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SEMINAR OF ECONOMIC AND SOCIAL STATISTICS
UNIVERSITY OF COLOGNE

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January 11, 2011

DISKUSSIONSBEITRÄGE ZUR STATISTIK UND ÖKONOMETRIE

SEMINAR FÜR WIRTSCHAFTS- UND SOZIALSTATISTIK
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On the Diversification of Portfolios of Risky Assets[1]

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Abstract

We introduce a measure of diversification for portfolios comprising d risky assets. This measure relates the smallest possible return variance among these d assets to the overall portfolio return variance, yielding the portion of non-diversifiable risk. In the context of normally distributed asset returns, its estimator and finite-sample properties are explored when being applied to the trivial asset allocation strategy. An overview of different previous approaches towards the measurement of diversification is provided, and the shortcomings of some of these approaches are illustrated. A categorization of tests regarding the portfolio return variance is given, especially for comparing naively allocated with minimum-variance portfolios. The empirical part of this work is carried out on monthly return data for the S&P500 constituents, with a return history spanning the last five decades. When measuring the diversification of naively allocated 40-asset portfolios, the average degree of diversification barely exceeds 60%. This result indicates that - for the mutual fund manager as well as for the private investor - well-founded selection of assets indeed leads to better portfolio diversification than naive allocation does.

JEL Classification: C13, C16, C58, G11.

Keywords: Diversification, Portfolio Management, Naive Portfolio, Variance Estimation, Finite-Sample Distribution, S&P500.

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1 Introduction

The benefits of diversification are well-known and vividly applied when investing into any kinds of risky assets. Nevertheless, it was not before the famous work of Markowitz (1952) who introduced the first thorough proof in favor of diversifying one's portfolio among all assets. Beyond that, this holds true even if it means sacrificing a seemingly higher expected return on certain assets. The result of his work, known as the *mean-variance framework*, has become a standard part of today's knowledge in finance.

However, direct application of the mean-variance approach towards portfolio optimization is prohibitive from the practitioner's as well as from the scientific point of view. To name only a two impacting empirical studies, Klein and Bawa (1976) and Chopra and Ziemba (1993) test the out-of-sample performance of the mean-variance framework and deem it inferior to its theoretical promise. From an analytical point of view, Jorion (1986) shows that superior out-of-sample performance can be obtained by applying a Bayesian approach, while Best and Grauer (1991) show that the composition of mean-variance efficient portfolios can dramatically change due to even small perturbations in the asset means.

Currently, the usefulness of quantitative portfolio allocation strategies in general is discussed in the literature. The question whether even the trivial asset allocation rule, that is, to invest one's wealth equally into a set of given assets, outperforms sophisticated approaches is not answered yet, cp. DeMiguel et al. (2009a) or Behr et al. (2010). The main problem of applying quantitative methods is the estimation error for the input parameters μ and Σ, that is, the unknown asset means and their variances and covariances, respectively.

While traditionally, estimation of the expected asset means, μ, is accomplished by analysts utilizing operating figures or balance sheet data rather than historical averages, estimation of variances and covariances among assets is more typically done using historical observations of stock prices. The reason covariance estimates are retrieved from historical data is twofold.

First, even for an asset universe of only 40 assets, there are 780 covariances and 40 variances to be estimated. Leaving the task of judging the co-movement of companies' stock prices to employees would create considerable cost, and would probably not even be possible if covariances would have to be estimated on a weekly or even daily basis.

Second, Chopra and Ziemba (1993) quantify (in terms of cash-equivalent loss) the error in estimating means about 10 times higher as errors in estimating the variances, and even about 20 times more costly when compared to estimation errors in covariances.

These observations have led to a shift in the attention of today's literature about quantitative portfolio strategies away from the mean-variance framework, focusing on the so-called minimum-variance strategies.

Minimum-variance strategies aim at minimizing the overall portfolio return variance, without explicitly paying attention to the estimation of the mean of asset returns. A growing stock of literature confirms superior performance of minimum-variance strategies as opposed to those originating from the mean-variance framework, cf. Jagannathan and Ma (2003), DeMiguel et al. (2009a), DeMiguel et al. (2009b), or Frahm and Memmel (2010). Seemingly, the benefits of combining the assets in a way that their return variance is minimized outweigh the loss due to the departure from the Markowitz model - even when asset returns are assumed to be normally distributed with mean μ and covariance matrix Σ. Put another way, it is the *diversification effect* among the different assets that seems to contribute to the portfolio performance. And even though this perception is common knowledge, diversification is mostly managed by ad-hoc constraints like lower bounds on the number of stocks held in a portfolio or other heuristics.

In line with the above, only little work can be found about the quantitative measurement of the diversification effect. While a qualitative definition can be found in Meucci (2009), who describes a portfolio as well-diversified *"if it is not heavily exposed to individual shocks,"* its implicit definition given by the Capital Asset Pricing Model (CAPM), as developed by Sharpe (1966), Lintner (1965) and Mossin (1966) is more precise, and will be reviewed later on.

The motivation for this paper is a direct consequence of the above considerations, and the following, central issues are addressed:

1. How can a quantitative measure for the diversification of portfolios of risky assets be defined in a way that it is consistent with the qualitative, common-sense definition? Which parameters should such a measure depend upon?

2. What are the existing approaches towards the quantitative measurement of diversification, and what are their strengths and weaknesses?

3. In the empirical section, can reliability of the measure be confirmed when being applied to portfolios of S&P500 constituents?

4. In connection to Question 3, does the proposed measure meet with the intuition that a portfolio containing stocks from a large number of industries is better-diversified than one that is concentrated on few assets only?

5. In terms of the measure, is the widely-spread intuition that the naive portfolio, defined by equal investment into all of its constituents, really well-diversified?

While the motivating questions are posed in the order of importance of each question, the design of this paper slightly deviates from this order. Sections 2 and 3 provide the framework of the analysis, a review on statistical properties of minimum-variance as well as naive portfolios, and give an overview on previous works regarding the measurement of diversification, respectively. In Section 4, the measure $\mathcal{D}_d(\cdot)$ is introduced. Also, the estimator of $\mathcal{D}_d(\cdot)$ is presented, and for the special case of the trivial portfolio, its finite-sample properties are described. Section 5 summarizes some statistical tests regarding the return variances of trivially allocated and minimum-variance portfolios, respectively. The empirical part of this work can be found in Section 6, and Section 7 concludes the paper.

2 Preliminaries

The asset returns $\bar{R}$ of d risky assets are assumed to follow a d-variate normal distribution with mean vector $\mu \in \mathbb{R}^d$ and covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$, viz.

$$\bar{R} \sim \mathcal{N}_d\Big(\mu, \Sigma\Big). \tag{1}$$

Let the entries of the covariance matrix Σ be denoted by σ_{ij} for $i, j = 1, \ldots, d$, that is, $\sigma_{ij} = \mathbb{C}\mathrm{ov}(\bar{R}_i, \bar{R}_j)$.

Given a finite sample $(R_1, \ldots, R_T)$ of independent copies of $\bar{R}$, the maximum-likelihood unbiased estimators of μ and Σ along with their distributions are given by

$$\hat{\mu} \;:=\; \frac{1}{T}\sum_{t=1}^{T} R_t \quad\sim\quad \mathcal{N}_d\left(\mu, \frac{1}{T}\,\Sigma\right) \quad \text{and} \tag{2}$$

$$\begin{aligned} \hat{\Sigma} \;:=\; \frac{1}{T-1}\sum_{t=1}^{T} \left(R_t - \hat{\mu}\right)\left(R_t - \hat{\mu}\right)', \\ \text{so } \; (T-1)\hat{\Sigma} \;\sim\; \mathcal{W}_d\left(\Sigma, T-1\right). \end{aligned} \tag{3}$$

Here, $\mathcal{W}_d\left(\Sigma, T-1\right)$ denotes a d-dimensional central Wishart distribution with scale matrix Σ and $T-1$ degrees of freedom.

As usual, any vector $v \in \mathbb{R}^d$ is supposed to be a column vector, whereas v' denotes the row vector arising by transposition of v. Furthermore, $\underline{1}$ denotes the column vector $(1, \ldots, 1)'$, and a portfolio weight vector $w \in \mathbb{R}^d$, or short, a portfolio, comprising the entries $w_1, \ldots, w_d$ describes the fractions of wealth invested into assets $1, \ldots, d$, respectively.

Note that, as $\bar{R}$ is a random variable, the portfolio return defined by the weighted sum of the asset returns, $w'\bar{R}$, is also a random variable. Its mean return is given by $\mu_P := \mathbb{E}(w'\bar{R}) = w'\mu$, whereas its return variance reads $\sigma_P^2 := \mathbb{V}\mathrm{ar}(w'\bar{R}) = w'\Sigma w$. As μ and Σ are unknown to the investor, both quantities may be estimated from the given return data $(R_1, \ldots, R_T)$ via replacing μ and Σ by their empirical counterparts (2) and (3).

2.1 The Naive Portfolio

In the special case of the naive portfolio, also called the *equally-weighted portfolio* and defined by $w = \frac{1}{d}\underline{1}$, equal fractions of wealth are allocated to each of the d assets. The return variance σ_d^2 of the naive portfolio reads

$$\sigma_d^2 = \left(\frac{1}{d}\underline{1}\right)' \Sigma \left(\frac{1}{d}\underline{1}\right) = \frac{1}{d^2}\,\underline{1}'\Sigma\underline{1} = \underbrace{\frac{1}{d}\,\overline{\mathbb{V}\mathrm{ar}}}_{\substack{\text{diversifiable}\\ \text{part}}} + \underbrace{\frac{d-1}{d}\,\overline{\mathbb{C}\mathrm{ov}}}_{\substack{\text{non-diversifiable}\\ \text{part}}}, \tag{4}$$

where

$$\overline{\mathbb{V}\mathrm{ar}} := \frac{1}{d}\sum_{i=1}^{d} \sigma_{ii} \quad \text{and} \quad \overline{\mathbb{C}\mathrm{ov}} := \frac{1}{d(d-1)} \sum_{i=1}^{d} \sum_{\substack{j=1\\ i\neq j}}^{d} \sigma_{ij}$$

denote the average of the variances and the average of the covariances of the assets, respectively.

Clearly, $\sigma_d^2 \overset{d\to\infty}{\longrightarrow} \overline{\mathbb{C}\mathrm{ov}}$, so for a growing number d of portfolio constituents, the impact of their individual variances vanishes, and the overall portfolio variance becomes the average of the constituents' covariances.

With the traditional sample covariance matrix estimator given by (3), the sample counterpart of σ_d^2, denoted by $\hat{\sigma}_d^2$, has the following finite-sample distribution. It can be obtained by application of well-known theorems for the Wishart distribution, see Chapter 3 in Muirhead (1982). It holds that

$$\hat{\sigma}_d^2 = \frac{1}{d^2}\,\underline{1}'\hat{\Sigma}\underline{1} \sim \sigma_d^2\,\frac{\chi^2_{T-1}}{T-1}, \tag{5}$$

that is, the sample variance of the naively allocated d-asset portfolio has a scaled χ^2-distribution around its true value σ_d^2.

Thus, in the context of normally distributed asset returns $\hat{\sigma}_d^2$ is an unbiased estimator of σ_d^2, and its variance amounts

$$\mathbb{V}\mathrm{ar}\Big(\hat{\sigma}_d^2 \mid \sigma_d^2\Big) \;=\; \mathbb{V}\mathrm{ar}\Big(\sigma_d^2\,\frac{\chi^2_{T-1}}{T-1} \mid \sigma_d^2\Big) \;=\; \sigma_d^4\,\frac{2}{(T-1)}. \tag{6}$$

As such, $\hat{\sigma}_d^2$ is an unbiased and also consistent estimator of σ_d^2.

2.2 The Global Minimum-Variance Portfolio

As mentioned in the introduction, minimum-variance portfolios have started to gain more attention in recent publications. A special portfolio in this context is the global minimum-variance portfolio (GMVP), which aims at minimizing the variance of its return. It is denoted by w_{GMVP} and defined by

$$w_{\mathrm{GMVP}} \;:=\; \arg\max_{w}\; w'\Sigma w \quad \text{ s.t. } w'\underline{1} = 1. \tag{7}$$

Without additional constraints, the analytical solution to (7) takes the form

$$w_{\mathrm{GMVP}} \;=\; \frac{\Sigma^{-1}\underline{1}}{\underline{1}'\Sigma^{-1}\underline{1}}, \tag{8}$$

resulting in the return variance of the GMVP given by

$$\sigma^2 \;:=\; w'_{\mathrm{GMVP}}\,\Sigma\,w_{\mathrm{GMVP}} \;=\; \left(\underline{1}'\Sigma^{-1}\underline{1}\right)^{-1}. \tag{9}$$

The distribution for the moment estimator of the GMVP return variance, denoted by $\hat{\sigma}^2$, is

$$\hat{\sigma}^2 \;:=\; \left(\underline{1}'\hat{\Sigma}^{-1}\underline{1}\right)^{-1} \quad\sim\quad \sigma^2\,\frac{\chi^2_{T-d}}{T-1}, \tag{10}$$

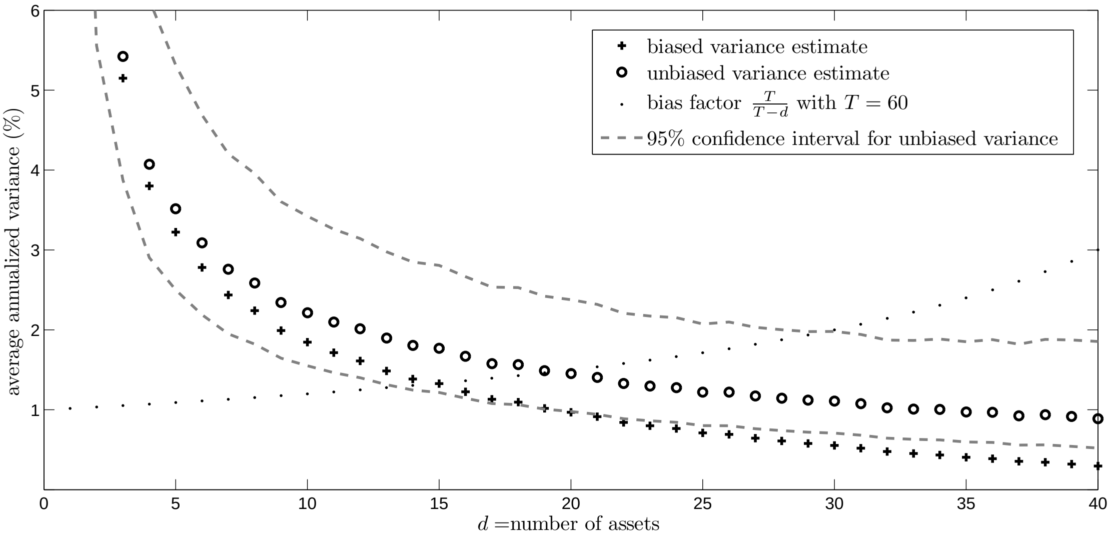
and it follows that

$$\mathbb{E}\left(\hat{\sigma}^2 \,|\, \sigma^2\right) \;=\; \sigma^2\,\frac{1}{T-1}\mathbb{E}\left(\chi^2_{T-d}\right) \;=\; \frac{T-d}{T-1}\,\sigma^2 \;\approx\; \left(1-\frac{d}{T}\right)\sigma^2. \tag{11}$$

With $Q := \frac{T}{d}$ defined as the *effective sample size*, and given an estimate $\hat{\sigma}^2$ of the return variance of the GMVP, it follows that the true variance σ^2 is underestimated by about the factor $1\,/\left(1-\frac{1}{Q}\right)$. As such, the estimator $\hat{\sigma}^2$ is biased, and furthermore it is consistent only if $\lim_{T\to\infty}\frac{d}{T} = 0$.

For example, when a medium-sized minimum-variance portfolio with $d = 20$ assets is estimated from $T = 60$ monthly return data, the true variance can be expected to

Figure 1: Bias of the GMVP variance estimator $\hat{\sigma}^2$



Biased and unbiased variance estimates of the d-asset GMVP when estimated from actual return data. 60 monthly observations of the excess returns of d assets are available, with the assets being randomly selected from the pool of S&P500 constituents in 2009. As d grows from 1 to 40, the bias of the variance estimate can be seen to sharply increase, as indicated by the dotted line. While the estimated GMVP variance $\hat{\sigma}^2$ decreases with growing d, it falls below the 95% confidence interval for the unbiased variance.

be about 50% above its estimate. Figure 1 demonstrates the importance of taking the effective sample size into account. With 60 months of return data at hand, denoted by $(R_1, \ldots, R_{60})$, the estimation of Σ and the d-asset GMVP weights, w_{GMVP}, is accomplished for various portfolio sizes d. In 250 repetitions, the estimated variance $\hat{\sigma}^2$ as well as its bias-corrected version are calculated and averaged. The confidence interval in Figure 1 is constructed using (10), and as a χ^2-distribution is involved, the interval encloses the true variance asymmetrically.

The confidence interval also shows - for all portfolio sizes - how variable the estimation of the GMVP variance even in the context of normally distributed asset returns is. For $d = 15$, the average bias-corrected variance estimate amounts 1.8% on an annual basis, while the 95% confidence interval allows for values between 1.25% and 2.9%. In more commonly used terms of annual portfolio standard deviations, these values allow the average 15-asset GMVP standard deviation σ_P to vary between 11.2% and as much as 17%.

3 Previous Approaches on Diversification Measurement

There exist different contributions towards how to measure the degree of diversification in a portfolio of d risky assets. However, there are different concepts of the term *diversification*, and a first aim is to motivate an understanding of that term.

Let there be a universe of risky assets, and let w be a portfolio of d of these assets, where, of course, not necessarily all assets from the asset universe must be included. When saying that w is a *well-diversified* portfolio, one would expect w to be immune against shocks created by a single or a few assets.

In turn, this does not mean that w is not subject to any shocks. As - by its very nature - the universe of risky assets is exposed to economic fluctuations, up- and downturns in the market will affect its value. As such, the task is to find some sensible benchmark which separates the level of variation induced by idiosyncratic shocks, defined as shocks generated by single assets, from the level of variation that is induced by the market and therefore is unavoidable.

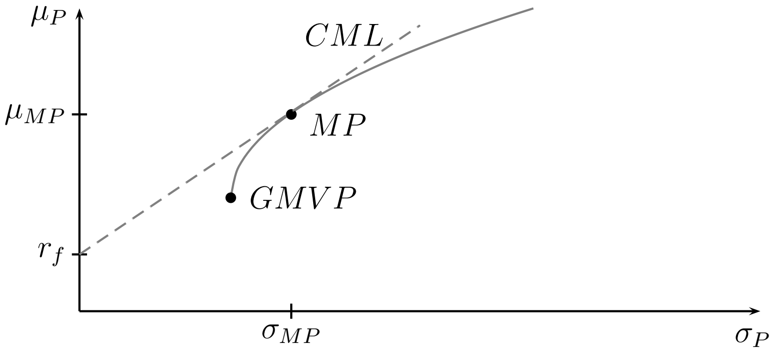
In this context, it is also important to clarify what data a measure of diversification should depend upon. Given the portfolio w of d risky assets, a reliable measurement of the degree of diversification can only be achieved by incorporating the dependence structure among these assets. Thus, not only d and w must be taken into account, but also the information of how the portfolio constituents interact. Commonly, and especially in the setting of normally distributed asset returns $\bar{R}$, this is done by evaluating the covariance matrix Σ.

Nevertheless, not all previous contributions towards diversification measurement are based on the above considerations. Indeed, some of the measures to be introduced reveal a very different perception of what diversification is.

On the other hand, there are many possibilities to measure the dispersion of the portfolio return caused by the interaction of the different assets' return characteristics. For example, Louton and Saraoglu (2008) examine portfolios with regard to the measures semivariance and expected shortfall. To evaluate the variance or the standard deviation is the most common approach, though. Using the variance of a portfolio alone as an indicator for its degree of diversification is not advisable. This is because the calculated variance needs to be compared to a benchmark, and it is not a priori clear which benchmark to use.

In the sequel of this section, a non-exhaustive overview of different methods is given and some main contributions are mentioned. Also, some weaknesses of the existing approaches are discussed.

Figure 2: The Capital Asset Pricing Model (CAPM)



The CAPM postulates a linear relationship between portfolio risk and portfolio return. In equilibrium, every investor holds the market portfolio (MP), leveraged by a factor according to his individual risk preference. The set of all these leveraged holdings constitutes the capital market line (CML). The global minimum-variance portfolio (GMVP) can be seen to have a smaller return variance than the MP.

3.1 Theoretical Considerations

The *Capital Asset Pricing Model* (CAPM), as developed independently by Sharpe (1966), Lintner (1965) and Mossin (1966), postulates (in the presence of a risk-free asset) a linear relationship between portfolio return μ_P and portfolio risk, measured by the portfolio return standard deviation, σ_P. In this context, the existence of a *market portfolio* (MP) consisting of all risky assets traded on the financial market is claimed.

The MP comprises all risky assets available in the asset universe, with each asset being weighted by its share of the total financial market's value. As a portfolio of risky assets only, the market portfolio is defined as being completely diversified, and the risk it bears is called *systematic* or *non-diversifiable*.

As the CAPM is an equilibrium model, it implies that all investors eventually allocate their wealth to the market portfolio. According to Tobin (1958), the fraction of wealth invested into the market portfolio depends on the individual risk behavior of the investor. A strongly risk-averse agent would prefer to invest only a small fraction of his wealth into the market portfolio, whereas a risk-loving agent might even take out a loan in order to obtain a higher leverage on the market portfolio. For each level of risk aversion, the combination of riskless asset and market portfolio constitutes the *capital market line* (CML), see Figure 2.

Unfortunately, as the market portfolio is a theoretical construct, it is unobservable. It is approximated by indices like the S&P500, or even worldwide indices like the MSCI World Index, comprising about 8 500 risky assets from more than 40 countries. Thus, the market portfolio cannot be used as a benchmark of diversification. At most, its proxies could be used, and these indices indeed serve as benchmarks for comparisons of levels of variations. Sharpe and Alexander (1999, p.654f.), for example, use the S&P500 index as proxy and compute R^2, the coefficient of determination, when explaining variation in fund returns by variations of the index over the period 1970-1974.

The above described framework constitutes the theoretical definition of diversification implied by the CAPM. It defines the MP to be totally diversified. But this does not mean that there is no other portfolio with smaller return variance, see Figure 2. By its very definition, the GMVP always allows for lower return variance, see (7). In contrast, the definition of diversification in connection with the CAPM stems from the fact that the CAPM is an equilibrium model, and that in equilibrium, every investor is expected to hold some fraction of the MP.

Furthermore, the investor or a manager of a mutual fund must choose among a given set of d risky assets, which - in practice - is bounded sharply by issues such as the investment policy or the resources needed to monitor these assets. Such an investor would want to know how much reduction in his portfolio return variation can theoretically be obtained, and how close to this bound his actual portfolio is.

The following approaches towards measuring the degree of diversification of a portfolio of risky assets therefore deal with the problem of measuring the degree of diversification of a portfolio that comprises of at most d assets.

3.2 Number of Assets

The most elementary approach to measure the diversification of a portfolio of risky assets is to count the number of its constituents. Numerous studies, with the two most impacting being the works of Evans and Archer (1968) and Fisher and Lorie (1970), have pursued this methodology. In detail, Evans and Archer (1968) build equally-weighted d-asset portfolios comprising randomly chosen assets from the S&P500 index for the year 1958. For each d between 1 and 40, the d-asset portfolio standard deviation is calculated and averaged over a total of 60 repetitions. Afterwards, the obtained average standard deviation of the d-asset portfolios is regressed against $1/d$.

Both works build upon the well-documented fact that the return variance of an equally-weighted portfolio declines with the number of its constituents. An explanation is already given by (4), indicating that the variance of such a portfolio eventually drops down towards the average of the covariances among all assets. Evans and Archer (1968), in their conclusion,

> *"raise doubts concerning the economic justification of increasing portfolio sizes beyond 10 or so securities, and indicate the need for analysts and private investors alike to include some form of marginal analysis in their portfolio selection models."*

With *marginal analysis*, an analysis of the trade-off between growing transaction costs on the one hand and the reduction of return standard deviation on the other hand is meant.

It should be noted, though, that the approach described above is not exactly in line with the theory suggested by (4), which was firstly pointed out by Elton and Gruber (1977, p. 418), and again by Bird and Tippett (1986). Actually, a linear relationship exists only between the return variance σ_d^2 of the naive portfolio and the inverse of the number of its constituents, $1/d$. In contrast, as described above, Evans and Archer (1968) and subsequent studies often regressed the estimated standard deviation $\hat{\sigma}_d$ on $1/d$, leading to inappropriate results.

Because of its simplicity, the number of securities still serves as a prominent measure of portfolio diversification. Tang (2004, p.156) gives an overview on textbooks' recommendations regarding the number of assets that constitute a well-diversified portfolio, yielding numbers between 10 and 40. It is interesting to see that most of these textbook recommendations still refer to either the study of Evans and Archer (1968) or to studies of comparable age.

There are three shortcomings in using the number of assets as measure for a portfolio's diversification.

First, this approach is only useful when an equally-weighted portfolio is under consideration, as it crucially depends on the relationship (4).

The second problem with using the number of assets as an indicator is the heterogeneity of the assets. In the idealized case where all asset returns arise from a multivariate normal distribution with equal means, equal variances and equal covariances, counting the number of assets is perfectly fine for measuring the reduction of its variance. But as these idealized assumptions do not meet with reality, the need for a different measure is apparent.

Third, even if the textbooks' recommendations of 'between 10 and 40 assets' to be held in a portfolio meets with reality for most investors, even institutional ones, the question still arises which stocks to choose.

Section 6 will reconsider the reduction of average portfolio variance for a growing number d of portfolio constituents in equally-weighted portfolios empirically and in more detail. Moreover, the statistical tests gathered in Section 5 might yield support for the investor in situations where the question of which asset to add to an existing portfolio is raised.

3.3 Information-Theoretic Approaches

Another approach towards assessing the degree of a portfolio's diversification stems from information theory. Loosely speaking, information theory is concerned with the quantification of the *disorder* of a random variable, with its most prominent measure being the Shannon entropy. These measures take the distribution of a random variable as the generic object, and as such, they are also applicable to non-negative weight vectors in portfolio theory. To apply this approach the portfolio weight vector w must not have negative entries and it must sum up to one.

Woerheide and Persson (1993) introduce measures from information theory as well as measures of economic concentration to portfolio theory in order to assess the concentration of weights on single assets. Thus, their approach of measuring the diversification of a portfolio w depends not only on the number of assets, d, but also on the fractions of wealth invested into the assets, $(w_1, \ldots, w_d)$. Their main point of criticism on former studies is that the mere number of portfolio constituents provides an adequate picture of a portfolio's degree of diversification only if it is equally-weighted. Also, they reach out for finding a measure which does not rely on the analysis of market data. A short outline of their methodology is presented.

With monthly return data covering the entire period of the years 1965 through 1985 from 483 American exchange-listed companies at hand, Woerheide and Persson (1993) evaluate the relationship between the standard deviation of randomly composed, and thus unequally-weighted d-asset-portfolios and the respective index of diversification.

The indices of diversification to be evaluated consist of 5 predetermined measures, which are called *diversification indices* (DI). These include the complements of the Herfindahl and the Rosenbluth indices, respectively, an entropy-based measure as well as two other measures, see Woerheide and Persson (1993, pp. 76-78). For example, the complement of the Herfindahl index (CHI) for a portfolio $w \in \mathbb{R}^d$ is given by

$$\mathrm{CHI}(w) \ := \ 1 - \underbrace{\sum_{i=1}^{d} w_i^2}_{\text{Herfindahl index}} \ . \tag{12}$$

The d-asset portfolios examined in their study are arranged by randomly choosing non-negative weights that sum up to one. Afterwards, d assets are randomly selected from the universe of 483 assets, and the weights $(w_1, \ldots, w_d)$ are assigned to these assets. Then, the standard deviation of each portfolio w is calculated utilizing the whole sample period of 240 months. For each d between 2 and 30, this procedure is repeated 60 times. This yields a series of 1 740 standard deviations $(\sigma_P)_{d,i}$ and a series of the portfolios' respective DI-measures, $\Big(\mathrm{DI}\,(k)\Big)_{d,i}$ for each measure $k = 1, \ldots, 5$, with $d = 2, \ldots, 30$ and $i = 1, \ldots, 60$.

Finally, for each of the 5 diversification indices, the portfolio return standard deviations $(\sigma_P)_{d,i}$ are regressed against the respective index values $\Big(\mathrm{DI}\,(k)\Big)_{d,i}$ via the 5 models

$$\sigma_P = \alpha_k \ + \ \gamma_k \, \mathrm{DI}(k) \ + \ \varepsilon_k, \quad k = 1, \ldots, 5. \tag{13}$$

The goodness-of-fit measure R^2 of each of the 5 regressions is then used as an indicator of how well the linear relationship between the standard deviations and the respective index of diversification fits.

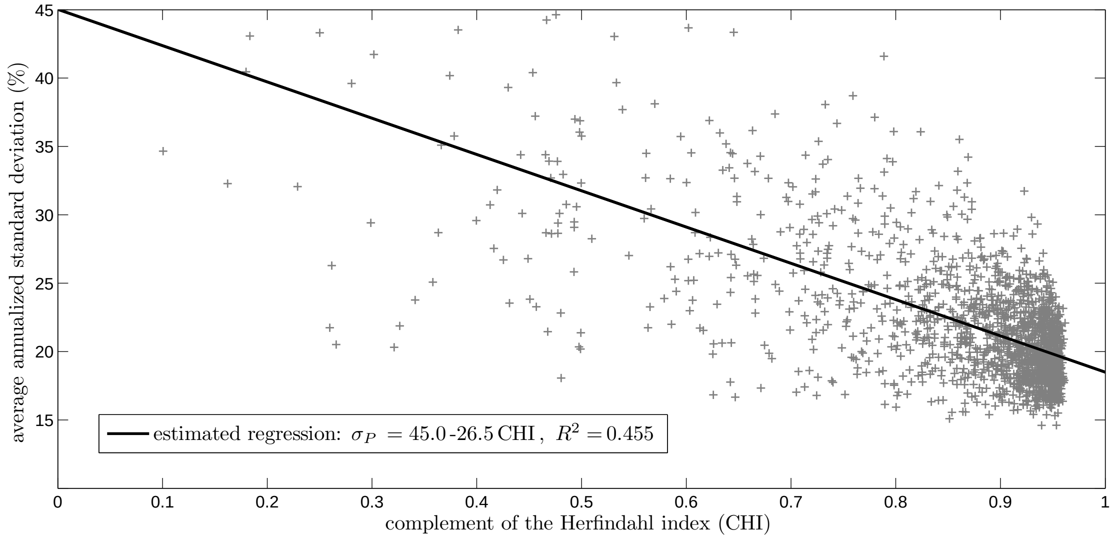
Woerheide and Persson (1993) find that - among the 5 indices of diversification they examine - the CHI, given by (12), yields the highest explanatory power with an R^2 of 0.548. Thus, they recommend the CHI as a means to assess the degree of diversification of a portfolio w.

When repeating the study with data from the 2009 CRSP database, including only companies with at least 12 years of continuous return data, the estimates of coefficients and the goodness-of-fit-measure R^2 were similar, see Figure 3.

Other studies that incorporate information-theoretical approaches are Bouchaud et al. (1997) and Bera and Park (2008), although these work are directed more onto the portfolio construction process itself. Nevertheless, both works propose the achievement of a certain level of diversification during the construction process of the optimal portfolio. While Bouchaud et al. (1997) entangle the additional postulation of a certain *level of entropy* among the portfolio weights with the reliability of the estimates of the means and covariance structure, Bera and Park (2008) maximize the entropy of the portfolio weights subject to constraints on the portfolio return and portfolio variance.

Even though entropy-based measures, when used in the portfolio optimization process, yield portfolios that are not concentrated on single assets, they should not be used as measures of its diversification. The main problem lies at the axiomatization of measures of concentration, especially in the axioms of symmetry and monotonicity, which are common to information-theoretic measures and measures of economic concentra-

Figure 3: The study of Woerheide and Persson (1993) repeated on 2009 data



The study of Woerheide and Persson (1993), conducted on 2009 data. For repeatedly, randomly generated portfolio weights $w = (w_1, \ldots, w_d)$, where d varies between 2 and 30, the CHI is calculated as well as the standard deviation of w when allocating randomly chosen assets from the CRSP database according to w. When assuming a linear relationship, the observed coefficients are proportional to those of the original study, and also a similar R^2 is obtained.

tion. These axioms imply that the portfolio weights can be exchanged without any alteration of the degree of diversification. Of course, this is in contrast to the non-exchangeability of the distribution of the asset returns.

Furthermore, while the approach in Bouchaud et al. (1997) might be a justified approach towards portfolio optimization from a decision theoretic point of view, the approach in Bera and Park (2008) seems to be - though interesting from the practitioner's perspective - rather heuristic.

3.4 Measurements via Principal Component Analysis

A general task of multivariate data analysis is to detect patterns among given data. In the case of d risky assets, the historical return data of which typically show correlations with each other, the idea is to transform the return data of these d risky assets into a set of k linear combinations of these assets, called *principal components*. The attractive feature about the principal components is that they are uncorrelated with each other. It also holds that $k \leq d$, so a principal component analysis can be interpreted as the search for a certain number of k factors which explain 'most' of the variance in the data.

Intuitively, whenever the original assets are close to being uncorrelated, the number of principal components will be close to d, indicating a high degree of heterogeneity.

Thus, skipping assets from such a portfolio might increase its variance. On the other hand, when there is much correlation among different assets, not much diversification potential would be lost in treating these assets as one single component.

Formally, the principal components are uncorrelated linear combinations of the original d assets, and as such, they cannot be interpreted as assets themselves. They can rather be thought of as uncorrelated portfolios, and as such they shall be called *principal portfolios* in the sequel.

It must be mentioned that the weights of such a principal portfolio need not sum up to one, and typically, it will include positive as well as negative weights. The name *principal portfolios* is given in the style of Partovi and Caputo (2004), who also deal with this concept; however, they use the principal portfolios in the context of constructing the efficient frontier.

Principal component analysis makes use of the spectral decomposition theorem from linear algebra, which ensures that the covariance matrix Σ can be written as the product

$$\Sigma \;=\; \Gamma\Lambda\,\Gamma', \tag{14}$$

where $\Lambda \;=\; diag(\lambda_1^2, \ldots, \lambda_d^2)$ is the diagonal matrix of eigenvalues of Σ. Without loss of generality, these eigenvalues can be ordered in descending order, that is, $\lambda_1^2 \geq \ldots \geq \lambda_d^2$. Moreover, Γ is an orthogonal matrix with $\Gamma'\Gamma = I_d$, and its columns are the standardized eigenvectors of Σ. These eigenvectors then constitute the principal portfolios, and for each principal portfolio i, λ_i^2 equals its variance.

Rudin and Morgan (2006) make use of principal component analysis in the following way. As they only examine equally-weighted d-asset portfolios, the portfolio weight vector w is fixed. They try to overcome the deficiency of only using the number of assets as a measure for diversification by defining their *Portfolio Diversification Index* as

$$\mathrm{PDI}_d \;:=\; 2\sum_{k=1}^{d} kW_i \quad \text{with } \; W_i \;:=\; \frac{\lambda_i^2}{\sum_{j=1}^{d}\lambda_j^2}.$$

The W_i are called the *relative strengths* of the ith principal portfolio, and each W_i can be interpreted as the fraction of the original portfolio's total return variance that is explained by the i-th principal portfolio, for $i = 1, \ldots, d$.

If the underlying assets contained in the original, equally-weighted portfolio show a strong correlation with each other, the first few principal portfolios account for nearly all the variability, and thus the PDI will be small. In this case, the same degree of diversification can be obtained with fewer assets. In the optimal case, where all d assets are uncorrelated, $W_i = \frac{1}{d}$ for all i, and thus the upper bound for PDI$_d$, d will be attained.

The interpretation of the number PDI for a d-asset equally-weighted portfolio therefore is as follows. Investment into these d assets yields the same degree of diversification as investing equally into PDI$_d$ uncorrelated assets.

Rudin and Morgan (2006) test their diversification index on equally-weighted portfolios of the S&P100 index and on a sample of hedge funds, and they find a sublinear relation between PDI$_d$ and portfolio size d. More exactly, in their study they find that the average PDI$_{40}$ for 40-asset portfolios randomly selected from the index equals about 20, the average PDI for the portfolio comprising of all 100 assets is only 40.

Although taking into account the return history of the underlying assets, Rudin and Morgan (2006) only consider equally-weighted portfolios of various sizes d. Clearly, institutional as well as private investors yearn for a measure that also works on non-naively allocated portfolios, and as such, the PDI might be of little use.

3.5 Meucci's Approach

A rather new methodology, which combines the previously introduced approaches, is presented in Meucci (2009). As in Rudin and Morgan (2006), the covariance matrix Σ of the d assets can be decomposed via

$$\Sigma \;=\; \Gamma\Lambda\,\Gamma',$$

where, as above, $\Lambda \;:=\; diag(\lambda_1^2, \ldots, \lambda_d^2)$, with $\lambda_1^2 \geq \ldots \geq \lambda_d^2$, is the diagonal matrix of eigenvalues of Σ, cp. (14). The columns $e_1, \ldots, e_d$ of the orthogonal matrix Γ are the eigenvectors corresponding to each eigenvalue $\lambda_1^2, \ldots, \lambda_d^2$. As above, these eigenvectors will be referred to as principal portfolios.

As Meucci deals with arbitrarily allocated d-asset portfolios w, though, some more terminology is needed.

The return on each of the principal portfolios $e_1, \ldots, e_d$, with $e_i \in \mathbb{R}^d$, is given by

$$\tilde{R} \;:=\; \left(e_1'\bar{R}, \ldots, e_d'\bar{R}\right)' \;=\; \Gamma'\bar{R}, \tag{15}$$

where, as above, $\bar{R}$ denotes the random returns on the d risky assets. Note that in this expression $\bar{R}$ does not need to be normally distributed. Again, the return variances of these d principal portfolios are $\lambda_1^2, \ldots, \lambda_d^2$, and their covariances are zero by construction.

The original portfolio w is now reconstructed as a linear combination of the principal portfolios via $\tilde{w} := \Gamma' w = (e_1' w, \ldots, e_d' w)'$, and $\tilde{w}$ is referred to as a *weighted principal portfolio*.

Using orthogonality of Γ, the random portfolio return can be written as

$$\tilde{w}'\tilde{R} \;=\; \Big(\Gamma' w\Big)'\Big(\Gamma'\bar{R}\Big) \;=\; w'\Gamma\Gamma^{-1}\bar{R} \;=\; w'\bar{R}. \tag{16}$$

Furthermore, each of the weighted principal portfolio return variances reads

$$\mathbb{V}\mathrm{ar}\left(\tilde{w}_i\tilde{R}_i\right) \;=\; \tilde{w}_i{}^2\; \mathbb{V}\mathrm{ar}\left(\tilde{R}_i\right) \;=\; \tilde{w}_i{}^2\lambda_i^2 \quad \text{for each } i = 1, \ldots, d, \tag{17}$$

and due to the fact that $\mathbb{C}\mathrm{ov}\left(\tilde{w}_i\tilde{R}_i,\, \tilde{w}_j\tilde{R}_j\right) \;=\; 0$ for $i \neq j$, the total portfolio return's variance can be expressed additively, as opposed to (4), as

$$\mathbb{V}\mathrm{ar}(w'\bar{R}) \;=\; \mathbb{V}\mathrm{ar}\left(\tilde{w}'\tilde{R}\right) \;=\; \sum_{i=1}^{d} \tilde{w}_i{}^2\, \lambda_i^2, \tag{18}$$

where the identity (16) is used.

The above observations give rise to the definition of what Meucci calls the *diversification distribution* $p(w) = \Big(p_1(w), \ldots, p_d(w)\Big) \in \mathbb{R}^d$ with

$$p_i(w) \;:=\; \frac{\tilde{w}_i{}^2\lambda_i^2}{\sum_{i=1}^{d} \tilde{w}_i{}^2\lambda_i^2}, \quad i = 1, \ldots, d, \tag{19}$$

which is the fraction of each weighted principal portfolio's return variance on the total variance of portfolio w.

Meucci then defines the portfolio w as well-diversified whenever its total variance, as given by the denominator of (19), is not concentrated in a few $p_i(w)$.

In turn, with this definition at hand, and with the uncorrelated principal portfolios, the application of measures of concentration introduced in (3.3) is justified, and gives a precise picture of what is meant by the phrase 'concentrated in a few $p_i(w)$'.

With the additive partition of total portfolio risk, each of the $p_i(w)$ in (19) measures a risk that arises from the i-th weighted principal portfolio. The next step of diversification measurement is to apply an entropy measure, as in Woerheide and Persson (1993) or in Bera and Park (2008), to the diversification distribution $p(w)$.

Meucci (2009) proposes to evaluate the exponential of the Shannon entropy measure on the diversification distribution, viz.

$$\mathcal{N}_{Ent}(w) \;:=\; \exp\;\underbrace{\left(-\sum_{i=1}^{d} p_i(w)\log\Big(p_i(w)\Big)\right)}_{\text{Shannon entropy of } p(w)}. \tag{20}$$

Put another way, this means that $\mathcal{N}_{Ent}(w)$ measures the number of truly independent sources of risk that is evident in the portfolio w consisting of d assets. A higher value of $\mathcal{N}_{Ent}$ would represent a more diversified portfolio, whereas a lower value indicates concentration on a few independent sources of risk only.

For any d-asset portfolio w, it holds that $1 \leq \mathcal{N}_{Ent}(w) \leq d$, where higher values indicate a better-diversified portfolio. Thus, the measure $\mathcal{N}_{Ent}(\cdot)$ can easily be normed by division by d. In Section 6, where the empirical part of this paper is carried out, the normed version of $\mathcal{N}_{Ent}$ will be used rather than the standard version for better comparability with the introduced measure.

Furthermore, $\mathcal{N}_{Ent}$ relies on all available information the investor has at his disposal, which are the number of assets d, the allocation w of wealth to these assets, and finally also the return characteristics of these assets stored in the covariance matrix Σ.

In the style of Bera and Park (2008), Meucci (2009) also proposes a new heuristic for asset selection. Instead of pursuing the Markowitz (1952) approach,

$$\max_{w}\; \mu' w \;-\; \frac{\lambda}{2}\, w'\Sigma w \quad \text{s.t. } w \in \mathcal{C}, \tag{21}$$

which theoretically yields some optimal solution w^{opt} that might be subject to a set of constraints $\mathcal{C}$ and is influenced by the investor-specific risk-aversion parameter λ, Meucci (2009) proposes to locate the optimal solution on what he names the *mean-diversification frontier*, calculated as follows.

For some investor-specific parameter $\phi \in [0,1]$, where ϕ reflects the degree of the investor's confidence in the asset return estimate, he selects the portfolio w^{div} which solves

$$\max_{w}\; \phi\, \mu' w \;-\; (1-\phi)\,\mathcal{N}_{Ent}(w) \quad \text{s.t. } w \in \mathcal{C}. \tag{22}$$

Meucci's way of measuring the diversification present in a portfolio of d risky assets is the most appealing one of the approaches summarized in this section. It combines information-theoretical approaches with the methodology of using principal component analysis, thus avoiding the problems of each of the approaches when applied separately to the portfolio weights w or when examining equally-weighted portfolios only.

The diversification distribution (19) shows how Meucci's approach can be seen as a generalization of the PDI introduced by Rudin and Morgan (2006). Instead of analyzing the assets' covariance matrix, which imputes an equally-weighted original portfolio, Meucci applies the weighting scheme via the weighted principal portfolio $\tilde{w}$ in his definition of the diversification distribution $p(w)$. The application of information-theoretic measures, which in Meucci (2009) is the exponential of the Shannon entropy, to the diversification distribution in this case is a useful and sensible approach. Therefore, his method also mitigates the shortcomings identified in the pure application of such measures to the original portfolio vector w.

It must be noted, though, that Meucci (2009) reveals a different perception of the term *diversification* than presented up to this point. In fact, he takes the portfolios w as given, and therefore, also its return variance. He then identifies the *risk drivers* of the portfolio w as the principal portfolios, which, by definition, are uncorrelated. His definition of a well-diversified portfolio can then be stated as a portfolio, the risk drivers of which are invested into equally. Put another way, the portfolio would be not well-diversified if not all risk drivers are invested into equally.

This definition of diversification is generalized by Tasche (2006, Definition 4.1). For an arbitrary risk measure ρ, Tasche defines a *diversification factor* of some risky position $v = \sum_{i=1}^{d} w_i \bar{R}_i$ as

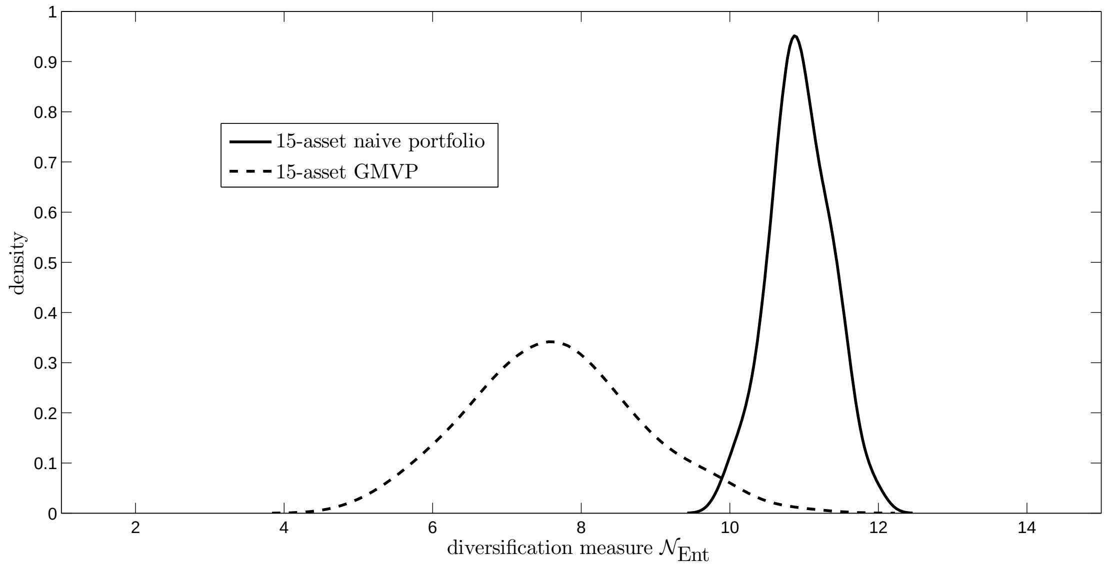
$$\mathrm{DF}_\rho(v) \;:=\; \frac{\rho(v)}{\sum_{i=1}^{d} \rho(w_i \bar{R}_i)}, \tag{23}$$

whenever all risks are properly defined. Here, the asset return $\bar{R}$ is modeled as a random variable which does not necessarily follow a multivariate distribution.

Clearly, as the variance is not a risk measure in the sense of Artzner et al. (1999), this definition is somehow inappropriate in the given context, but it meets with the understanding of diversification as Meucci defines it.

To illustrate the above point, which is in contrast to the described and also to previous perceptions of what a well-diversified portfolio should be, the following example is presented. An evaluation of the d-asset GMVP and the d-asset naive portfolio is carried out via randomly selecting $d = 15$ assets from the S&P500 constituents from 2009, which possess a continuous return history of at least 12 months. The GMVP weights are estimated, and afterwards, for both portfolios, the respective indices of diversification $\mathcal{N}_{Ent}$ are calculated. In 500 repetitions, this yields a series of 500 measured diversification numbers for both allocation strategies. As in each run, the same assets are used, the

Figure 4: The measure $\mathcal{N}_{Ent}$ for different portfolios



Simulation of the measure of diversification introduced in Meucci (2009). For 15 randomly selected assets from the S&P500 in 2009, the naive portfolio (solid line) and the GMVP (dashed line) is calculated. $\mathcal{N}_{Ent}$ is evaluated on these portfolios, and the resulting numbers $\mathcal{N}_{Ent}$ for 100 repetitions are shown. As in each run, the GMVP and the naive portfolio for the same 15 assets is constructed, the measure in this case acts somehow contraintuitive. The GMVP can be expected to possess the lower return variance, which should be expressed in a larger degree of diversification.

'better-diversified' portfolio should be expected to also have the lower return variance. The results are depicted in Figure 4.

The 100 equally-weighted portfolios, with an average diversification number of 10.9, were found to have a mean annual estimated standard deviation of 18.7%. By contrast, the average diversification number of the 100 portfolios allocated via the minimum-variance strategy possess only a mean diversification number of 7.6 and a mean annual estimated (and bias-corrected) standard deviation of 14.4%, which indicates that the return of the minimum-variance portfolios is much less volatile than the return of the equally-weighted portfolios. The understanding of the term *diversification* would expect the portfolio with a higher diversification index to have a lower return variance, though.

4 A Measure of Diversification

This section presents the measure of diversification motivated in the preceding sections. First, the theoretical construct is presented, making use of the concepts introduced in Section 2. As already shown, estimation error must not be ignored when dealing with the variance, and thus, the finite-sample properties of the measure will be reviewed in detail, along with its asymptotic properties.

4.1 Theoretical Construct

Given a portfolio w of d risky assets traded on some market, the following simple measure of its degree of diversification is proposed:

$$\begin{aligned}\mathcal{D}_d(w) &:= \frac{\text{smallest possible variance among } d \text{ assets}}{\text{actual variance of } w} \\ &= \frac{\text{variance of the } d\text{-asset-GMVP}}{\text{actual variance of } w} = \frac{\left(\underline{1}'\Sigma^{-1}\underline{1}\right)^{-1}}{w'\Sigma w}\end{aligned} \tag{24}$$

$\mathcal{D}_d$ is a natural measure, as it yields the ratio of non-avoidable return variation to overall return variation. In practice, the portfolio selection process is often a combination of qualitative and quantitative analyses, resulting in portfolios that are subject to investor-specific constraints like weight restrictions or even legal constraints. For portfolios constructed in this manner, the information of how much removable variation is still contained when compared to a non-restricted portfolio might be valuable to the investor.

It is noteworthy, though, that at this stage, the measure $\mathcal{D}_d(\cdot)$ is a theoretical construct. This is because the true distributional parameter Σ is unknown to the investor. Estimating the parameter Σ introduces estimation error, as well as estimation of the portfolio weights w. In contrast to the previous studies, this paper explicitly accounts for the estimation error when estimating the measure from observed return data.

Also, there are no restrictions towards the portfolio weights w in the denominator of (24), except that they must sum up to one. Especially, at this stage they are allowed to be negative. It should also be noted that the smallest achievable variance is given by $\sigma^2 = (\underline{1}'\Sigma^{-1}\underline{1})^{-1}$, in which case the GMVP may have negative weights.

Extensions of $\mathcal{D}_d$ should include a definition of $\mathcal{D}_d$ for long-only portfolios, i.e., for portfolios with short sale restriction. For practical purposes, using the long-only GMVP as the benchmark might be even more interesting than using the unrestricted version, as short sale restrictions are a natural restriction for the private investor as well as for mutual funds; in the light of the current financial crisis, some Euroland countries even consider short sale restrictions for all market participants.

As such a short sale constrained measure is used in Section 6, it is defined as

$$\mathcal{D}_d^+(w) \;:=\; \frac{\text{variance of the short sale restricted } d\text{-asset GMVP}}{\text{actual variance of } w}\,. \tag{25}$$

Clearly, it holds that $\mathcal{D}_d^+ \;\geq\; \mathcal{D}_d$, as the variance of the short sale restricted minimum-variance portfolio with d assets is always larger than the variance of the unconstrained d-asset-GMVP.

4.2 Estimation of $\mathcal{D}_d$

Estimation error is a prominent phenomenon in portfolio optimization, mostly in connection with the expected returns of the assets. Chopra and Ziemba (1993) find that - in terms of the performance measure cash equivalence loss - that errors in estimating the mean is up to four times more harmful than errors in estimating the variance of certain assets, and up to ten times more harmful than estimation error in covariances.

More recent research has revealed that also variances estimated from historical observations can contain large errors, see, for example, Ledoit and Wolf (2003), Pafka and Kondor (2003) or Jagannathan and Ma (2003). In Section 2.2, even under the assumption of normally distributed asset returns, it is shown that the basic level of return variation can be drastically underestimated by the traditional estimator $\hat{\sigma}^2$. The reliability of the estimator crucially depends on the effective sample size $Q = d/T$, see (11). Thus, examination of the nominator of the measure $\mathcal{D}_d$ with regard to its susceptibility towards estimation error is essential.

Estimation error in the denominator of (24) is even more difficult to handle, as it depends on the strategy chosen. For a strategy that relies heavily on the observed data, there may be large deviations, while for a data-independent strategy, the estimation error in the denominator stems only from uncertainty in $\hat{\Sigma}$.

Nevertheless, as estimation error is present in both, the nominator and the denominator of (24), the estimator of the measure of diversification is defined by

$$\hat{\mathcal{D}}_d(\hat{w}) \;:=\; \frac{(\underline{1}'\hat{\Sigma}^{-1}\underline{1})^{-1}}{\hat{w}'\hat{\Sigma}\hat{w}}. \tag{26}$$

To calculate the bias-free version (26), it must be noted that estimation error is not only prominent for the covariance matrix estimator $\hat{\Sigma}$, but, as stated above, even more so in the estimator for the portfolio weights, $\hat{w}$.

The same holds for the short sale restricted version of the introduced measure of diversification, $\mathcal{D}_d^+(\hat{w})$, as defined by (25). Its estimator is defined in exact analogy to (26), but in the nominator, the estimate of the variance of the short sale restricted GMVP is used.

4.2.1 Diversification of the Naive Portfolio

In the case of an equally-weighted d-asset portfolio, the estimator of the measure of diversification reads

$$\hat{\mathcal{D}}_d(\frac{1}{d}\underline{1}) \;=\; \frac{\hat{\sigma}^2}{\hat{\sigma}_d^2} \;=\; \frac{\hat{\sigma}^2}{(\frac{1}{d}\underline{1})'\hat{\Sigma}(\frac{1}{d}\underline{1})} \;=\; \frac{\hat{\sigma}^2}{\frac{1}{d^2}\underline{1}'\hat{\Sigma}\underline{1}} \;=\; d^2\,\frac{\left(\underline{1}'\hat{\Sigma}^{-1}\underline{1}\right)^{-1}}{\underline{1}'\hat{\Sigma}\underline{1}}. \tag{27}$$

Now, with the distributions from (5) and (10), it holds that

$$\begin{aligned} \hat{\sigma}^2 \quad &\sim \quad \frac{1}{\underline{1}'\Sigma^{-1}\underline{1}}\,\frac{\chi^2_{T-d}}{T-1} \;=\; \sigma^2\,\frac{\chi^2_{T-d}}{T-1} \qquad \text{and} \\ \hat{\sigma}_d^2 \quad &\sim \quad \frac{\underline{1}'\Sigma\underline{1}}{d^2}\,\frac{\chi^2_{T-1}}{T-1} \;=\; \sigma_d^2\,\frac{\chi^2_{T-1}}{T-1}. \end{aligned}$$

If the respective χ^2-distributions in the above equations were independent, the estimator of the measure of diversification was distributed as some multiple of a $F_{T-d,T-1}$-distribution. In the case of (27), though, nominator and denominator are not independent, as both are formed using the sample covariance matrix $\hat{\Sigma}$.

Fortunately, the distribution of (27) can directly be obtained as a byproduct of Frahm and Memmel (2010, Theorem 9), as they examine a similar statistic.

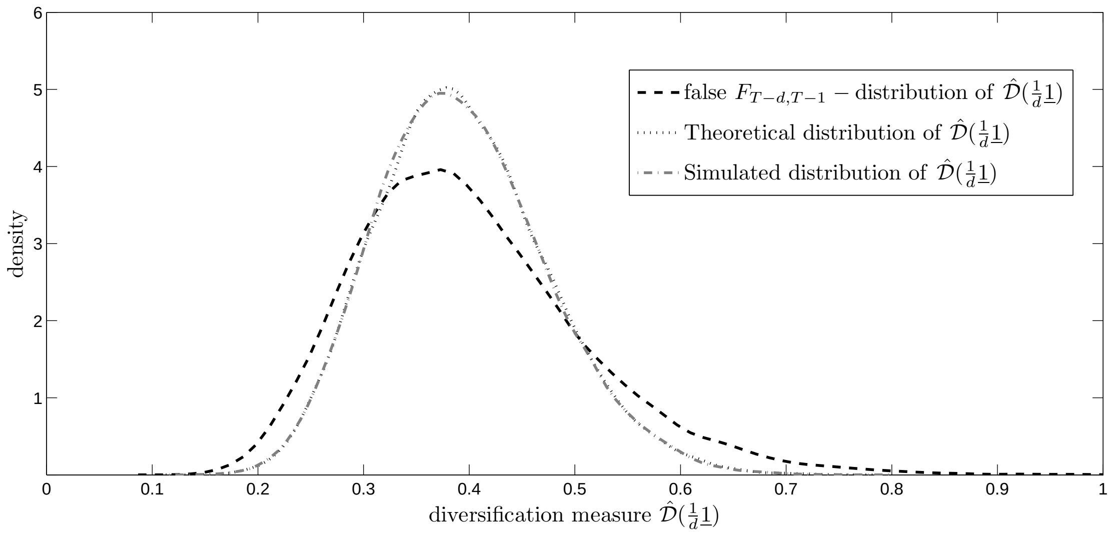
It holds that

$$\hat{\mathcal{D}}_d\Big(\frac{1}{d}\underline{1}\Big) \quad\sim\quad \left[1 \,+\, \frac{d-1}{T-d}\,F_{d-1,T-d}\Big((\mathcal{D}_d^{-1}(\frac{1}{d}\underline{1}) - 1)\,\chi^2_{T-1}\Big)\right]^{-1}, \tag{28}$$

where $F_{\nu_1,\nu_2}(\lambda)$ denotes a noncentral F-distribution with noncentrality parameter λ and ν_1 and ν_2 degrees of freedom in the nominator and denominator, respectively.

To illustrate the effect of the dependence of the nominator and denominator through $\hat{\Sigma}$ in the expression of $\hat{\mathcal{D}}_d\big(\frac{1}{d}\underline{1}\big)$, its theoretical distribution (28) is plotted in Figure 5, on the basis of a data-based simulation. The inappropriate $F_{T-d,T-1}$-distribution is also plotted. In a simulation with $d = 5$ assets, the respective distribution means are nearly identical with $0.383,\;0.382$ and 0.392, respectively, but the $F_{T-d,T-1}$-distribution has an inferior fit. Specifically, the $F_{T-d,T-1}$-distribution has heavier tails, leading to inconclusions or even mistakes whenever confidence intervals are constructed or hypothesis

Figure 5: Distribution of $\hat{\mathcal{D}}_d$ for the naive portfolio



Distribution of $\hat{\mathcal{D}}_d\big(\frac{1}{d}\underline{1}\big)$ for $T = 40$, $d = 5$ and $50\,000$ data-based simulation runs. The underlying data consists of 10 years of monthly returns of 5 stocks randomly selected from the S&P500 stocks from 2009. The departure of the finite-sample distribution of $\hat{\mathcal{D}}_d\big(\frac{1}{d}\underline{1}\big)$ from the false $F_{T-d,T-1}$-distribution can be seen to be substantial, while the simulated and the theoretical distributions nearly coincide.

tests are carried out.

Nevertheless, the above allows for the calculation of an unbiased estimator of $\mathcal{D}_d\Big(\frac{1}{d}\underline{1}\Big)$. In (6) and (10), unbiased estimators for the naive portfolio's and the GMVP's variance were deduced. Following this course of action, an unbiased estimator of the degree of diversification of the naive portfolio can be derived via taking the expectation of the right side of (28), yielding

$$\mathbb{E}\Big(\hat{\mathcal{D}}_d\Big(\frac{1}{d}\underline{1}\Big) \,|\, \mathcal{D}_d\Big(\frac{1}{d}\underline{1}\Big)\Big) \;\approx\; \frac{T-d}{T}\,\mathcal{D}_d\Big(\frac{1}{d}\underline{1}\Big) \;=\; \frac{T-d}{T}\,\frac{\sigma^2}{\sigma_d^2}. \tag{29}$$

The factor of bias-correction for $\hat{\mathcal{D}}_d\Big(\frac{1}{d}\Big)$ is the same as for the variance of the GMVP in (11) and equals $\frac{T-d}{T}$. This is in line with the intuition about the fraction (28), as there are no additional errors introduced by estimating portfolio weights for the naive portfolio.

Of course, for the measure $\hat{\mathcal{D}}_d$, the underestimation of the GMVP variance by the traditional estimator means that the true degree of diversification is higher than estimated by its biased version.

It is a well-known fact that imposing short sale restrictions on the GMVP mitigates the estimation error, see, e.g., Jagannathan and Ma (2003). Thus, $\mathcal{D}_d^+$ can be expected to be less biased than $\mathcal{D}_d$. Nevertheless, the measure $\mathcal{D}_d^+$, when applied to the naive d-asset portfolio, will also be bias-corrected, with the same correction term used for $\hat{\mathcal{D}}_d(\frac{1}{d}\underline{1})$.

Firstly, this will make comparison between $\hat{\mathcal{D}}_d$ and $\hat{\mathcal{D}}_d^+$ easier when applied to naively allocated portfolios.

Secondly, doing so makes $\hat{\mathcal{D}}_d^+$ an *optimistic* measure when assessing the diversification of an equally-weighted portfolio.

4.2.2 Diversification of any Constant Portfolio

Extending the result of the preceding paragraph, the distribution of $\hat{\mathcal{D}}_d(\underline{c})$ can be calculated whenever the portfolio weights $\underline{c}$ are constant and sum up to one, i.e., when $\underline{c}'\underline{1} = 1$ holds true. This is the case whenever these weights are not estimated from the data, implying that they are not subject to any estimation risk. Investors who wish to allocate certain but not necessarily equal amounts of their wealth to certain industries or sectors, might be characterized by this behavior.

In the same way as above, utilizing the rules for the Wishart distribution, it holds that

$$\hat{\sigma}_{\underline{c}}^2 \;:=\; \underline{c}\hat{\Sigma}\underline{c} \quad\sim\quad \underline{c}'\Sigma\underline{c}\,\frac{\chi_{T-1}^2}{T-1} \;=\; \sigma_{\underline{c}}^2\,\frac{\chi_{T-1}^2}{T-1},$$

where $\sigma_{\underline{c}}^2$ denotes the true return variance of the constant portfolio $\underline{c}$.

Following again Frahm and Memmel (2010, Theorem 9), the distribution of $\hat{\mathcal{D}}_d(\underline{c})$ can be derived as

$$\hat{\mathcal{D}}_d(\underline{c}) \quad\sim\quad \left[1 + \frac{d-1}{T-d}\, F_{d-1,T-d}\Big(\big(\mathcal{D}_d^{-1}(\underline{c}) - 1\big)\,\chi_{T-1}^2\Big)\right]^{-1}, \tag{30}$$

where, as above, $F_{\nu_1,\nu_2}(\lambda)$ denotes a noncentral F-distribution with noncentrality parameter λ and ν_1 and ν_2 degrees of freedom in the nominator and denominator, respectively.

Clearly, the bias correction is the same as for the estimated measure of diversification for the equally-weighted portfolio.

4.3 Asymptotic Properties of the Estimator of $\mathcal{D}_d$

Whenever a large number T of observations of asset returns is available for a fixed set of d assets, or at least when the number of observations relative to the number of assets is fixed, the asymptotic behavior of the measure $\hat{\mathcal{D}}_d(\frac{1}{d}\underline{1})$ is of interest.

In the case of the naive portfolio, the asymptotic behavior of $\hat{\mathcal{D}}_d(\frac{1}{d}\underline{1})$ in the case where the number of observations grows to infinity is as expected, i.e., the estimator is asymptotically unbiased.

For fixed d and $T \longrightarrow \infty$, it holds that

$$\hat{\mathcal{D}}_d\Big(\frac{1}{d}\underline{1}\Big) \quad \sim \quad \Big[1 + \frac{d-1}{T-d}\, F_{d-1,T-d}\Big(\big(\mathcal{D}_d^{-1}(\frac{1}{d}\underline{1}) - 1\big)\, \chi^2_{T-1}\Big)\Big]^{-1} \xrightarrow{\text{dist.}} \mathcal{D}_d\Big(\frac{1}{d}\underline{1}\Big). \tag{31}$$

The rather theoretical case $T \to \infty,\ d \to \infty,\ \frac{T}{d} \to Q < \infty$ results in

$$\hat{\mathcal{D}}_d\Big(\frac{1}{d}\underline{1}\Big) \quad \xrightarrow{\text{dist.}} \quad (1 - \frac{1}{Q})\, \mathcal{D}_d\Big(\frac{1}{d}\underline{1}\Big).$$

This second result shows that whenever both, the number of observations and the number of assets grow in a constant proportion, the estimator stays asymptotically biased by the factor $(1 - \frac{1}{Q})$. The bias in this case can be viewed as an heritage from the estimator of the GMVP variance $\hat{\sigma}^2$, and needs to be taken care of whenever large investment universes are under consideration, even if the number T of observations of historical returns is large.

The asymptotic properties of the diversification measure for any constant portfolio $\underline{c}$ are in exact analogy.

5 Testing for Variance and Diversification

Most of the tests given in this section are basic tests from the theory of univariate statistics. For an investor caring about not exceeding some prespecified level of variance, however, they might be useful instruments. Moreover, testing the variance of an equally-weighted portfolio against the variance of the GMVP is not a standard method. Thus, this section can be viewed as a *toolbox* for controlling and testing investment decisions.

5.1 Variance Tests

A comprehensive overview of some well-known statistical hypothesis tests for variances is given. For all of these tests, let α denote some significance level, on which the tests are based. Furthermore, it is assumed that the investor has historical return data of length T at hand, from which he estimates the expected portfolio variance. Moreover, the historical return data $(R_1, \ldots, R_T)$ is assumed to stem from a multivariate normal distribution, cp. (1).

5.1.1 Naive Portfolio Variance

To test whether the variance σ_d^2 of the naive d-asset portfolio exceeds some constant threshold value $\bar{\sigma}^2$, the following alternatives are to be tested:

$$H_0 \ : \ \sigma_d^2 \ \geq \ \bar{\sigma}_d^2 \qquad \text{vs.} \qquad H_1 \ : \ \sigma_d^2 \ < \ \bar{\sigma}_d^2.$$

The valid test statistic for this setting and its distribution under H_0 is

$$S_d \ := \ (T-1)\frac{\hat{\sigma}_d^2}{\bar{\sigma}^2} \quad \sim \quad \chi^2_{T-1}. \tag{32}$$

Thus, to test the hypothesis whether the variance of a given portfolio exceeds $\bar{\sigma}^2$ on a given confidence level α, H_0 can be rejected whenever

$$S_d < \chi^{2\,-1}_{T-1}(1-\alpha). \tag{33}$$

A more interesting test for the investor who believes in naive asset allocation might be as follows. He might ask wheter for his naive portfolio of d assets, has adding k more assets to this portfolio and rebalancing the weights towards $\frac{1}{d+k}\underline{1}$ a reducing impact on its variance, or whether he can safely keep his portfolio of d assets.

To this end, the alternatives are given by

$$H_0 \ : \ \sigma_{d+k}^2 \ \geq \ \sigma_d^2 \qquad \text{vs.} \qquad H_1 \ : \ \sigma_{d+k}^2 \ < \ \sigma_d^2, \tag{34}$$

where σ_d^2 and σ_{d+k}^2 denote the variances of the naively allocated d-asset portfolio and of the naive portfolio with k new assets added, respectively. In a survey paper, Zhang (1998) gives the test statistic for the differences of the two variances and its distribution under H_0 as

$$t \ = \ \frac{\hat{\sigma}_{d+k}^2 - \hat{\sigma}_d^2}{\sqrt{4(\hat{\sigma}_d^2\hat{\sigma}_{d+k}^2 - \hat{\sigma}_{d,d+k}^2)/(T-2)}} \quad \sim \quad t_{T-2}, \tag{35}$$

where $\hat{\sigma}_{d,d+k}$ is the sample covariance of the portfolios' returns, and t_{T-2} denotes a t-distribution with $T-2$ degrees of freedom.

Thus, H_0 can be rejected whenever $t \ < \ t_{T-2}^{-1}(1-\alpha)$.

A derivation of the test statistic (35) is given in Memmel (2004, Appendix 8). Seeking a possibility to compare two empirical return variances σ_i^2 and σ_j^2, Memmel uses the fact that when two assets i and j have equal variances, their weight in a 2-asset global minimum-variance portfolio equals $\frac{1}{2}$, respectively. With the finite-sample distribution of the GMVP weights in the context of normally distributed asset returns at hand, the test is readily comprehensible. For further details, the reader might also be interested in Kempf and Memmel (2006) or Frahm (2010).

5.1.2 GMVP Variance

In the same style as for the naive portfolio variance, the question arises whether the d-asset GMVP return variance σ^2 exceeds some prespecified benchmark $\bar{\sigma}^2$. The alternatives are given by

$$H_0 \;:\; \sigma^2 \;\geq\; \bar{\sigma}^2 \quad \text{vs.} \quad H_1 \;:\; \sigma^2 \;<\; \bar{\sigma}^2,$$

for which the well-known test statistic is $S_{\mathrm{GMVP}} \;:=\; (T-1)\,\hat{\sigma}^2/\bar{\sigma}^2$, which under H_0 is χ^2_{T-d}-distributed. Thus, to reject the null hypothesis, it must hold that

$$S_{\mathrm{GMVP}} \;<\; \chi^{2\;-1}_{T-d}(1-\alpha).$$

Note the crucial role of d, the number of assets, in the critical value $\chi^{2\;-1}_{T-d}(1-\alpha)$.

5.1.3 Comparing the Naive Portfolio to the GMVP

Another interesting question to ask for an investor might be whether - in terms of variance - it is fruitful to allocate his wealth equally among d assets or to apply some variance minimization technique to historical data and to invest into the GMVP constituted by these d assets. To this end, the alternatives

$$H_0 \;:\; \sigma^2 \;\geq\; \sigma^2_d \quad \text{vs.} \quad H_1 \;:\; \sigma^2 \;<\; \sigma^2_d \tag{36}$$

are tested, where, as above, σ^2 denotes the variance of the d-asset-GVMP, and σ^2_d denotes the naive d-asset-portfolio's variance. As mentioned above, Kempf and Memmel (2006) derive the finite-sample distribution for the GMVP weight vector by proving that the weights of the GMVP can be obtained as the coefficients of an ordinary least squares regression. Thus, the above test can be stated as the alternative whether the regression coefficients significantly deviate from $\frac{1}{d}$. In turn, this question can be cast in the well-known framework of testing the restricted regression model versus the original model.

The test statistic for the above alternatives reads

$$S_{\text{GMVP}}^{\text{naive}} \;=\; \frac{T-d}{d-1}\left(\frac{\hat{\sigma}_d^2}{\hat{\sigma}^2} \;-\; 1\right) \tag{37}$$

and, under H_0, has a $F_{d-1,T-d}$-distribution.

As such, H_0 can be rejected whenever

$$S_{\text{GMVP}}^{\text{naive}} \;<\; F_{d-1,T-d}^{-1}(1-\alpha),$$

in which case a significant reduction in variance is obtained by allocating the d assets according to the GMVP strategy.

5.2 Testing with the Measure $\mathcal{D}_d$

With the finite-sample distribution of the measure of diversification $\mathcal{D}_d$ at hand, it is possible to test whether a naively allocated portfolio attains a certain degree of diversification. For $\gamma \in [0,1]$, the alternatives are

$$H_0 \;:\; \mathcal{D}_d\Big(\frac{1}{d}\underline{1}\Big) \;<\; \gamma \qquad \text{vs.} \qquad H_1 \;:\; \mathcal{D}_d\Big(\frac{1}{d}\underline{1}\Big) \;\geq\; \gamma. \tag{38}$$

Whenever H_0 can be rejected, the investor can be confident that $\gamma \cdot 100\%$ of diversifiable risk is eliminated by naively allocating his wealth among his assets of choice.

Given the finite-sample distribution $F_{\mathcal{D}}$ of the diversification measure for the case of the equally-weighted d-asset portfolio as in (28), it is possible to calculate the appropriate critical value $F_{\mathcal{D}}^{-1}(1-\alpha)$ that must be exceeded in order to reject H_0.

Even though these critical values cannot be obtained from a table, for they are a mixture of two distributions, they can be simulated for any given γ. The value $F_{\mathcal{D}}^{-1}(1-\alpha)$ then depends on the diversification level γ to be tested as well as on the chosen confidence level α.

6 Empirical Study

This section gives empirical results for the levels of variance measured in naively allocated as well as minimum-variance portfolios. Also, the different measures of diversification, as introduced in the preceding section, are evaluated on actual return data. More precisely, the estimators for $\mathcal{D}_d$, $\mathcal{D}_d^+$ and $\mathcal{N}_{Ent}/d$ are tested on equally-weighted portfolios.

The broad basis of S&P500 constituents are considered, which have been obtained from the CRSP database. For each year between 1965 and 2009, all constituents with a continuous return history of 120 months were downloaded, yielding a maximal number d of assets between 335 (in 1965) and 461 (in 1983).

Despite these large numbers of available assets, one must keep in mind that estimation of the covariance matrix becomes meaningless for practical purposes once the number of assets exceeds the number of monthly observations of returns. Even for the case $d < T$, the variance estimated for the GMVP turns to be highly unreliable as d approaches T, cp. Section 2.

After having retrieved the data from CRSP, the monthly asset returns are converted into excess returns using the 3-month treasury bills for the corresponding month. The interest data for the calculation of excess returns are obtained from the Federal Reserve System (2010).

6.1 Evolution of the Naive Portfolio Return Variance over Time

Figures 6 and 7 show the average variances of the naively allocated d-asset-portfolios plotted against d, with d ranging between 2 and 40.

For each period's end, 1965 and 2009, and for each d between 1 and 40, a total number of 100 naively allocated d-asset portfolios are built. The constituents of each of these portfolios are drawn randomly from the S&P500 stocks of the respective period. The average variance $\overline{\sigma}_d^2$ belonging to each period is computed as the simple average of these 100 d-asset portfolios' individual variances. Afterwards, the theoretically correct model (4) for normally distributed asset returns,

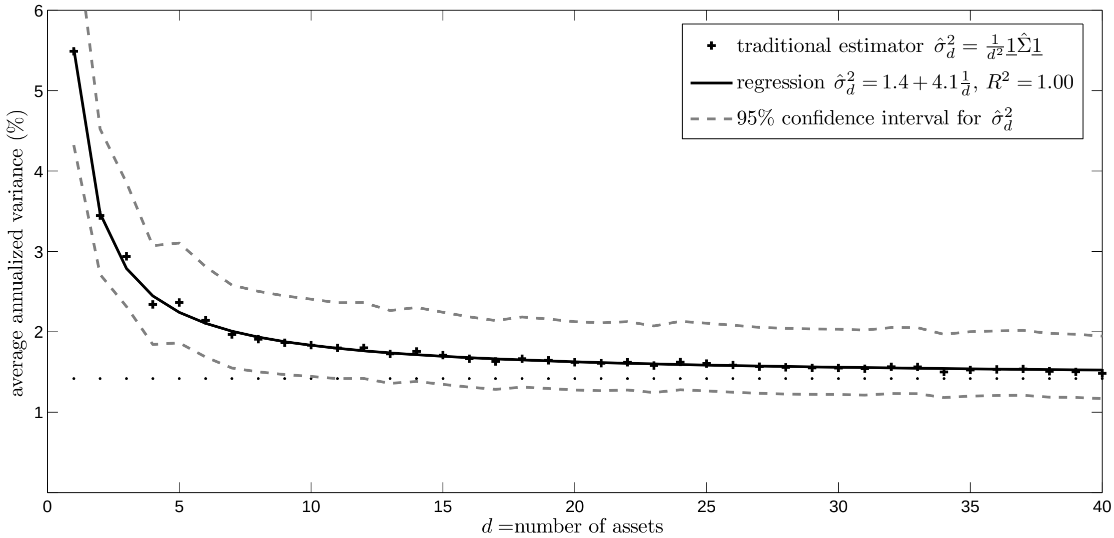
$$\overline{\sigma}_d^2 \,=\, \alpha \,+\, \beta \,\frac{1}{d} \,+\, \varepsilon,$$

is fitted to the data for each period.

The asymmetric confidence intervals displayed around the variance estimate are due to its χ^2-distribution, see (5). Both figures show that on average, the decomposition model (4) for the equally-weighted d-asset portfolio return variance gives a good fit, as supported by large R^2 for both regressions.

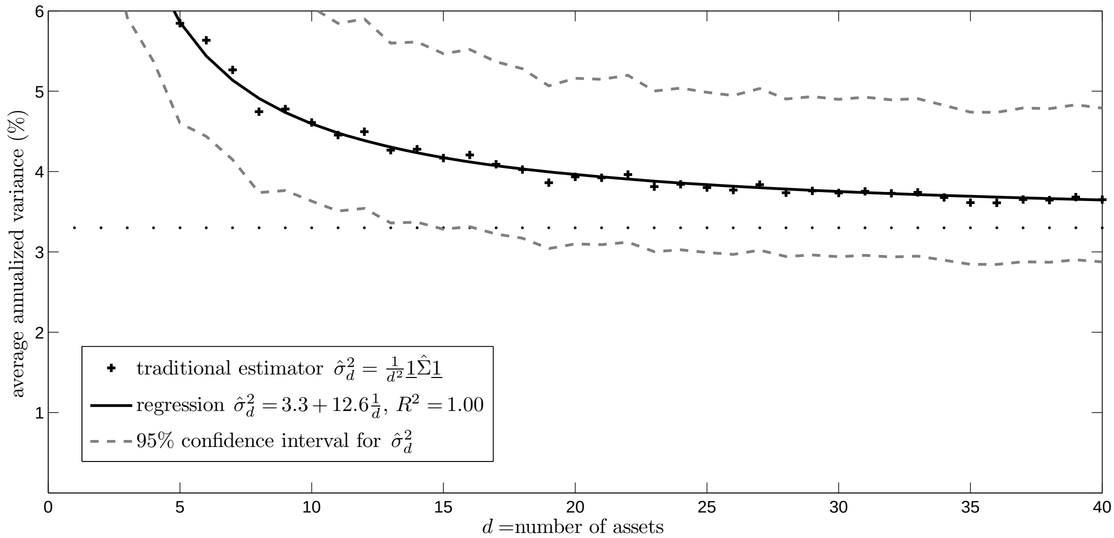
Another well-documented fact in finance literature is the time-varying basic level of return variation. When comparing the two decades, this fact can also be reconfirmed. In the 10-year-period ending 1965, this basic level is 1.4%, while in the period ending 2009, it amounts to about 3.3%.

Figure 6: Variance of the naive d-asset portfolio in 1965



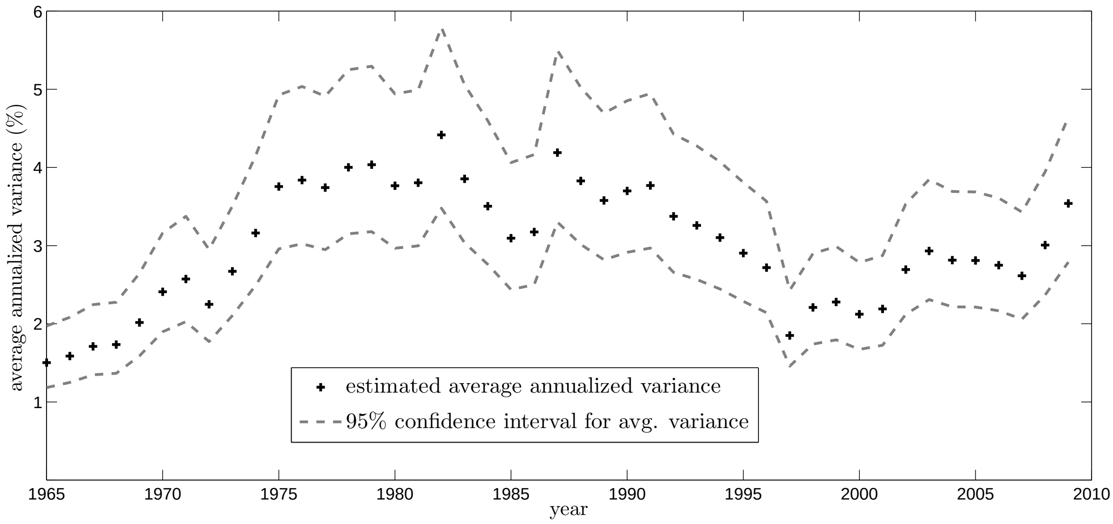
Average return variance of naively allocated d-asset portfolios plotted against d. In line with the theory, the average variance quickly drops towards some basic level of variance. S&P500 constituents from 1965 are used to randomly build equally-weighted portfolios of size d, for d between 1 and 40. The data used for the variance estimation consists of 120 monthly excess returns.

Figure 7: Variance of the naive d-asset portfolio in 2009



Average return variance of naively allocated d-asset portfolios plotted against d. In line with the theory, the average variance quickly drops towards some basic level of variance. S&P500 constituents from 2009 are used to randomly build equally-weighted portfolios of size d, for d between 1 and 40. The data used for the variance estimation consists of 120 monthly excess returns.

Figure 8: The average naive 40-asset portfolio return variance



Average basic level of return variance of naively allocated portfolios over the last 5 decades. The estimates are obtained using 120 months of excess return data for each year. For example, the variance estimate for 1965 is obtained using the excess returns from Jan 1956 through Dec 1965.

The evolution of the basic levels of the average estimated basic levels of variation for naively allocated portfolios for the years 1965 - 2009 is shown in Figure 8. The methodology used to obtain the variance estimates as well as the confidence bounds is the same as for Figures 6 and 7.

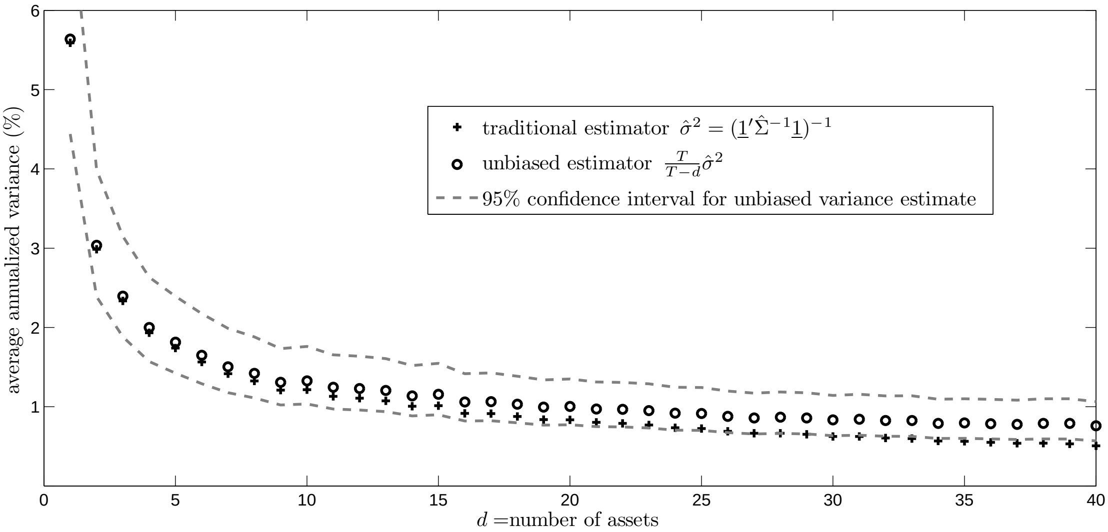
6.2 Evolution of the GMVP Return Variance over Time

For growing portfolio sizes d, the average estimated variance of the GMVP can be expected to be decreasing. As the estimated GMVP is the result of an optimization process, see (7), the estimated variance of a $(d+1)$-asset GMVP must be smaller than that of the GMVP without the additional asset.

Even though, as pointed out in (11), the estimated variance is heavily biased, especially when the effective sample size $Q = \frac{T}{d}$ is small, see also Figure 1. Thus, the gap between the average estimated return variance of the d-asset GMVP and its unbiased counterpart increases for growing d.

Similarly, the confidence bounds for the true average variance become larger with growing d, while the asymmetry of the confidence interval again is due to the χ^2-distribution of the variance estimator, see (10).

Figure 9: Average return variance of the d-asset GMVP in 1965



Average return variance of the d-asset GMVP plotted against d. The estimated average variance quickly drops towards some low level of variance. Randomly selected S&P500 constituents from 1965 are used to estimate the d-asset GMVPs, for d between 1 and 40. The data used for the variance estimation consists of 120 monthly excess returns. The growing discrepancy between the variance estimates for higher d stems from the bias which depends crucially on the portfolio size d.

Comparing the periods 1965 and 2009, as carried out in Figures 9 and 10, the basic levels of variation are different. While in 1965, a 40-asset GMVP had an average estimated variance of 0.75%, the 2009 unbiased estimate is about 1.2%.

Figure 11 gives the estimated unbiased levels of the GMVP variance for all years between 1965 and 2009.

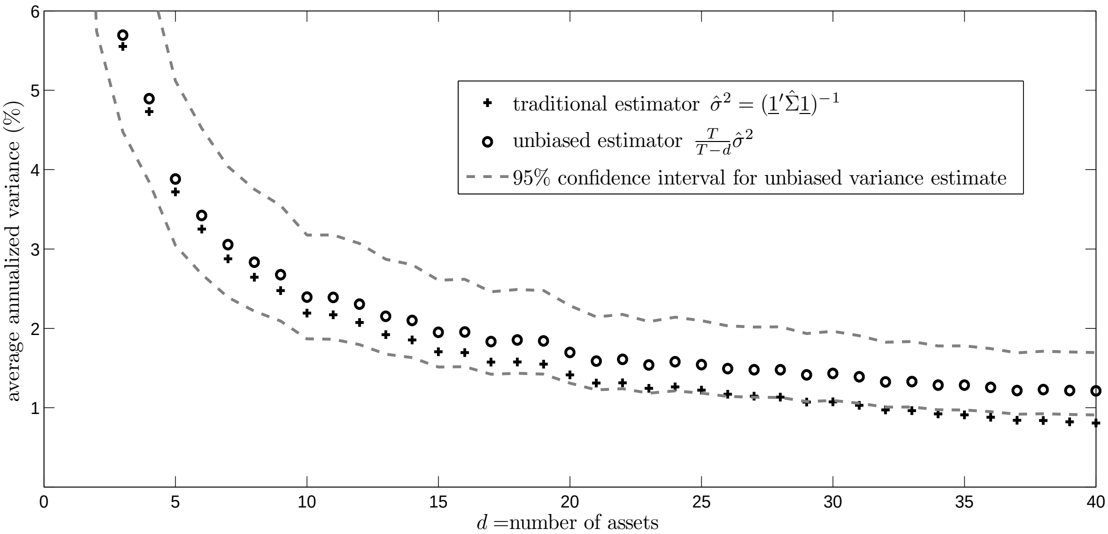
6.3 Diversification of Naively Allocated Portfolios

The diversification measure (24) introduced in this paper relates the variance of a given portfolio w, comprising of at most d assets, to the d-asset GMVP composed of the same d assets.

Figure 12 shows the average values of $\hat{\mathcal{D}}_{20}(\frac{1}{i}\underline{1})$, that is, the average degree of diversification of the naive i-asset portfolio as i is varied from 1 to 20. In spite of the general perception that a portfolio with many constituents can be safely taken to be well-diversified, it is apparent that for 20 assets, on average only about 40% of the removable return variation is eliminated by allocating naively.

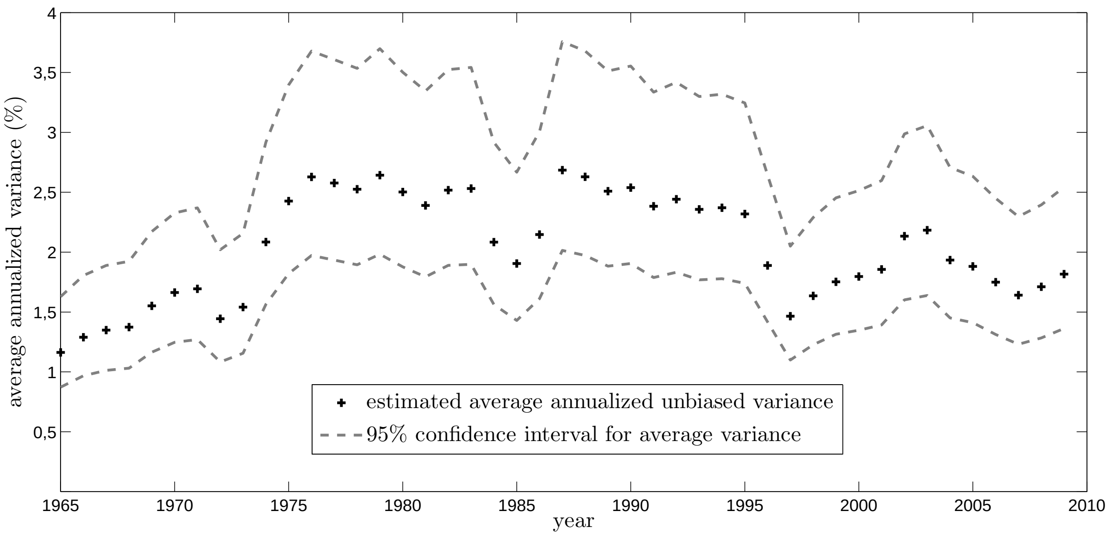
Also, it is observed that the diversification degree increases for growing i. This is in line with the general perception of the fact that a portfolio concentrated on few assets has an inferior diversification than a portfolio spread among more assets. Note,

Figure 10: Average return variance of the d-asset GMVP in 2009



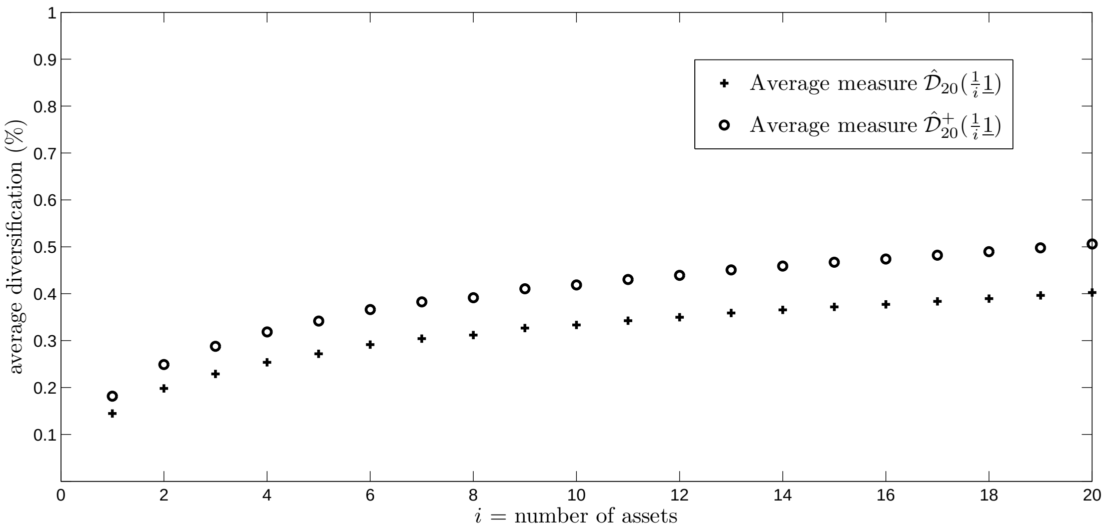
Average return variance of the d-asset GMVP plotted against d. The estimated average variance quickly drops towards some low level of variance. Randomly selected S&P500 constituents from 2009 are used to estimate the d-asset GMVPs, for d between 1 and 40. The data used for the variance estimation consists of 120 monthly excess returns. The growing discrepancy between the variance estimates for higher d stems from the bias which depends crucially on the portfolio size d.

Figure 11: The GMVP variance evolution from 1965 to 2009



Average basic level of return variance of the GMVP over the last 5 decades. The estimates are obtained using 120 months of excess return data for each year. For example, the variance estimate for 1965 is obtained using the excess returns from Jan 1956 through Dec 1965.

Figure 12: Equally-weighted portfolios evaluated by $\hat{\mathcal{D}}_{20}$ and $\hat{\mathcal{D}}_{20}^{+}$ in 2009



For each portfolio size i, the average measure of diversification of the naive portfolio is given. It relates the variance of the i-asset naive portfolio to the return variance of the 20-asset GMVP. Estimation data are 120 monthly excess returns from randomly chosen S&P500 constituents in 2009.

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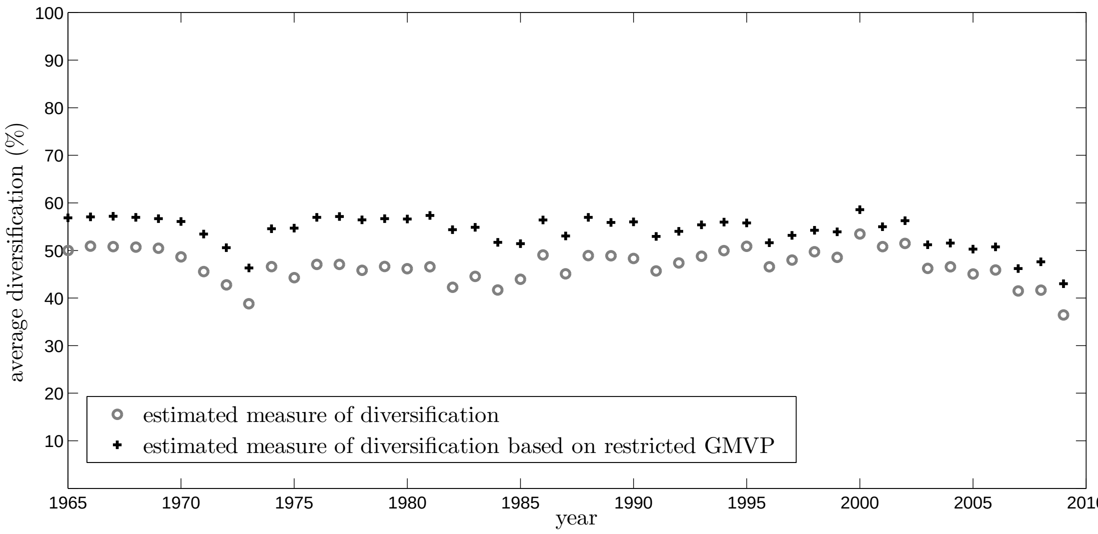
though, that in this case, comparison is always made to the 20-asset GMVP, and not to the respective i-asset GMVP.

In Section 4, it was already mentioned that the GMVP estimator $\hat{w}_{\mathrm{GMVP}}$ used to evaluate $\mathcal{D}_d$ might be restricted to carry positive weights only. This restriction is common to the private investor as well as to the manager of a mutual fund, and as such it constitutes a more realistic *benchmark* to which a portfolio return's variation should be compared. As this has a heightening effect on $\mathcal{D}_d$, the upper curve in Figure 12 shows the average diversification when the measure $\mathcal{D}_d^{+}$ is applied.

Nevertheless, even with this modification, the naive portfolio only uses less than half of the diversification potential that exists among the assets.

Another interesting fact is displayed in Figure 13. The average value of diversification of naively allocated 20-asset portfolios can be seen to lie between 30% and 50%, with a sharp decline in the most recent years 2008 and 2009.

Figure 13: Equally-weighted portfolios evaluated by $\hat{\mathcal{D}}_{20}$ for 1965-2009



For each year between 1965 and 2009, the average measure of diversification of the 20-asset naive portfolio is given. It relates the variance of the 20-asset naive portfolio to the return variance of the 20-asset GMVP, and to obtain an average value, this procedure is repeated 100 times. In each of these trials, 20 assets are selected randomly from the S&P500 in the respective year. Estimation data consists of 120 monthly excess returns for these assets.

6.4 Empirical Evaluation of the Different Measures

A comparison of the different measures of diversification introduced in Section 3 shall be given now. Thus, leading Question 4 shall be answered, whether - empirically - the measures are in line with the economic intuition that a d-asset portfolio comprising assets from different sectors is more diversified than a d-asset portfolio comprising of assets from only a few sectors.

For this purpose, the CRSP dataset of S&P500 constituents from 2009 will be used, divided by sectors. Again, only companies with a continuous monthly return history of at least 10 years are considered. Table 1 gives an overview of the sectors and the number of assets grouped to each sector, given that they fulfill the constraint on their return history.

The first comparison is related to two equally-weighted portfolios of sizes $d = 10$ and $d = 20$, respectively. While the 10-asset portfolio comprises of randomly selected assets from each of the S&P500 sectors, the 20-asset portfolio comprises of the same assets as the 10-asset portfolio plus again one randomly selected asset from each sector's residual assets. Thus, both portfolios should be rather well-diversified. The actual composition of the portfolios are given in Table 2. When applying the different measures of diversification to these portfolios, the obtained results are displayed in Table

Table 1: S&P500 stocks grouped by sectors

S&P500 Sector	**Abbreviation**	**Number of Assets**
Consumer Discretionary	CD	69
Consumer Staples	CS	35
Energy	EN	36
Financials	FIN	68
Health Care	HC	46
Industrials	IND	57
Information Technology	IT	64
Materials	MAT	30
Telecommunications	TEL	7
Utilities	UTIL	34
total		**446**

Number of constituents of the S&P500 index at the end of 2009, grouped by sectors. The total number is less than 500, as the requirement of a continuous monthly return history of at least 10 years is not met by every company in the S&P500.

3. For an equally-weighted portfolio, the complement of the Herfindahl index (CHI), defined by (12), yields the numbers 0.9 for the 10-asset portfolio and 0.975 for the 20-asset portfolio as diversification indices. The reason is that this measure depends on the weights of the assets, but not on their return characteristics.

The estimate $\hat{\mathcal{N}}_{Ent}$ of the measure introduced by Meucci (2009) shows large values of diversification for both portfolios. The fraction $\hat{\mathcal{N}}_{Ent}/d$ is introduced to make Meucci's index of portfolio diversification comparable to the other indices. As $0 \leq \mathcal{N}_{Ent} \leq d$, it holds that $\mathcal{N}_{Ent}/d$ takes on values between zero and one, and can thus be interpreted as percentage of diversification achieved by some portfolio as measured by Meucci (2009).

The measure $\hat{\mathcal{D}}_d^+$ equals the extension of the measure $\hat{\mathcal{D}}_d$ introduced in Section 4, but instead of using the estimated variance of the unconstrained GMVP in the nominator of the definition of $\mathcal{D}_d$, given by (24), it utilizes the estimated variance of the GMVP with short sale constraint.

Finally, the variance of the two portfolios is reported in order to assess the absolute reduction in variation of the portfolio returns. The shortcoming of the variance alone as a measure of diversification was already discussed in Section 3.

As the portfolio is equally-weighted, the measure CHI does not contribute any more information than the very number of its constituents already does.

Table 2: Composition of the two equally-weighted portfolios

Sector	**10-asset portfolio**	**20-asset portfolio**
CD	Gannett Inc	Stanley Works
CS	Procter & Gamble Co	Pepsico Inc
EN	Nabors Industries Ltd	Baker Hughes Inc
FIN	Kimco Realty Corp	Morgan Stanley Dean Witter & Co
HC	Bristol Myers Squibb Co	Dentsply International Inc New
IND	Norfolk Southern Corp	Dover Corp
IT	C A Inc	Advanced Micro Devices Inc
MAT	Airgas Inc	Weyerhaeuser Co
TEL	Verizon Communications Inc	Centurytel Inc
UTIL	Duke Energy Corp New	Ameren Corp

Constituents of the equally-weighted portfolios. Both include assets from all S&P500 sectors effective 2009, indicated by the left column. The portfolios are equally-weighted on these assets.

Table 3: Resulting estimates for the diversification measures

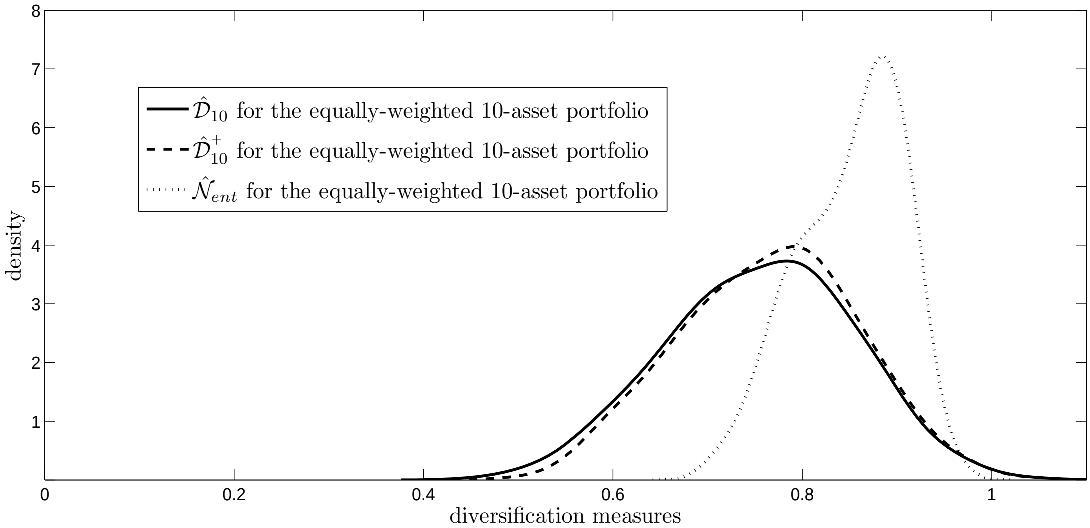
Measure	**10-asset portfolio**	**20-asset portfolio**	**reduction (%)**
CHI	0.9	0.975	-8.33
$\mathcal{N}_{Ent}$	8.247	14.545	
$\mathcal{N}_{Ent}/d$	0.825	0.727	11.88
$\mathcal{D}_d$	0.560	0.494	11.79
$\mathcal{D}_d^+$	0.591	0.533	9.81
$\mathbb{V}$ar	0.00325	0.00294	9.54

The different measures introduced in Section 3 as well as the proposed measures of diversification are used for the evaluation of the equally-weighted portfolios described in Table 2. The percentage of reduction for $\mathcal{N}_{Ent}$ is not reported as this measure is not normed.

The measure $\hat{\mathcal{N}}_{Ent}/d$ ascribes high degrees of diversification to both, the 10-asset portfolio and the 20-asset portfolio, with 82.5% and 72.7%, respectively. As expected, the measures $\hat{\mathcal{D}}_d$ and its short sale constrained version $\hat{\mathcal{D}}_d^+$ yield much more conservative estimates, as they relate the actual variance estimates to the respective GMVP variance estimate. In that case, the measure states that for both portfolios, only 50% to 60% of the diversification potential is exploited.

With regard to leading Question 4, it must be conceded that the measure $\mathcal{D}_d$ does not express the wide-spread intuition that a portfolio of a large number of different

Figure 14: Different measures applied to a one-sector portfolio



The measures $\hat{\mathcal{D}}_d$, $\hat{\mathcal{D}}_d^+$ and $\hat{\mathcal{N}}_{Ent}/d$, applied to 10-asset equally-weighted portfolios, respectively. In a row of 500 trials, $d = 10$ assets are randomly chosen from the constituents of the sector *Health Care* in the 2009 S&P500, and afterwards, the measures are estimated. While the difference between $\mathcal{D}_d$ and $\mathcal{D}_d^+$ is small, and both measures on average yield 0.78, the measure $\mathcal{N}_{Ent}$ attests a high degree of diversification among the assets.

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industries is automatically well-diversified. This is because when a portfolio comprises stocks from a large number of industries, idiosyncratic shocks can better be smoothed by allocating the stocks unevenly, according to some minimum-variance strategy.

Next, the effect of the measures on portfolios that are allocated equally among a single sector of the S&P500 only is investigated, see Figure 14. For the measure $\mathcal{N}_{Ent}$, astonishingly high degrees of diversification are estimated.

But also the estimators of the measures $\mathcal{D}_d$ and $\mathcal{D}_d^+$, which are rather conservative as compared to $\mathcal{N}_{Ent}$, yield comparatively high numbers.

Of course, when comparing the results for the sector-wide diversified portfolio, and for the average one-sector portfolios as in Figure 14, the results are contra-intuitive in the sense that the sector-wide diversified portfolios should be assigned a higher degree of diversification.

But as both types of portfolios are benchmarked differently, the 10-asset portfolio from one sector only does not allow for as much reduction in the return variance as the 10-asset portfolio comprising assets from all sectors. Thus, the reduction by allocating 10 assets from one sector equally accounts for a larger reduction towards the lowest possible variance among these assets than in the case where 10 assets are allocated from 10 different sectors.

7 Conclusion

The measurement of volatility of portfolios, and even more of movements in entire financial markets is of great interest for the investment industry. The focus of quantitative portfolio optimization has shifted away from the traditional sample-based Markowitz approach, and the estimation of means is often left to analysts who generate qualitative forecasts based on forward-looking figures rather than on historical return data.

The fact that there is only little work available about how to measure the diversification effect itself, which is the basic tenet of Markowitz's work, is surprising when realizing that in recent years, minimum-variance strategies have gained attraction not only from the researcher's perspective, but also from the practitioner's point of view.

This paper gives a comprehensive overview of previous approaches towards determining this quantity. The shortcomings of some of these approaches are pointed out, and an own measure is introduced, relating the acutal portfolio risk to the minimal risk that cannot be avoided when combining the given assets.

In the context of normally distributed asset returns, the problem of estimating this measure is considered. Moreover, the finite-sample distribution of this measure, when applied to the constant portfolio, is given, with the special case of the measure's finite-sample distribution when the diversification of an equally-weighted portfolio is examined.

Empirically, the validity of former studies on average diversification of the equally-weighted portfolio and of the change of regimes of overall market volatility can be reconfirmed. The basis for the empirical part is provided by monthly return data of the S&P500 constituents from the last five decades.

For the recently advocated strategy of equally-weighted portfolios, see DeMiguel et al. (2009a) or DeMiguel et al. (2009b), the presented study shows that the degree of diversification crucially depends on the assets chosen. When choosing assets with similar return characteristics, like assets being allocated in the same sector, the measure yields a satisfactory level of diversification. When investing into different sectors, historically, only about 40% to 60% of the diversifiable risk is removed, indicated by the corresponding values for $\hat{\mathcal{D}}_d$. This fact is due to the possibility of allocating the assets according to the minimum-variance rule, which results in drastically lower return variance.

Even for the setting where the equally-weighted portfolio return variance is related to the variance of a short sale constrained minimum-variance portfolio, measured by $\hat{\mathcal{D}}_d^+$, the degree diversification potential for such portfolios can scarcely be judged to have increased.

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