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UNIVERSITY OF COLOGNE

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Nonparametric Tests based on Area-Statistics

by

Stefan Kraft and Friedrich Schmid

August 2000

DISKUSSIONSBEITRÄGE ZUR STATISTIK UND ÖKONOMETRIE

SEMINAR FÜR WIRTSCHAFTS- UND SOZIALSTATISTIK
UNIVERSITÄT ZU KÖLN

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Abstract: Area statistics are sample versions of areas occuring in a probability plot of two distribution functions F and G. This paper gives a unified basis for five statistics of this type. They can be used for various testing problems in the framework of the two sample problem for independent observations such as testing equality of distributions against inequality or testing stochastic dominance in one or either direction against nondominance. Though three of the statistics considered have already been suggested in literature, two of them are new and deserve our interest. The finite sample distribution of these statistics can be calculated via recursion formulae. Two tables with critical values of the new statistics are added. The asymptotic distribution of the properly normalized versions of the area statistics are functionals of the Brownian Bridge. The distribution functions and quantiles thereof are obtained by Monte-Carlo-Simulation. Finally, the power of two new tests based on area statistics is compared to the power of tests based on corresponding supremum statistics, i.e. statistics of the Kolmogorov-Smirnov type.

Keywords: Area Statistics, $P-P-$Plot, Functionals of Brownian Bridge, Monte Carlo Simulation, Nonparametric Tests, Recursion Formulae

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1 Introduction

Using areas in graphs as a basis for the development of statistical measures and indicators has a long tradition in statistics. The most prominent example is, of course, the Gini measure, which can be interpreted as two times the area between the Lorenz curve and the diagonal in a unit square. This paper deals with the comparison of two distribution functions F and G and the well known $p - p$ plot is a pictorial tool for that purpose.

If F and G are equal the $p - p$ plot is just the diagonal in the unit square and deviations from the diagonal can easily be interpreted. Therefore areas occuring in the $p - p$ plot can be taken as a basis for measures of the amount and type of dissimilarity of F and G. The sample versions of these areas which we call area-statistics can be used as test statistics for various testing problems for independent observations.

This paper gives a unified treatment of five area statistics. Though three of them have already been considered in literature, two of them are new and deserve our interest.

The paper is structured as follows. Section 2 gives a detailed derivation of the area statistics and discusses their usefulness for various testing problems, such as testing equality of distributions against inequality or testing stochastic dominance in one or either direction against nondominance. For each of these testing problems a supremum-statistic (i.e. a statistic of the Kolmogorov-Smirnov-type) is available and area statistics can be viewed as their natural competitors.

Section 3 presents a recursive scheme for the computation of the finite sample distributions of the test statistics under $F = G$. We include tables with critical values for the two statistics which are introduced in this paper and for which tables are not available in the literature.

Section 4 summarises the asymptotic distributions of the properly normalized statistics. These are functionals of a Brownian Bridge whose analytical treatment is difficult. Therefore careful Monte-Carlo simulations are used to get reliable estimates of the distribution functions, moments and critical values.

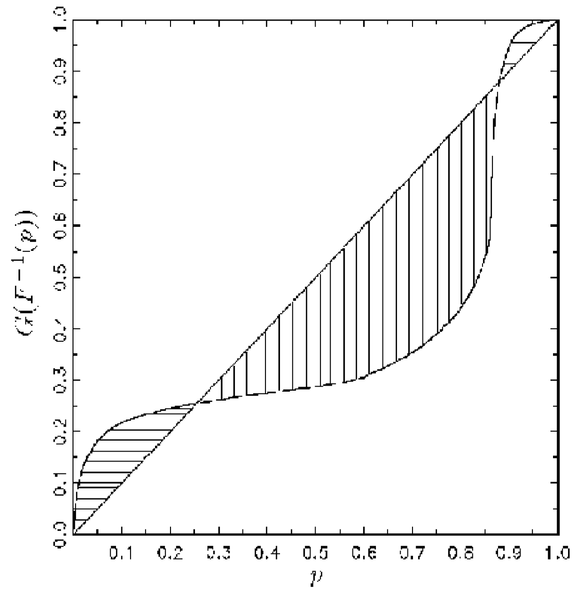
Section 5 presents some power results for tests based on those two area statistics which appear for the first time. They are compared to the corresponding supremum statistics for selected alternatives.

2 Derivation of Test Statistics

Let X and Y denote two random variables with continous distribution functions F and G and quantile functions F^{-1} and G^{-1}, respectively, where $F^{-1}(p) = \inf\{x|F(x) \geq p\}$ for $0 < p < 1$ and G^{-1} is defined analogously.

The probability plot of F and G is defined by $p \mapsto G(F^{-1}(p))$ for $p \in (0,1)$ (see Figure 1 for an example). The $p-p-$plot is an excellent tool for our purpose because it summarizes in a simple but pictorial way in a unit square the information necessary for a comparison of F and G. Obviously $F = G$ if and only if $G(F^{-1}(p)) = p$ for $p \in (0,1)$. Deviations from the diagonal can easily be interpreted, e.g. $G(F^{-1}(p)) > p$ for $p \in (0,1)$ indicates stochastic dominance of F over G i.e. $F(x) < G(x)$ for $x \in I\!R$. If $G(F^{-1}(p))$ crosses the diagonal at

Figure 1: Probability Plot of two Continous Distribution Functions F and G



least once then G and F cross at least once which indicates that there is no dominance of F over G or vice versa.

It can be shown that under appropriate conditions the $p-p-$plot is a maximal invariant of F and G with respect to a group of strictly increasing transformations, see Holmgren (1995) for details and further discussion of this point. Therefore the $p-p-$plot should be a good basis for the derivation of test statistics as long as ordinal information is used.

The basic quantities of interest in this paper are the areas

$$A^+ = \int_0^1 (G(F^{-1}(p)) - p)^+ \, dp = \int_{-\infty}^{\infty} (G(x) - F(x))^+ \, dF(x)$$

and

$$A^- = \int_0^1 (G(F^{-1}(p)) - p)^- \, dp = \int_{-\infty}^{\infty} (G(x) - F(x))^- \, dF(x) \ .$$

As usual z^+ denotes the nonnegative part of a real number z, i.e., $z^+ = \max\{z, 0\}$ and $z^- = \max\{0, -z\}$. Therefore $z = z^+ - z^-$ and $|z| = z^+ + z^-$. In Figure 1 A^+ and A^- denote the horizontally shaded and vertically shaded areas, respectively.

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ denote two independent samples from X and Y. The sample versions of A^+ and A^- are

$$\begin{aligned} A^+_{m,n} &= \int_{-\infty}^{\infty} (\hat{G}_n(x) - \hat{F}_m(x))^+ \, d\hat{F}_m(x) \\ &= \frac{1}{m} \sum_{i=1}^{m} \left(\hat{G}_n(X_{(i)}) - \frac{i}{m} \right)^+ \end{aligned}$$

and

$$
\begin{aligned}
A^-_{m,n} &= \int_{-\infty}^{\infty} (\hat{G}_n(x) - \hat{F}_m(x))^- \, d\hat{F}_m(x) \\
&= \frac{1}{m} \sum_{i=1}^{m} \left(\hat{G}_n(X_{(i)}) - \frac{i}{m} \right)^-
\end{aligned}
$$

where $\hat{F}_m$ and $\hat{G}_n$ are the empirical distribution functions of the two samples and $X_{(1)} \leq \ldots \leq X_{(m)}$ and $Y_{(1)} \leq \ldots \leq Y_{(n)}$ denote the corresponding order statistics. Using ranks $R(X_{(i)})$ of $X_{(i)}$ in the combined sample we arrive at

$$
\begin{aligned}
A^+_{m,n} &= \frac{1}{m} \sum_{i=1}^{m} \left(\frac{R(X_{(i)}) - i}{n} - \frac{i}{m} \right)^+ \\
A^-_{m,n} &= \frac{1}{m} \sum_{i=1}^{m} \left(\frac{R(X_{(i)}) - i}{n} - \frac{i}{m} \right)^-
\end{aligned}
$$

which are nonlinear rank statistics.

$A^+_{m,n}$ and $A^-_{m,n}$ and some simple functions thereof can be used as test statistics for various testing problems as summarized in the second column of table 1.

Table 1: Testing Problems and Suitable Area and Supremum Statistics

Nullhypotheses and Alternative Hypotheses	Area-Statistics	Supremum-Statistics
(1) Equality		
$H_0 : F(x) = G(x) \quad \forall x \in I\!R$	$A^+_{m,n} + A^-_{m,n}$	$D^+_{m,n} + D^-_{m,n}$
H_1 : not H_0	(Schmid and Trede (1995))	(Kuiper (1960))
	$A^+_{m,n} - A^-_{m,n}$	$D^+_{m,n} - D^-_{m,n}$
	(Wilcoxon (1945))	(Weichselberger (1993))
	$\max\{A^+_{m,n}, A^-_{m,n}\}$	$\max\{D^+_{m,n}, D^-_{m,n}\}$
		(Kolmogorov (1933))
(2) One sided stochastic dominance		
$H_0 : F(x) \geq G(x) \quad \forall x \in I\!R$	$A^+_{m,n}$	$D^+_{m,n}$
H_1 : not H_0	(Schmid and Trede (1996))	(Kolmogorov (1933))
(3) Stochastic dominance in either direction		
$H_0 : F(x) \geq G(x) \quad \forall x \in I\!R$ or $G(x) \geq F(x) \quad \forall x \in I\!R$	$\min\{A^+_{m,n}, A^-_{m,n}\}$	$\min\{D^+_{m,n}, D^-_{m,n}\}$ (Schmid and Trede (1996a))
H_1 : not H_0		

For testing problem (1), i.e. for testing equality of F and G against two sided deviations, the test statistic

$$\begin{aligned} A^+_{m,n} + A^-_{m,n} &= \int_{-\infty}^{\infty} |(\hat{G}_n(x) - \hat{F}_m(x))| \, d\hat{F}_m(x) \\ &= \frac{1}{m} \sum_{i=1}^{m} \left| \hat{G}_n(X_{(i)}) - \frac{i}{m} \right| \\ &= \frac{1}{m} \sum_{i=1}^{m} \left| \frac{R(X_{(i)}) - i}{n} - \frac{i}{m} \right| \end{aligned}$$

can be used. It is the sample version of the sum of the two shaded areas and can be viewed as the L_1-version of the Cramer-von Mises test. This test was suggested by Schmid and Trede (1995).

Another suitable test statistic for testing (1) is

$$A^+_{m,n} - A^-_{m,n} = \frac{1}{m} \sum_{i=1}^{m} \left(\frac{R(X_{(i)}) - i}{n} - \frac{i}{m} \right)$$

This is simply an affine transformation of the Wilcoxon test statistic $\sum_{i=1}^{m} R(X_{(i)})$.

Finally a suitable test statistic for (1) is

$$\max\{A^+_{m,n}, A^-_{m,n}\}$$

i.e the sample version of the maximum of the two shaded areas. This statistic appears for the first time and will be considered further in the next sections.

Concerning testing problem (2) i.e. testing one-sided dominance, Schmid and Trede (1996) suggested $A^+_{m,n}$ as a suitable test statistic.

Testing problem (3) i.e testing dominance in either direction against nondominance (i.e. intersection of distribution functions) requires a different test statistic. If H_0 is true either A^+ or A^- is zero, i.e. $\min\{A^+, A^-\} = 0$. Therefore a suitable test statistic seems to be the sample version of $\min\{A^+, A^-\}$ i.e.

$$\min\{A^+_{m,n}, A^-_{m,n}\}\,.$$

This test statistic was not yet considered in literature and will be further investigated in the next sections.

Let us close this section by mentioning that the $p-p-$plot can also serve as the common basis for the derivation of test statistics of the Kolmogorov-Smirnov-type which are based on distances. Indeed let

$$D^{+} = \sup_{x \in \mathbb{R}} (G(x) - F(x)) = \sup_{p \in (0,1)} (G(F^{-1}(p)) - p)$$

and

$$D^{-} = \sup_{x \in \mathbb{R}} (F(x) - G(x)) = \sup_{p \in (0,1)} (p - G(F^{-1}(p)))$$

which are the vertical distances between $G(F^{-1}(p))$ and p and p and $G(F^{-1}(p))$, respectively.

The empirical versions are

$$D^{+}_{m,n} = \sup_{x \in \mathbb{R}} \left(\hat{G}_n(x) - \hat{F}_m(x) \right)$$

and

$$D^{-}_{m,n} = \sup_{x \in \mathbb{R}} \left(\hat{F}_m(x) - \hat{G}_n(x) \right) \ .$$

Statistics $D^{+}_{m,n}$, $D^{-}_{m,n}$ and $\max\{D^{+}_{m,n}, D^{-}_{m,n}\} = \sup_{x \in \mathbb{R}} |\hat{F}_m(x) - \hat{G}_n(x)|$ are the well known K-S-statistics for one and two sided testing problems which are contained in every textbook. $D^{+}_{m,n} + D^{-}_{m,n}$ was introduced by Kuiper (1960) and $D^{+}_{m,n} - D^{-}_{m,n}$ was comprehensively studied by Weichselberger (1993). Finally $\min\{D^{+}_{m,n}, D^{-}_{m,n}\}$ was introduced by Mosler (1995) and studied by Schmid and Trede (1996a).

3 Finite Sample Distribution of Test Statistics under $F = G$

Practical application of the area statistics for testing requires critical values, i.e. quantiles of the test statistic under $F = G$. As all statistics are simple functions of $A^{+}_{m,n}$ and $A^{-}_{m,n}$ it is useful to have a procedure for the computation of the joint probabilities

$$P(A^{+}_{m,n} \leq \frac{c^{(+)}}{m^2 n}, A^{-}_{m,n} \leq \frac{c^{(-)}}{m^2 n})$$

where

$$c^{(+)} = 0, 1, \ldots, \frac{m(m-1)n}{2} \quad \text{and} \quad c^{(-)} = 0, 1, \ldots, \frac{m(m+1)n}{2} \ .$$

Remember that

$$A^{+}_{m,n} = \frac{1}{m} \sum_{i=1}^{m} \left(\frac{R(X_{(i)}) - i}{n} - \frac{i}{m} \right)^{+} \quad \text{and} \quad A^{-}_{m,n} = \frac{1}{m} \sum_{i=1}^{m} \left(\frac{R(X_{(i)}) - i}{n} - \frac{i}{m} \right)^{-} \ .$$

For $k, l \in \mathbb{N} \cup \{0\}$ and $k \leq m$ and $l \leq n$ let $N(k, l, c^{(+)}, c^{(-)})$ be the number of combinations of k $x-$observations and l $y-$observations for which

$$A^{+}_{m,n} \leq \frac{c^{(+)}}{m^2 n} \quad \text{and} \quad A^{-}_{m,n} \leq \frac{c^{(-)}}{m^2 n} \ .$$

Inspecting the structure of $A^+_{m,n}$ and $A^-_{m,n}$ we can see that

$$\begin{aligned} N(k,l,c^{(+)},c^{(-)}) &= N(k,l-1,c^{(+)},c^{(-)}) \\ &+ N(k-1,l,c^{(+)},c^{(-)}) \cdot 1_{\{lm-kn=0\}} \\ &+ N(k-1,l,c^{(+)}-|lm-kn|,c^{(-)}) \cdot 1_{\{lm-kn>0 \wedge c^{(+)}-|lm-kn|\geq 0\}} \\ &+ N(k-1,l,c^{(+)},c^{(-)}-|lm-kn|) \cdot 1_{\{lm-kn<0 \wedge c^{(-)}-|lm-kn|\geq 0\}} \end{aligned}$$

with

$$1_{\{A\}} = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{else} \end{cases}$$

being the usual indicator function.

The initial conditions are

$$\begin{aligned} &N(k,l,c^{(+)},c^{(-)}) = 0 \quad \text{for } c^{(+)} < 0 \vee c^{(-)} < 0 \\ &N(0,l,c^{(+)},c^{(-)}) = 1 \quad \text{for } 0 \leq c^{(+)} \leq \tfrac{m(m-1)n}{2}\,,\ 0 \leq c^{(-)} \leq \tfrac{m(m+1)n}{2} \\ &N(k,0,c^{(+)},c^{(-)}) = 1 \quad \text{for } 0 \leq c^{(+)} \leq \tfrac{m(m-1)n}{2}\,,\ c^{(-)} \geq \tfrac{k(k+1)n}{2} \end{aligned}$$

$N(m,n,c^{(+)},c^{(-)})$ can therefore be computed recursively and we arrive at

$$P(A^+_{m,n} \leq \frac{c^{(+)}}{m^2 n}\,,\ A^-_{m,n} \leq \frac{c^{(-)}}{m^2 n}) = \frac{N(m,n,c^{(+)},c^{(-)})}{\binom{m+n}{m}}\,.$$

To avoid possible numerical difficulties we propose the following transformation:

$$U(k,l,c^{(+)},c^{(-)}) := \frac{N(k,l,c^{(+)},c^{(-)})}{\binom{k+n}{k}}$$

The recursive scheme can now be rewritten as

$$\begin{aligned} U(k,l,c^{(+)},c^{(-)}) &= U(k,l-1,c^{(+)},c^{(-)}) \\ &+ \frac{k}{k+n} \cdot U(k-1,l,c^{(+)},c^{(-)}) \cdot 1_{\{lm-kn=0\}} \\ &+ \frac{k}{k+n} \cdot U(k-1,l,c^{(+)}-|lm-kn|,c^{(-)}) \cdot 1_{\{lm-kn>0 \wedge c^{(+)}-|lm-kn|\geq 0\}} \\ &+ \frac{k}{k+n} \cdot U(k-1,l,c^{(+)},c^{(-)}-|lm-kn|) \cdot 1_{\{lm-kn<0 \wedge c^{(-)}-|lm-kn|\geq 0\}} \end{aligned}$$

The initial conditions are

$$\begin{aligned} &U(k,l,c^{(+)},c^{(-)}) = 0 \quad && \text{for } c^{(+)} < 0 \vee c^{(-)} < 0 \\ &U(0,l,c^{(+)},c^{(-)}) = 1 \quad && \text{for } 0 \leq c^{(+)} \leq \tfrac{m(m-1)n}{2}\,,\ 0 \leq c^{(-)} \leq \tfrac{m(m+1)n}{2} \\ &U(k,0,c^{(+)},c^{(-)}) = \tfrac{1}{\binom{k+n}{k}} \quad && \text{for } 0 \leq c^{(+)} \leq \tfrac{m(m-1)n}{2}\,,\ c^{(-)} \geq \tfrac{k(k+1)n}{2} \end{aligned}$$

The marginal distributions of $A^+_{m,n}$ and $A^-_{m,n}$ can be derived by setting $c^{(-)} = \frac{m(m+1)n}{2}$ and $c^{(+)} = \frac{m(m-1)n}{2}$, repectively.

The distributions of $A^+_{m,n} - A^-_{m,n}$ and $A^-_{m,n} - A^-_{m,n}$ can be derived via convolution.

The distribution of $\max\{A^+_{m,n}\,,\,A^-_{m,n}\}$ can be obtained by setting $c^{(+)} = c^{(-)} = c$ $(0 \le c \le \frac{m(m-1)n}{2})$.

Finally the distribution of $\min\{A^+_{m,n}\,,\,A^-_{m,n}\}$ is obtained by

$$P(\min\{A^+_{m,n}\,,\,A^-_{m,n}\} \le c) = P(A^+_{m,n} \le c) + P(A^-_{m,n} \le c) - P(\max\{A^+_{m,n}\,,\,A^-_{m,n}\} \le c)$$

Tables with critical values (at the 5%− and 10%− level) for the $\max\{A^+_{m,n}, A^-_{m,n}\}$ and $\min\{A^+_{m,n}, A^-_{m,n}\}$ statistic are added in the appendix because these statistics appear in literature for the first time and critical values are not yet published.

4 Asymptotic Distribution of Area Statistics

The standard tool for the derivation of asymptotic distributions of test statistics under $F = G$ in nonparametric statistics is the invariance principle for the two sample empirical process

$$\sqrt{\frac{mn}{m+n}}(\hat{F}_m(p) - \hat{G}_n(p)) \qquad p \in [0,1]$$

which states weak convergence of the process to a Brownian Bridge $(B(p))_{0 \le p \le 1}$ on [0,1] under $F = G$ where F denotes the distribution function of a uniform distribution on [0,1]. Continuous functionals of the empirical process converge weakly to the corresponding functional of the Brownian Bridge. The asymptotic distributions of the normalized test statistics considered so far can therefore be easily expressed as functionals of $B(\cdot)$ (see table 2).

Table 2: Asymptotic Distributions of Normalized Area Statistics

Statistic	Asymptotic Distribution
$(\frac{mn}{m+n})^{\frac{1}{2}}(A^+_{m,n}, A^-_{m,n})$	$\left(\int_0^1 B^+(p)\,dp, \int_0^1 B^-(p)\,dp\right)$
$(\frac{mn}{m+n})^{\frac{1}{2}}(A^+_{m,n} + A^-_{m,n})$	$\int_0^1 \lvert B(p)\rvert\,dp$
$(\frac{mn}{m+n})^{\frac{1}{2}}(A^+_{m,n} - A^-_{m,n})$	$\int_0^1 B(p)\,dp \sim N(0, \sigma^2 = \frac{1}{12})$
$(\frac{mn}{m+n})^{\frac{1}{2}} \max\{A^+_{m,n}, A^-_{m,n}\}$	$\max\left\{\int_0^1 B^+(p)\,dp, \int_0^1 B^-(p)\,dp\right\}$
$(\frac{mn}{m+n})^{\frac{1}{2}} \min\{A^+_{m,n}, A^-_{m,n}\}$	$\min\left\{\int_0^1 B^+(p)\,dp, \int_0^1 B^-(p)\,dp\right\}$

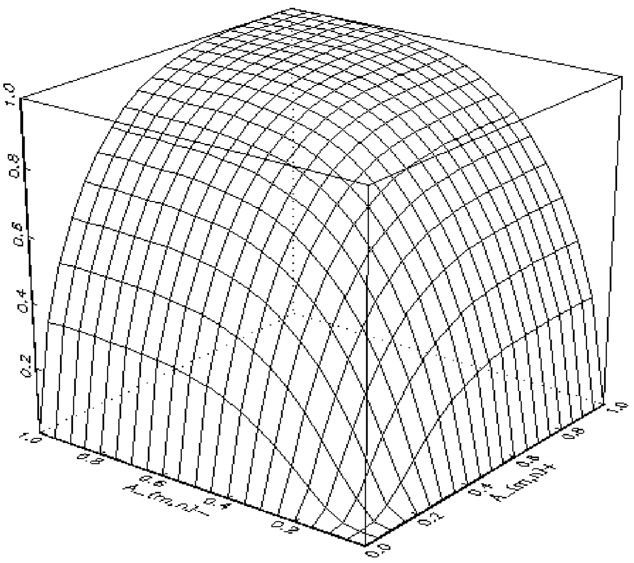
The limiting distribution of $(\frac{mn}{m+n})^{\frac{1}{2}}(A^+_{m,n} + A^-_{m,n})$ which is $\int_0^1 |B(p)|\, dp$ has been investigated analytically by Shepp(1982) and Rice (1982). The asymptotic distribution of $(\frac{mn}{m+n})^{\frac{1}{2}} A^+_{m,n}$ i.e. $\int_0^1 B^+(p)\, dp$ was investigated by Monte-Carlo Simulation by Schmid and Trede (1996). Perman and Wellner (1996) succeeded in computing the double Laplace transform but did not succeed in inverting analytically this transform.

Table 3: Moments and Quantiles of the Asymptotic Distributions of Test Statistics

Asymptotic Distribution	Monte-Carlo Estimates				
	mean	st. dev.	0,9 qu.	0,95 qu.	0,99 qu.
$\int_0^1 B^+(p)\, dp$	0,15659	0,15925	0,38513	0,48288	0,67319
$\int_0^1 B^-(p)\, dp$	0,15663	0,15963	0,38612	0,48671	0,67410
$\int_0^1 \lvert B(p)\rvert\, dp$	0,31322	0,13584	0,49914	0,58214	0,74847
$\int_0^1 B(p)\, dp$	-0,00004	0,28850	0,36915	0,47350	0,66807
$\max\left\{\int_0^1 B^+(p)\, dp, \int_0^1 B^-(p)\, dp\right\}$	0,27176	0,15154	0,48479	0,57294	0,74432
$\min\left\{\int_0^1 B^+(p)\, dp, \int_0^1 B^-(p)\, dp\right\}$	0,04146	0,03684	0,09313	0,11235	0,15257

The joint distribution of $\int_0^1 B^+(p)\, dp$ and $\int_0^1 B^-(p)\, dp$ seems to be totally unknown, the same is true for the functionals $\max\left\{\int_0^1 B^+(p)\, dp, \int_0^1 B^-(p)\, dp\right\}$ and $\min\left\{\int_0^1 B^+(p)\, dp, \int_0^1 B^-(p)\, dp\right\}$. These distributions have been investigated by Monte-Carlo-Simulation in the following way. A path of a Brownian Bridge was approximated at 30000 equidistant points in [0,1] and $\int_0^1 B^+(p)\, dp$ and $\int_0^1 B^-(p)\, dp$ were calculated approximately for that path by a simple quadrature rule. This procedure was repeated independently 100000 times thus obtaining 100000 independent realisations of the joint distribution of $\left(\int_0^1 B^+(p)\, dp, \int_0^1 B^-(p)\, dp\right)$. These realisations were used to approximate moments and quantiles of the asymptotic distributions of

Figure 2: Joint Distribution Function of $\int_0^1 B^+(p)\,dp$ and $\int_0^1 B^-(p)\,dp$



test statistics (see table 3). The Monte-Carlo estimate of the joint distribution function of $\int_0^1 B^+(p)\,dp$ and $\int_0^1 B^-(p)\,dp$ is displayed in figure 2.

The results in the third and fourth row of table 3 can be used to check the correctness of the Monte Carlo simulation. $\int_0^1 B(p)\,dp \sim N(\mu = 0, \sigma = 0,28867)$ and the expectation and standard error of $\int_0^1 |B(p)|\,dp$ is 0,3133 and 0,1360 (see Johnson and Killeen (1983) which conforms well with our simulation.

The distribution of $\int_0^1 B^+(p)$ and $\int_0^1 B^+(p)$ are identical. Perman and Wellner (1996) succeeded in computing numerically their expectation and standard deviation which are 0,15666 and 0,15955 respectively. This, again, is in good agreement with our simulations. Distribution functions of the min- and max- statistics are displayed in figures 3 and 4.

Figure 3: Distribution Function of $\max\{\int_0^1 B^+(p)\,dp, \int_0^1 B^-(p)\,dp\}$

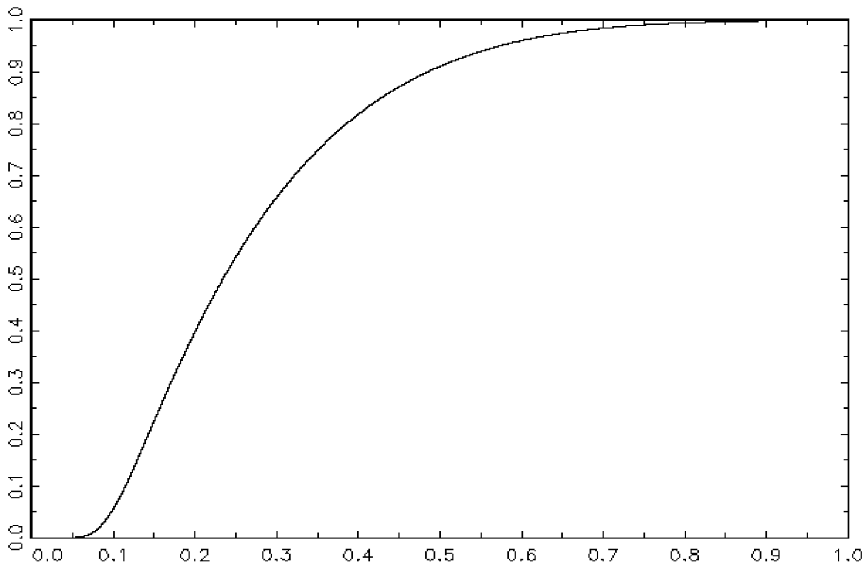
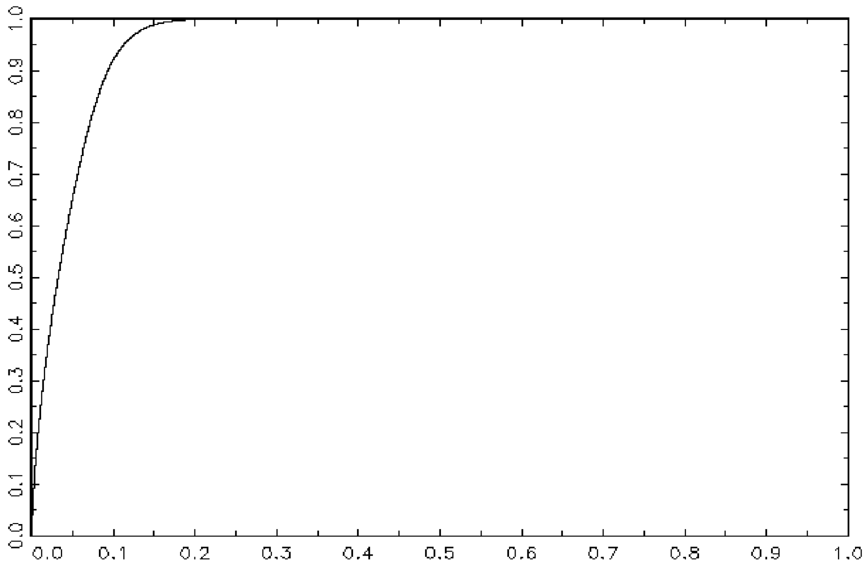


Figure 4: Distribution Function of $\min\{\int_0^1 B^+(p)\,dp, \int_0^1 B^-(p)\,dp\}$



5 Properties of the new tests: Comparing power of area and sup statistics

Our attempt to give a unified basis for area statistics resulted in five different statistics. Three of them have already been suggested in the literature. Two of them are new and will be considered further in this section. These are $\max\{A^+_{m,n}, A^-_{m,n}\}$ for testing problem (1) in table 1 and $\min\{A^+_{m,n}, A^-_{m,n}\}$ for testing problem (3). As supremum statistics (i.e. statistics of the K-S-type) are natural competitors to area statistics it is tempting to compare their power in testing problems. Though these tests are nonparametric we compare their power within one parameter families of distributions which allow for power graphs as a function of the parameter. The examples to be presented below show that there are instances when tests based on area statistics perform better than their competitors. A detailed investigation of power of the new test is, however, outside the scope of this paper.

5.1 The statistic $\max\{A^+_{m,n}, A^-_{m,n}\}$

In this section we consider testing problem (1)

$$H_0 : F = G \text{ against } H_1 : F \neq G\,.$$

Of course H_0 is rejected when $\max\{A^+_{m,n}, A^-_{m,n}\} > c$, where c denotes the appropriate critical value.

Area statistic $\max\{A^+_{m,n}, A^-_{m,n}\}$ is compared to the corresponding statistic $\max\{D^+_{m,n}, D^-_{m,n}\}$ for (see Figure 5)

$$F(x) = 1 - \left(1 + x^{\beta}\right)^{-2} \quad x > 0 \quad \text{and} \quad \beta = 1,5$$
$$G(x) = 1 - \left(1 + x^{\beta}\right)^{-2} \quad x > 0 \quad \text{for} \quad \beta > 1,5$$

and (see Figure 6)

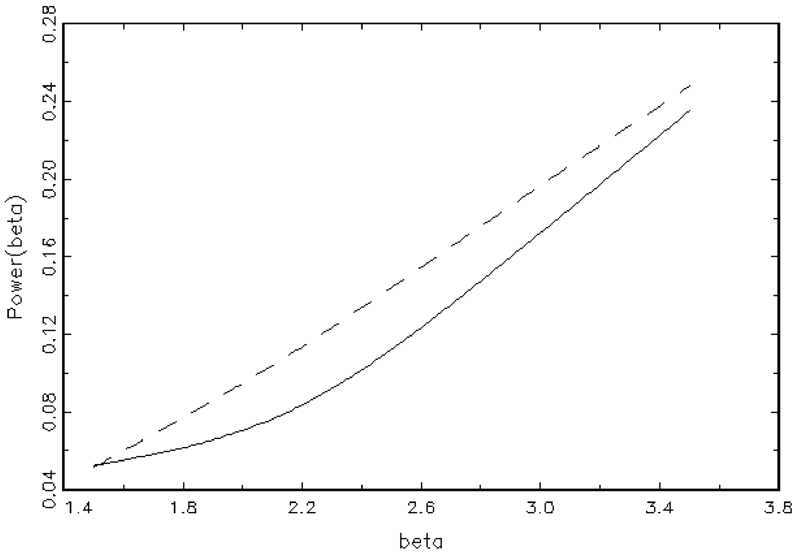
$$F(x) = \Phi(x - \mu) \quad \text{and} \quad \mu = 0$$
$$G(x) = \Phi(x - \mu) \quad \text{for} \quad \mu \neq 0\,.$$

Figure 5 and 6 display power curves (as a function of β and μ respectively) for

$$\max\{A^+_{m,n}, A^-_{m,n}\} \qquad \text{(dashed line)}\ ,$$
$$\max\{D^+_{m,n}, D^-_{m,n}\} \qquad \text{(solid line)}\ .$$

The area statistic has clearly higher power for the Singh-Maddala-alternatives (see Figure 5). There is a certain asymmetry in the power curve for the normal-alternatives (see Figure 6) which is probably due to the asymmetry in the distributions of $A^+_{m,n}$ and $A^-_{m,n}$ in $\min\{A^+_{m,n}, A^-_{m,n}\}$ in finite samples.

Figure 5: Power Curves for the Singh-Maddala Distribution ($m = n = 10, \alpha = 0,05$)



It is well known that the classical Kolmogorov-test based on $\max\{D^+_{m,n}, D^-_{m,n}\}$ is consistent on the whole set of alternatives H_1 (see e.g. Büning/Trenkler (1978)). The same is true for the test based on the area statistic $\max\{A^+_{m,n}, A^-_{m,n}\}$. Indeed for $F \neq G$ we have

$$A := \max\{A^+, A^-\} > 0$$

Further we have with probability 1

$$\lim_{n,m\to\infty} \max\{A^+_{m,n}, A^-_{m,n}\} = \max\{A^+, A^-\} = A > 0$$

and we can conclude

$$\lim_{n,m\to\infty} P\left(\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} \max\{A^+_{m,n}, A^-_{m,n}\} > c\right) = 1 \ .$$

5.2 The statistic $\min\{A^+_{m,n}, A^-_{m,n}\}$

In this section we consider testing problem (3)

$$H_0 : F(x) \geq G(x) \ \forall x \in I\!R \text{ or } G(x) \geq F(x) \ \forall x \in I\!R$$

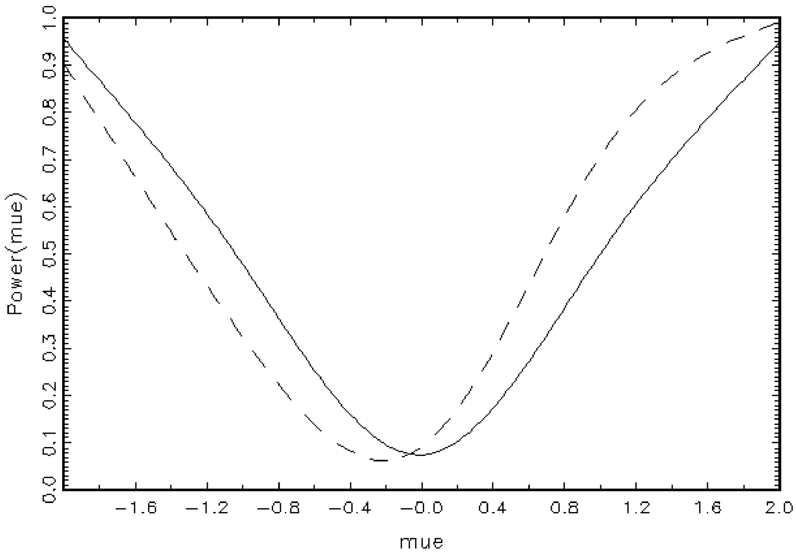
against

$$H_1 : \text{ not } H_0 \ .$$

H_0 is tantamount to equality or dominance in either direction of F and G whereas H_1 entails absence of dominance, i.e at least one intersection of F and G.

$\min\{A^+_{m,n}, A^-_{m,n}\}$ is a suitable test statistic and H_0 should be rejected if $\min\{A^+_{m,n}, A^-_{m,n}\} > c$ where c is an appropriate quantile of the distribution of $\min\{A^+_{m,n}, A^-_{m,n}\}$ under $F = G$.

Figure 6: Power Curves for Normal Distribution ($m = 8, n = 9, \alpha = 0,1$)



It should be pointed out that the test keeps the level α on the whole null hypotheses because one can see that

$$\begin{aligned}\alpha \geq \alpha^* &= P(\min\{A^+_{m,n}, A^-_{m,n}\} > c||F = G) \\ &= \sup_{F,G \text{ in } H_0} P(\min\{A^+_{m,n}, A^-_{m,n}\} > c||F, G) \ .\end{aligned}$$

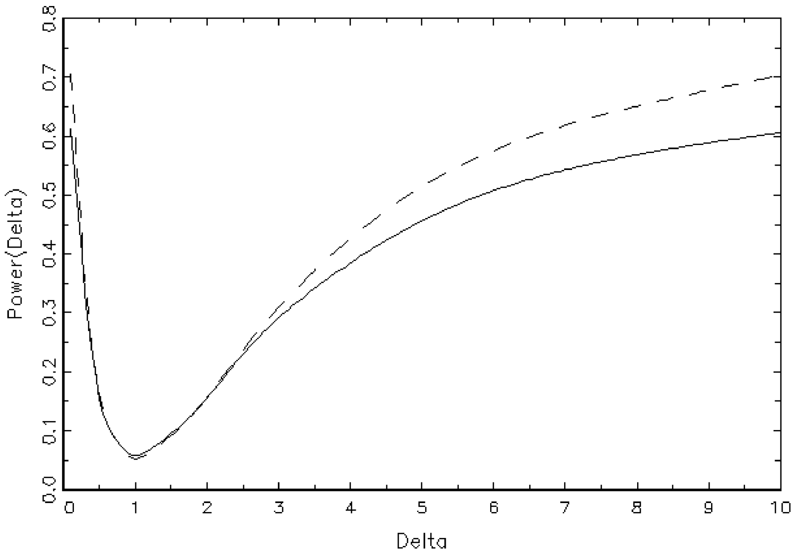
Area statistic $\min\{A^+_{m,n}, A^-_{m,n}\}$ is compared to the corresponding statistic $\min\{D^+_{m,n}, D^-_{m,n}\}$ (see Figure 7) for

$$F(x) = x \qquad 0 \leq x \leq 1$$

$$G_\delta(x) = \begin{cases} (2x)^\delta & 0 \leq x \leq 0,5 \\ 1 - \dfrac{(2(1-x))^\delta}{2} & 0,5 < x \leq 1 \end{cases} \qquad \text{and } \delta \neq 1$$

and (see Figure 8)

Figure 7: Power curves for the Delta-distribution ($m = n = 10, \alpha = 0,05$)

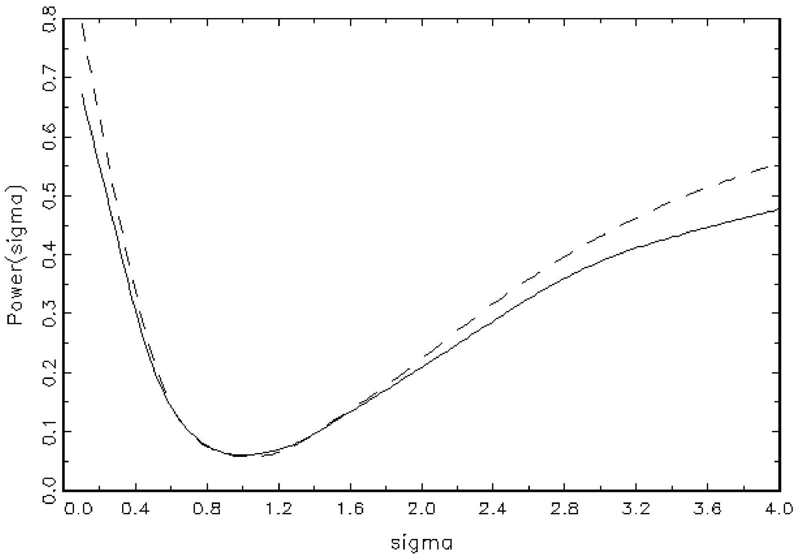


$$F(x) = \Phi(\tfrac{x}{\sigma}) \quad \text{and} \quad \sigma = 0$$
$$G(x) = \Phi(\tfrac{x}{\sigma}) \quad \text{for} \quad \sigma \neq 1 \ .$$

Figure 7 and 8 display power curves (as a function of δ and σ respectively) for

$$\min\{A^+_{m,n}, A^-_{m,n}\} \qquad \text{(dashed line)} \ ,$$
$$\min\{D^+_{m,n}, D^-_{m,n}\} \qquad \text{(solid line)} \ .$$

Figure 8: Power curves for Normal distribution



For both sets of alternatives we can conclude that the area-statistic outperforms the sup-statistic. It has been shown in Schmid and Trede (1996a) that the test based on $\min\{D^+_{m,n}, D^-_{m,n}\}$ is consistent on the whole set of alternatives. The same is true for the test based on the area-statistic $\max\{A^+_{m,n}, A^-_{m,n}\}$. Indeed for $F \neq G$ we have

$$B := \min\{A^+, A^-\} > 0$$

Further, with probability 1

$$\lim_{n,m\to\infty} \min\{A^+_{m,n}, A^-_{m,n}\} = B > 0$$

and therefore

$$\lim_{n,m\to\infty} P\left(\left(\frac{mn}{m+n}\right)^{\frac{1}{2}} \min\{A^+_{m,n}, A^-_{m,n}\} > c\right) = 1\ .$$

Appendix

Selected critical values of $\max\{A^+_{m,n}, A^-_{m,n}\}$ at the nominal level α=**0,05**.

The first entry gives the smallest integer c satisfying

$$P(m^2 n \max\{A^+_{m,n}, A^-_{m,n}\} \geq c) \leq \alpha$$

The second entry gives the true probabiltity of rejection

$$P(m^2 n \max\{A^+_{m,n}, A^-_{m,n}\} \geq c)$$

n \ m	3	4	5	6	7	8	9
2	-	-	26 / 0.0476	37 / 0.0357	50 / 0.0278	57 / 0.0444	73 / 0.0364
3	16 / 0.0500	27 / 0.0286	36 / 0.0357	46 / 0.0476	64 / 0.0333	77 / 0.0485	100 / 0.0364
4	22 / 0.0286	33 / 0.0286	46 / 0.0317	61 / 0.0333	78 / 0.0394	97 / 0.0404	118 / 0.0434
5	25 / 0.0357	39 / 0.0317	51 / 0.0476	70 / 0.0455	92 / 0.0404	117 / 0.0365	136 / 0.0490
6	28 / 0.0476	45 / 0.0333	61 / 0.0411	79 / 0.0487	106 / 0.0402	129 / 0.0456	160 / 0.0500
7	34 / 0.0333	51 / 0.0364	71 / 0.0366	94 / 0.0379	120 / 0.0385	149 / 0.0401	181 / 0.0408
8	37 / 0.0424	57 / 0.0364	76 / 0.0474	103 / 0.0420	133 / 0.0499	161 / 0.0453	199 / 0.0428
9	40 / 0.0500	63 / 0.0378	86 / 0.0425	112 / 0.0456	141 / 0.0483	181 / 0.0404	217 / 0.0426
10	46 / 0.0385	69 / 0.0380	96 / 0.0380	121 / 0.0486	155 / 0.0466	193 / 0.0452	235 / 0.0440
11	49 / 0.0440	75 / 0.0388	101 / 0.0453	136 / 0.0405	169 / 0.0452	205 / 0.0496	253 / 0.0447
12	55 / 0.0352	81 / 0.0390	111 / 0.0412	145 / 0.0427	183 / 0.0440	225 / 0.0446	271 / 0.0452
13	58 / 0.0411	87 / 0.0395	116 / 0.0479	154 / 0.0449	197 / 0.0429	237 / 0.0483	289 / 0.0457
14	61 / 0.0456	93 / 0.0395	126 / 0.0440	163 / 0.0470	204 / 0.0493	257 / 0.0441	307 / 0.0459
15	67 / 0.0380	99 / 0.0400	131 / 0.0496	172 / 0.0489	218 / 0.0478	269 / 0.0473	325 / 0.0462

n \ m	10	11	12	13	14	15
2	91 / 0.0303	111 / 0.0256	133 / 0.0220	144 / 0.0476	169 / 0.0417	196 / 0.0368
3	116 / 0.0455	144 / 0.0357	163 / 0.0484	196 / 0.0393	218 / 0.0471	256 / 0.0392
4	141 / 0.0470	166 / 0.0491	197 / 0.0489	235 / 0.0391	267 / 0.0415	301 / 0.0436
5	166 / 0.0450	199 / 0.0426	235 / 0.0401	261 / 0.0498	302 / 0.0473	346 / 0.0444
6	191 / 0.0430	221 / 0.0482	265 / 0.0415	300 / 0.0463	341 / 0.0497	391 / 0.0447
7	216 / 0.0411	254 / 0.0417	295 / 0.0422	339 / 0.0427	379 / 0.0500	430 / 0.0499
8	231 / 0.0492	276 / 0.0454	321 / 0.0495	365 / 0.0478	421 / 0.0448	466 / 0.0498
9	256 / 0.0465	298 / 0.0484	346 / 0.0491	404 / 0.0441	456 / 0.0460	511 / 0.0476
10	281 / 0.0424	321 / 0.0497	373 / 0.0492	430 / 0.0479	491 / 0.0467	556 / 0.0454
11	296 / 0.0485	353 / 0.0436	403 / 0.0483	460 / 0.0499	526 / 0.0471	586 / 0.0498
12	321 / 0.0457	375 / 0.0461	433 / 0.0457	495 / 0.0474	561 / 0.0474	631 / 0.0475
13	339 / 0.0498	397 / 0.0477	463 / 0.0455	521 / 0.0484	596 / 0.0476	665 / 0.0499
14	361 / 0.0476	419 / 0.0491	483 / 0.0498	560 / 0.0459	631 / 0.0462	706 / 0.0486
15	386 / 0.0452	445 / 0.0497	511 / 0.0491	586 / 0.0480	666 / 0.0470	736 / 0.0500

Selected critical values of $\max\{A^+_{m,n}, A^-_{m,n}\}$ at the nominal level α=**0,10**.

n \ m	3	4	5	6	7	8	9
2	8 / 0.1000	11 / 0.0667	12 / 0.0952	15 / 0.0714	18 / 0.0556	19 / 0.0889	22 / 0.0727
3	13 / 0.1000	19 / 0.0571	22 / 0.0714	25 / 0.0833	28 / 0.0917	31 / 0.0970	37 / 0.0727
4	23 / 0.0571	25 / 0.1000	31 / 0.0952	37 / 0.0857	43 / 0.0818	49 / 0.0768	55 / 0.0741
5	31 / 0.0714	41 / 0.0556	46 / 0.0794	51 / 0.0931	61 / 0.0770	66 / 0.0894	71 / 0.0999
6	40 / 0.0952	49 / 0.1000	59 / 0.0974	67 / 0.0974	76 / 0.0985	85 / 0.0982	94 / 0.0979
7	57 / 0.0667	64 / 0.0970	78 / 0.0859	92 / 0.0787	99 / 0.0924	113 / 0.0847	120 / 0.0975
8	69 / 0.0788	81 / 0.0970	101 / 0.0761	113 / 0.0846	125 / 0.0912	137 / 0.0915	149 / 0.0969
9	82 / 0.1000	100 / 0.0965	118 / 0.0919	136 / 0.0891	154 / 0.0865	165 / 0.0996	181 / 0.0945
10	106 / 0.0769	121 / 0.0949	146 / 0.0816	161 / 0.0942	179 / 0.0998	201 / 0.0912	216 / 0.0980
11	122 / 0.0879	144 / 0.0960	166 / 0.0984	188 / 0.0986	210 / 0.0986	232 / 0.0980	254 / 0.0972
12	142 / 0.0923	169 / 0.0940	199 / 0.0894	223 / 0.0965	247 / 0.0948	273 / 0.0993	295 / 0.0967
13	170 / 0.0821	196 / 0.0945	226 / 0.0999	259 / 0.0999	287 / 0.0916	313 / 0.0944	339 / 0.0965
14	190 / 0.0926	225 / 0.0941	260 / 0.0944	295 / 0.0920	323 / 0.0989	353 / 0.0993	386 / 0.0961
15	214 / 0.0956	256 / 0.0942	291 / 0.0997	331 / 0.0956	366 / 0.0999	403 / 0.1000	436 / 0.0956

n \ m	10	11	12	13	14	15
2	23 / 0.0909	26 / 0.0769	27 / 0.0989	30 / 0.0857	31 / 0.1000	34 / 0.0882
3	40 / 0.0804	43 / 0.0852	46 / 0.0901	49 / 0.0946	52 / 0.0985	58 / 0.0821
4	57 / 0.0949	63 / 0.0901	69 / 0.0857	75 / 0.0819	77 / 0.0971	83 / 0.0931
5	81 / 0.0846	86 / 0.0934	96 / 0.0824	101 / 0.0898	106 / 0.0967	116 / 0.0861
6	103 / 0.0977	112 / 0.0975	121 / 0.0953	130 / 0.0953	139 / 0.0949	148 / 0.0945
7	134 / 0.0894	141 / 0.0998	155 / 0.0926	163 / 0.1000	176 / 0.0932	190 / 0.0881
8	161 / 0.0999	178 / 0.0999	193 / 0.0887	205 / 0.0914	217 / 0.0936	229 / 0.0957
9	199 / 0.0926	217 / 0.0890	226 / 0.0996	244 / 0.0960	262 / 0.0924	274 / 0.0995
10	231 / 0.0991	256 / 0.0917	271 / 0.0960	287 / 0.0997	311 / 0.0916	326 / 0.0947
11	276 / 0.0964	298 / 0.0917	320 / 0.0929	342 / 0.0918	358 / 0.0999	376 / 0.0998
12	321 / 0.0999	343 / 0.0971	361 / 0.0992	391 / 0.0955	411 / 0.0990	439 / 0.0946
13	365 / 0.0977	391 / 0.0987	417 / 0.0994	443 / 0.0963	469 / 0.0990	495 / 0.0991
14	419 / 0.0997	444 / 0.0999	477 / 0.0961	501 / 0.0999	533 / 0.0953	567 / 0.0999
15	466 / 0.0983	500 / 0.0993	535 / 0.0999	571 / 0.0954	601 / 0.0972	631 / 0.0957

Selected critical values of $\min\{A^+_{m,n}, A^-_{m,n}\}$ at the nominal level α=**0,05**.

n \ m	3	4	5	6	7	8	9
2	-	-	2 / 0.0476	-	3 / 0.0278	3 / 0.0222	4 / 0.0182
3	-	4 / 0.0286	5 / 0.0179	4 / 0.0238	5 / 0.0500	6 / 0.0364	7 / 0.0091
4	4 / 0.0286	-	6 / 0.0238	7 / 0.0143	8 / 0.0273	9 / 0.0121	10 / 0.0252
5	8 / 0.0179	8 / 0.0476	6 / 0.0397	11 / 0.0390	12 / 0.0366	13 / 0.0420	14 / 0.0420
6	-	-	13 / 0.0346	13 / 0.0325	16 / 0.0361	19 / 0.0213	19 / 0.0412
7	12 / 0.0417	13 / 0.0485	16 / 0.0480	19 / 0.0402	22 / 0.0117	23 / 0.0476	26 / 0.0430
8	15 / 0.0485	17 / 0.0242	19 / 0.0490	23 / 0.0376	27 / 0.0385	25 / 0.0379	31 / 0.0432
9	19 / 0.0273	23 / 0.0490	25 / 0.0465	28 / 0.0460	33 / 0.0485	35 / 0.0491	37 / 0.0305
10	22 / 0.0455	25 / 0.0470	31 / 0.0240	35 / 0.0480	40 / 0.0472	43 / 0.0395	46 / 0.0474
11	27 / 0.0467	32 / 0.0491	38 / 0.0492	42 / 0.0444	47 / 0.0448	51 / 0.0465	55 / 0.0464
12	31 / 0.0374	37 / 0.0462	43 / 0.0486	49 / 0.0378	53 / 0.0491	61 / 0.0369	64 / 0.0455
13	36 / 0.0482	43 / 0.0466	51 / 0.0434	57 / 0.0471	63 / 0.0480	68 / 0.0488	74 / 0.0480
14	43 / 0.0456	49 / 0.0461	57 / 0.0489	63 / 0.0467	71 / 0.0398	77 / 0.0500	84 / 0.0498
15	49 / 0.0257	58 / 0.0454	66 / 0.0372	73 / 0.0478	80 / 0.0495	89 / 0.0499	94 / 0.0499

n \ m	10	11	12	13	14	15
2	3 / 0.0455	4 / 0.0385	5 / 0.0110	5 / 0.0286	5 / 0.0250	5 / 0.0441
3	7 / 0.0420	8 / 0.0330	8 / 0.0418	9 / 0.0375	9 / 0.0471	10 / 0.0233
4	11 / 0.0180	12 / 0.0264	13 / 0.0165	14 / 0.0252	15 / 0.0196	15 / 0.0490
5	16 / 0.0236	17 / 0.0449	18 / 0.0451	19 / 0.0472	20 / 0.0482	21 / 0.0355
6	19 / 0.0485	22 / 0.0433	25 / 0.0226	25 / 0.0491	27 / 0.0468	28 / 0.0439
7	27 / 0.0451	30 / 0.0443	31 / 0.0485	33 / 0.0494	36 / 0.0307	38 / 0.0474
8	35 / 0.0467	36 / 0.0473	41 / 0.0296	42 / 0.0482	45 / 0.0385	47 / 0.0441
9	42 / 0.0498	44 / 0.0491	46 / 0.0476	51 / 0.0484	53 / 0.0498	55 / 0.0494
10	51 / 0.0295	54 / 0.0496	57 / 0.0492	61 / 0.0462	65 / 0.0456	67 / 0.0500
11	60 / 0.0462	67 / 0.0232	68 / 0.0473	72 / 0.0478	76 / 0.0488	81 / 0.0484
12	69 / 0.0433	74 / 0.0496	73 / 0.0433	83 / 0.0478	89 / 0.0492	94 / 0.0432
13	79 / 0.0496	85 / 0.0500	91 / 0.0482	92 / 0.0405	102 / 0.0477	108 / 0.0479
14	89 / 0.0499	97 / 0.0493	103 / 0.0485	109 / 0.0497	113 / 0.0395	122 / 0.0487
15	101 / 0.0498	111 / 0.0477	118 / 0.0476	124 / 0.0485	131 / 0.0487	136 / 0.0389

Selected critical values of $\min\{A^+_{m,n}, A^-_{m,n}\}$ at the nominal level α=**0,10**.

n \ m	3	4	5	6	7	8	9
2	1 / 0.1000	1 / 0.0667	2 / 0.0476	2 / 0.0714	2 / 0.0833	3 / 0.0222	3 / 0.0545
3	1 / 0.1000	3 / 0.0857	4 / 0.0536	4 / 0.0238	4 / 0.1000	5 / 0.0727	4 / 0.0864
4	4 / 0.0286	5 / 0.0000	6 / 0.0238	7 / 0.0143	7 / 0.0848	9 / 0.0121	8 / 0.0867
5	6 / 0.0536	7 / 0.0714	6 / 0.0397	9 / 0.0671	10 / 0.0707	10 / 0.0995	12 / 0.0784
6	7 / 0.0833	9 / 0.0667	10 / 0.0758	13 / 0.0325	13 / 0.0897	13 / 0.0929	16 / 0.0603
7	10 / 0.0833	12 / 0.1000	14 / 0.0859	15 / 0.0967	15 / 0.0728	19 / 0.0894	21 / 0.0827
8	12 / 0.0909	13 / 0.0949	17 / 0.0894	19 / 0.0972	22 / 0.0876	25 / 0.0379	26 / 0.0977
9	13 / 0.0955	19 / 0.0797	22 / 0.0804	22 / 0.0931	26 / 0.0934	29 / 0.0967	28 / 0.0760
10	19 / 0.0874	23 / 0.0839	26 / 0.0789	29 / 0.0943	31 / 0.0988	35 / 0.0945	37 / 0.0989
11	22 / 0.0989	27 / 0.0952	31 / 0.0902	34 / 0.0986	38 / 0.0968	42 / 0.0956	46 / 0.0905
12	28 / 0.0769	29 / 0.0819	35 / 0.0957	37 / 0.0925	44 / 0.0992	49 / 0.0836	55 / 0.0759
13	31 / 0.0982	37 / 0.0895	42 / 0.0928	47 / 0.0980	51 / 0.0994	56 / 0.0962	61 / 0.0970
14	36 / 0.0779	41 / 0.0997	48 / 0.0963	53 / 0.0951	57 / 0.0883	65 / 0.0905	69 / 0.0989
15	40 / 0.0846	47 / 0.0962	51 / 0.0926	61 / 0.0902	67 / 0.0983	73 / 0.0984	79 / 0.0929

n \ m	10	11	12	13	14	15
2	3 / 0.0455	3 / 0.0769	3 / 0.0659	3 / 0.0952	3 / 0.0833	4 / 0.0735
3	6 / 0.0734	6 / 0.0879	7 / 0.0418	7 / 0.0839	7 / 0.1000	7 / 0.0797
4	9 / 0.0809	10 / 0.0689	9 / 0.0747	11 / 0.0895	11 / 0.0797	12 / 0.0885
5	11 / 0.0666	13 / 0.0991	15 / 0.0772	15 / 0.0962	16 / 0.0987	16 / 0.0754
6	17 / 0.0979	18 / 0.0976	19 / 0.0699	22 / 0.0740	23 / 0.0854	22 / 0.0913
7	22 / 0.0980	24 / 0.0937	25 / 0.0978	27 / 0.0967	29 / 0.0629	31 / 0.0944
8	29 / 0.0771	30 / 0.0996	33 / 0.0732	34 / 0.0997	37 / 0.0844	38 / 0.0990
9	34 / 0.0913	37 / 0.0920	37 / 0.0973	42 / 0.0957	44 / 0.0952	46 / 0.0896
10	41 / 0.0588	44 / 0.0983	47 / 0.0960	50 / 0.0980	53 / 0.0934	56 / 0.0879
11	49 / 0.0949	45 / 0.0945	55 / 0.0983	59 / 0.0955	63 / 0.0939	66 / 0.0962
12	57 / 0.0909	61 / 0.0965	61 / 0.0829	69 / 0.0980	73 / 0.0928	76 / 0.0986
13	65 / 0.0975	70 / 0.0981	74 / 0.0996	79 / 0.0691	84 / 0.0975	88 / 0.0980
14	75 / 0.0923	80 / 0.0974	85 / 0.0958	90 / 0.0990	85 / 0.0990	100 / 0.0970
15	86 / 0.0882	91 / 0.0956	94 / 0.0995	102 / 0.0994	107 / 0.0990	106 / 0.0894

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