $N \in \mathbb{N}$ such that $r'_{\alpha_n}(1) \le M < \infty$ for all $n \ge N$. Therefore,

$$|Q_{p'X}(t) r'_{\alpha_n}(t)| \le M\, |Q_{p'X}(t)| \quad \text{for all } n \ge N.$$

Since $M|Q_{p'X}(t)|$ is integrable, it follows from the dominated convergence theorem that

$$\begin{aligned}\lim_{n\to\infty} \int_0^1 Q_{p'X}(t)\, dr_{\alpha_n}(t) &= \lim_{n\to\infty} \int_0^1 Q_{p'X}(t) r'_{\alpha_n}(t)\, dt \\ &= \int_0^1 Q_{p'X}(t) r'_\alpha(t)\, dt = \int_0^1 Q_{p'X}(t)\, dr_\alpha(t)\,.\end{aligned}$$

Therefore the support function converges pointwise on the sphere S^{d-1}, which is equivalent to convergence of the associated convex bodies in the Hausdorff sense. □

Proposition 7 (Continuity in the distribution). *Assume that* (X_n) *is a sequence of random vectors with finite first moments that converges in distribution to* X. *If the sequence* (X_n) *is uniformly integrable, then* $\mathrm{D}_\alpha(X_n)$ *converges to* $\mathrm{D}_\alpha(X)$ *in the Hausdorff metric.*

Proof: We have to show pointwise convergence of the support functions.

Note that the sequence (X_n) converges in distribution to X if and only if the sequence of linear combinations $(p'X_n)$ converges in distribution to $p'X$ for every $p \in S^{d-1}$. Further, (X_n) is uniformly integrable if and only if the sequence of one-dimensional projections $(p'X_n)$ is uniformly integrable for every $p \in S^{d-1}$.

According to Skorohod's representation theorem there are random variables $Z, Z_1, Z_2, \dots$ defined on a common probability space such that Z_n (Z) has the same distribution as $p'X_n$ $(p'X)$ and $Z_n(\omega) \xrightarrow{n\to\infty} Z(\omega)$ for every $\omega \in \Omega$. Note that the sequence (Z_n) is uniformly integrable as well.

Since $(p'X_n)$ converges in distribution to $p'X$, it follows that $F_{p'X_n}$ converges to $F_{p'X}$ at all continuity points of $F_{p'X}$. It can be shown that the same holds for the quantile functions, i.e., $Q_{p'X_n}$ converges to $Q_{p'X}$ at all continuity points of $Q_{p'X}$. Since $Q_{p'X}$ is monotone, it has at most countable many discontinuities.

In the next step we show that the sequence of quantile functions Q_{Z_n} is uniformly integrable. First, it follows from the uniform integrability of the sequence (Z_n) that $\lim_{n\to\infty} E[|Z_n|] = E[|Z|]$. It is well-known that $E[|Z_n|] = \int_0^1 |Q_{Z_n}(t)|\, dt$. Thus,

$$\lim_{n\to\infty} \int_0^1 |Q_{Z_n}(t)|\, dt = \int_0^1 |Q_Z(t)|\, dt\,.$$

Since Q_{Z_n} converges to Q_Z almost everywhere, this implies uniform integrability of the sequence Q_{Z_n}. Now it follows from uniform integrability of the Q_{Z_n} that

$$\lim_{n\to\infty} \int_0^1 |Q_{Z_n}(t) - Q_Z(t)|\ dt = 0\,.$$

For the last step, note that r'_α is bounded so that, for all t, $|r'_\alpha(t)| < M$ with some $M > 0$. Now,

$$\begin{aligned}\left|h_{\mathrm{D}_\alpha(X_n)}(p) - h_{\mathrm{D}_\alpha(X)}(p)\right| &= \left|\int_0^1 Q_{p'X_n}(t)\, dr_\alpha(t) - \int_0^1 Q_{p'X}(t)\, dr_\alpha(t)\right| \\ &= \left|\int_0^1 (Q_{Z_n}(t) - Q_Z(t))\, r'_\alpha(t)\, dt\right|\end{aligned}$$

$$\leq \int_0^1 \Big|Q_{Z_n}(t) - Q_Z(t)\Big| \cdot \big|r'_\alpha(t)\big| \; dt$$
$$\leq M \int_0^1 \Big|Q_{Z_n}(t) - Q_Z(t)\Big| \, dt$$

Since the right hand side goes to zero when n goes to infinity, we conclude that the support functions converge pointwise, which was to be shown. □

4. Weighted-mean orderings

Stochastic orderings of d-variate distributions have many applications; see Shaked and Shanthikumar (2007). A large and flexible class of orderings can be based on weighted-mean regions.

Definition 2 (Weighted-mean ordering)**.** *Let $r = \{r_\alpha\}_{\alpha \in I}$ be a family satisfying (i) to (iv). We define an ordering of d-variate random vectors (and their probability distributions) by $X \preceq_r Y$ if*

$$D_\alpha(X) \subset D_\alpha(Y) \quad \textit{for all } \alpha \in I\,.$$

Obviously, this relation is transitive and reflexive.

The weighted-mean orderings can be seen as variability orderings between random vectors as is apparent from the following results.

Proposition 8. *For every random vector X with finite first moment holds*

$$E[X] \preceq_r X\,.$$

Proof: Note that the identity $id : t \mapsto t$ is in $\mathcal{F}_{wg}$. Further, for every function $r_\alpha \in \mathcal{F}_{wg}$ holds $r_\alpha \leq id$ where the inequality is to be understood pointwise. Now, as in the proof of Proposition 3, it follows that

$$h_{\mathrm{D}_\alpha(X)}(p) = \int_0^1 Q_{p'X}(t)\, dr_\alpha(t) \geq \int_0^1 Q_{p'X}(t)\, dt = E[p'X] = p'E[X]\,.$$

Therefore, $\mathrm{D}_\alpha(E[X]) = \{E[X]\} \subset \mathrm{D}_\alpha(X)$ and the assertion follows. □

Proposition 9. *For every random vector X with finite first moment and every $\lambda \geq 1$ holds*

$$X - E[X] \preceq_r \lambda(X - E[X])\,.$$

Proof: From the affine equivariance of the WM regions it follows that

$$\mathrm{D}_\alpha\big(\lambda(X - E[X])\big) = \lambda\left(\mathrm{D}_\alpha(X) - E[X]\right) .$$

The set $\mathrm{D}_\alpha(X) - E[X]$ is convex and, since $E[X] \in \mathrm{D}_\alpha(X)$, contains the origin. Therefore, for every $\lambda \geq 1$,

$$(\mathrm{D}_\alpha(X) - E[X]) \subset \lambda\left(\mathrm{D}_\alpha(X) - E[X]\right)$$

which completes the proof. □

The weighted-mean ordering is preserved under arbitrary affine transformations.

Proposition 10. *Let X and Y be random vectors with finite first moments. Then, for every matrix $A \in \mathbb{R}^{m\times d}$ and every $b \in \mathbb{R}^m$ it holds*

$$X \preceq_r Y \qquad \Longrightarrow \qquad AX + b \preceq_r AY + b\,.$$

Proof: The proposition follows immediately from Proposition 2. □

As a corollary we note that the weighted-mean ordering is preserved under marginalization. Let X_J denote the marginal of X regarding the coordinate set $J \subset \{1, \ldots, d\}$.

Corollary 1. *Let $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ be random vectors with finite first moments. Then,*

$$X \preceq_r Y \qquad \Longrightarrow \qquad X_J \preceq_r Y_J\,, \quad J \subset \{1, \ldots, d\},\ J \neq \emptyset\,.$$

Proof: The proposition follows immediately from the preceding proposition by choosing A as the projection matrix on the coordinate set J. □

Consider the special case of the zonoid regions, where r_α is given by (3). The weighted-mean order based on the zonoid regions is known as the zonoid order; see Mosler (2002). The zonoid order plays a special role as is shown in the following proposition.

Proposition 11. *The zonoid order $\preceq_Z$ implies any weighted-mean ordering, i.e.,*

$$X \preceq_Z Y \quad \Longrightarrow \quad X \preceq_r Y \text{ for any such family } r.$$

Proof: It holds (Mosler, 2002) that $X \preceq_Z Y$ iff $p'X \leq_{cx} p'Y$ for all $p \in \mathbb{R}^d$. By Theorem 4.A.4 in Shaked and Shanthikumar (2007) this implies

$$h_{D_\alpha(X)}(p) = \int_0^1 Q_{p'X}(t)\, dr_\alpha(t) \leq \int_0^1 Q_{p'Y}(t)\, dr_\alpha(t) = h_{D_\alpha(Y)}(p)\,,$$

which means $D_\alpha(X) \subset D_\alpha(Y)$ for all α. □

As we will see below in Example 1 the converse of Proposition 11 is in general wrong, i.e., there exist weighted-mean orderings that are not equivalent to the zonoid order.

One of the most important multivariate variability orders is the convex order (see Shaked and Shanthikumar, 2007, chap. 7). The following proposition clarifies the relation between the convex order and the weighted-mean orderings.

Proposition 12. *The convex order $\leq_{cx}$ implies any weighted-mean ordering, i.e.,*

$$X \leq_{cx} Y \quad \Longrightarrow \quad X \preceq_r Y \text{ for any such family } r.$$

In the case $d > 1$ the weighted-mean orderings are strictly weaker than the convex order, i.e., for any weighted-mean ordering

$$X \preceq_r Y \quad \not\Longrightarrow \quad X \leq_{cx} Y\,.$$

Proof: In Mosler (2002) it was shown that the zonoid order is implied by the convex order, but is different from the convex order when $d > 1$. The assertion then follows from Proposition 11. □

Proposition 13. *Let $\{r_\alpha\}_{\alpha\in I}$ satisfy $\sup_{\alpha\in I} r_\alpha = id$, where id denotes the identity. Then,*

$$X \preceq_r Y \quad \Longrightarrow \quad E[X] = E[Y]\,.$$

Proof: Under the condition given above, there is a sequence (α_n) such that (r_{α_n}) converges pointwise to the identity. From Proposition 6 it follows that the sequences $(\mathrm{D}_{\alpha_n}(X))$ and $(\mathrm{D}_{\alpha_n}(Y))$ both converge in the Hausdorff metric to the singletons $\{E[X]\}$ and $\{E[Y]\}$. Since $\mathrm{D}_\alpha(X) \subset \mathrm{D}_\alpha(Y)$ for all α, this implies $E[X] = E[Y]$. □

We conclude this section with an example that shows that the converse of Proposition 11 is in general wrong. In particular we show that the ordering $\preceq_{\mathrm{CECH}^*}$, which is defined by the *continuous* ECH* regions, does not imply the zonoid order.

Example 1. *Let X be a random variable that takes on the values $x_1 = -1.05$, $x_2 = -0.05$, $x_3 = 0.05$ and $x_4 = 1.05$ with equal probability, and let Y be a random variable that takes on the values $y_1 = -1$, $y_2 = -0.15$,*

$y_3 = 0.15$ and $y_4 = 1$ with equal probability. For the zonoid regions we get

$$h_{\mathrm{ZD}_{1/4}(X)}(1) = \frac{1}{1/4}\int_{3/4}^{1} Q_X(t)\,dt = 4\cdot\frac{1}{4}\cdot 1.05 = 1.05\,,$$
$$h_{\mathrm{ZD}_{1/2}(X)}(1) = \frac{1}{1/2}\int_{1/2}^{1} Q_X(t)\,dt = 2\cdot\left(\frac{1}{4}\cdot 0.05 + \frac{1}{4}\cdot 1.05\right) = 0.55\,,$$

and

$$h_{\mathrm{ZD}_{1/4}(Y)}(1) = \frac{1}{1/4}\int_{3/4}^{1} Q_Y(t)\,dt = 4\cdot\frac{1}{4}\cdot 1 = 1\,,$$
$$h_{\mathrm{ZD}_{1/2}(Y)}(1) = \frac{1}{1/2}\int_{1/2}^{1} Q_Y(t)\,dt = 2\cdot\left(\frac{1}{4}\cdot 0.15 + \frac{1}{4}\cdot 1\right) = 0.575\,.$$

Therefore, neither $X \preceq_Z Y$ nor $Y \preceq_Z X$ holds.

Now consider the continuous ECH* *regions. For the sake of simplicity we set $\beta = 1/\alpha$ and write $r_\alpha(t) = t^\beta$. Then we get*

$$\begin{aligned} h_{\mathrm{CECH}^*_\alpha(X)}(1) &= -1.05\cdot\left[\left(\frac{1}{4}\right)^\beta - \left(\frac{0}{4}\right)^\beta\right] - 0.05\cdot\left[\left(\frac{2}{4}\right)^\beta - \left(\frac{1}{4}\right)^\beta\right] \\ &\quad + 0.05\cdot\left[\left(\frac{3}{4}\right)^\beta - \left(\frac{2}{4}\right)^\beta\right] + 1.05\cdot\left[\left(\frac{4}{4}\right)^\beta - \left(\frac{3}{4}\right)^\beta\right] \\ &= \frac{1}{4^\beta}\left[1.05\cdot 4^\beta - 3^\beta - 0.1\cdot 2^\beta - 1\right] \end{aligned}$$

Analogously, we get

$$h_{\mathrm{CECH}^*_\alpha(Y)}(1) = \frac{1}{4^\beta}\left[4^\beta - 0.85\cdot 3^\beta - 0.3\cdot 2^\beta - 0.85\right]\,.$$

The difference is given by

$$h_{\mathrm{CECH}^*_\alpha(X)}(1) - h_{\mathrm{CECH}^*_\alpha(Y)}(1) = \frac{1}{4^\beta}\left[0.05\cdot 4^\beta - 0.15\cdot 3^\beta + 0.2\cdot 2^\beta - 0.15\right]\,.$$

Now consider the function $f : [1,\infty) \to \mathbb{R}$,

$$f(x) = 0.05\cdot 4^x - 0.15\cdot 3^x + 0.2\cdot 2^x - 0.15\,.$$

We have to show that $f(x) \geq 0$ for all $x \in [1,\infty)$. For $x = 1$ obtain $f(1) = 0$. The derivative is given by

$$f'(x) = 0.05\cdot\ln 4\cdot 4^x - 0.15\cdot\ln 3\cdot 3^x + 0.2\cdot\ln 2\cdot 2^x\,.$$

Obviously, $f'(x) > 0$ *iff*

$$g(x) := 0.05 \cdot \ln 4 \cdot \left(\frac{4}{3}\right)^x + 0.2 \cdot \ln 2 \cdot \left(\frac{2}{3}\right)^x > 0.15 \cdot \ln 3\,.$$

It can be seen by routine calculations that g *is convex on* $[1, \infty)$ *and has a unique global minimum at*

$$x^* = 1 + \ln\left(\frac{\ln 3/2}{\ln 4/3}\right) \Big/ \ln 2 \approx 1.49510245\,.$$

The value at x^* *is*

$$g(x^*) \approx 0.18217704 > 0.16479184 \approx 0.15 \cdot \ln 3\,.$$

Thus, $f'(x) > 0$ *on* $[1, \infty)$ *and therefore* $f(x) \geq 0$ *on* $[1, \infty)$*. This shows that* $h_{\mathrm{CECH}^*_\alpha(X)}(1) \geq h_{\mathrm{CECH}^*_\alpha(Y)}(1)$ *for all* $\alpha \in (0, 1]$ *and therefore* $X \succeq_{\mathrm{CECH}^*} Y$*, which proves that the orderings* $\preceq_Z$ *and* $\preceq_{\mathrm{CECH}^*}$ *are in fact different.*

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