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DISKUSSIONSBEITRÄGE ZUR STATISTIK UND ÖKONOMETRIE

SEMINAR FÜR WIRTSCHAFTS- UND SOZIALSTATISTIK
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A Power Comparison of Homogeneity Tests in Mixtures of Exponentials

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October 1997

Abstract

The empirical power of several test procedures is studied which test for homogeneity against mixtures of exponential distributions.

AMS 1991 *subject classifications.* Primary 62G10; secondary 62N05.
Key words and phrases. Mixture components, mixture diagnosis, survival analysis, hazard models, overdispersion, goodness-of-fit, likelihood ratio tests.

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1 Homogeneity Tests in Mixtures of Exponentials

In survival analysis the distribution of a lifetime Y can often be modeled by a mixture of exponential distributions; for some theory and applications see the recent survey by McLachlan (1995). Then, with some unknown mixing distribution π on $\mathbb{R}_+$, the survival function reads

$$S(y) = P[Y > y] = \int_{\mathbb{R}_+} e^{-\frac{y}{u}} d\pi(u), \quad y \geq 0. \tag{1}$$

We assume that π has finite fourth moment and $\int_{\mathbb{R}_+} u d\pi(u) > 0$. An important problem in many applications is whether, given the data, π is concentrated at some single point $u_0 > 0$, in other words, whether Y follows a non-mixed exponential distribution with mean u_0, or not.

For testing the null hypothesis,

$$H_0 : \pi(\{u_0\}) = 1 \text{ at some } u_0 > 0, \tag{2}$$

i.e., "$H_0 : Y$ has density $\frac{1}{u_0}\exp(-\frac{y}{u_0})$", several approaches are practical, among them tests for overdispersion, likelihood procedures and goodness-of-fit tests. In this paper we investigate the power of five such tests and, in addition, two hybrid procedures, which are combinations of a goodness-of-fit test with another test. All these tests are invariant against scale transformations of the data.

The power of the tests is compared on the following alternative hypotheses.
1. Two-component mixtures having density

$$f_Y(y) = (1-\epsilon)\frac{1}{u}\exp(-\frac{y}{u}) + \epsilon\frac{1}{v}\exp(-\frac{y}{v}), \quad y \geq 0. \tag{3}$$

As the tests statistics are scale invariant, u is fixed to $u = 1$, while ϵ and v are systematically varied.
2. Three-component mixtures generated by a mixing distribution having mass ϵ_j at three points u_j, $j = 1, 2, 3$, where $u_1 = 1/\beta$, $u_2 = 1$ and $u_3 = \beta$, for various values of β and ϵ_j.
3. Five-component mixtures generated by a discrete uniform mixing distribution on $u_1 = 1/\beta^2$, $u_2 = 1/\beta$, $u_3 = 1$, $u_4 = \beta$ and $u_5 = \beta^2$, for various values of β.
4. Continuous mixtures: Here π is (4a) exponential with mean at 1, (4b) continuous uniform on $[0, 1]$, (4c) the distribution of a random variable U where U^{-1} has continuous uniform distribution on $[0, 1]$, in other words, π is an inverse rectangle distribution.

Given a sample $Y_1, Y_2, \ldots, Y_n$ that is i.i.d. by (1), we consider the following test statistics. First an overdispersion statistic is used

$$O_n = \frac{\frac{1}{n-1}\sum_{i=1}^n (Y_i - \overline{Y})^2 - \frac{1}{2n}\sum_{i=1}^n Y_i^2}{\overline{Y}^2} \left(\frac{n(n-1)}{n+1}\right)^{\frac{1}{2}}, \tag{4}$$

whose quantiles under H_0 have been obtained by simulation; see Table 4. All quantile tables are found in Appendix B.

The second test statistic is a moment likelihood ratio statistic. Let

$$m_1 = \overline{Y}, \quad m_2 = \frac{1}{2}\,\frac{1}{n}\sum_{i=1}^{n} Y_i^2, \quad m_3 = \frac{1}{6}\,\frac{1}{n}\sum_{i=1}^{n} Y_i^3,$$

$$u_{1,2} = \frac{m_3 - m_1 m_2 \mp \sqrt{4(m_2 - m_1^2)(m_2^2 - m_1 m_3) + (m_1 m_2 - m_3)^2}}{2(m_2 - m_1)^2},$$

$\gamma = (m_1 - u_1)/(u_2 - u_1)$ and $M = m_2 - m_1^2$. Let the symbol log denote the natural logarithm.

The moment likelihood ratio statistic R_n^{mom} is given by

$$R_n^{mom} = \begin{cases} \sum_{i=1}^n \left[\log\left(\frac{\gamma}{u_1}\exp(-\frac{Y_i}{u_1}) + \frac{1-\gamma}{u_2}\exp(-\frac{Y_i}{u_2})\right) - \log\left(\frac{1}{m_1}\exp(-\frac{Y_i}{m_1})\right)\right] \\ \qquad \text{if } M > 0 \text{ and } u_1 > 0, \\ \sum_{i=1}^n \left[\log\left(\frac{m_1 m_2 - m_1^3}{m_2^2}\exp(-\frac{m_1 Y_i}{m_2})\right) - \log\left(\frac{1}{m_1}\exp(-\frac{Y_i}{m_1})\right)\right] \\ \qquad \text{if } M > 0 \text{ and } u_1 \le 0, \\ -\infty \quad \text{if } M \le 0. \end{cases} \tag{5}$$

Quantiles of R_n^{mom}, calculated by simulation, are collected in Table 5.

Further, three goodness-of-fit procedures are used: a one-sided Kolmogorov-Smirnov test, an Anderson-Darling test, and a test by Tiku (1980).

Let $Y_{(1)} \le Y_{(2)} \le \dots Y_{(n)}$ be the ordered sample, $Y_{(0)} = 0$. Then

$$V_i = \frac{\sum_{k=1}^{i}(n+1-k)(Y_{(k)} - Y_{(k-1)})}{\sum_{k=1}^{n}(n+1-k)(Y_{(k)} - Y_{(k-1)})}, \qquad i = 1, 2, \dots, n-1, \tag{6}$$

are the order statistics of $n-1$ variables i.i.d. uniformly on $[0,1]$. The one-sided Kolmogorov-Smirnov statistic of them reads (with finite sample correction)

$$D_n^+ = \max_{i=1,\dots,n-1}\left\{\frac{i}{n-1} - V_i\right\} \cdot \left(\sqrt{n-1} + 0.12 + \frac{0.11}{\sqrt{n-1}}\right). \tag{7}$$

The Anderson-Darling statistic is

$$A_n^2 = \left\{-n - \frac{1}{n}\sum_{i=1}^{n}\left[(2i-1)\log\left(1 - \exp(-\frac{Y_{(i)}}{\overline{Y}})\right) - (2n+1-2i)\frac{Y_{(i)}}{\overline{Y}}\right]\right\} \cdot \left(1 + \frac{0.6}{n}\right). \tag{8}$$

For these versions of Kolmogorov-Smirnov and Anderson-Darling, see D'Agostino and Stephens (1986). Their critical values are reproduced in Table 6.

Tiku's (1980) test statistic is given by

$$T_n = \sum_{i=1}^{n-2} \frac{n-1-i}{n-2}\,\frac{(n-i)(Y_{(i+1)} - Y_{(i)})}{\sum_{k=1}^{n-1}(n-k)(Y_{(k+1)} - Y_{(k)})}. \tag{9}$$

The statistic is two-sided, and its distribution converges very rapidly to a normal distribution with mean $\frac{1}{2}$ and variance $(12(n-2))^{-1}$.

Finally, we employ a combination of the overdispersion test and the Kolmogorov-Smirnov test:

$$\text{Reject } H_0 \text{ if } D_n^+ \geq t_1 \text{ or } O_n \geq t_2.$$

For given size α, we choose t_1 and t_2 such that the probability of rejection $P[D_n^+ \geq t_1 \text{ or } O_n \geq t_2]$ is equal to α and that both $P[D_n^+ \geq t_1]$ and $P[O_n \geq t_2]$ are approximately equal to some $\tilde{\alpha} < \alpha$. Table 7 presents critical values t_1 and t_2 for various sample sizes n and test sizes α.

A similar combination of the moment likelihood test and the Kolmogorov-Smirnov test,

$$\text{Reject } H_0 \text{ if } D_n^+ \geq s_1 \text{ or } R_n^{mom} \geq s_2,$$

has been investigated. For critical values see Table 8.

In the next two sections the power of these tests is investigated on various alternatives, which are two-component mixtures in Section 2 and higher mixtures in Section 3. It comes out that the moment likelihood test is not recommendable and that, among the above tests, the combination of the overdispersion test with the Kolmogorov-Smirnov test develops the best overall power. In addition we introduce a likelihood ratio test (Section 4) and show that its power exceeds the power of the combined test on lower contaminations and is the same on other alternatives. However this likelihood ratio test is difficult to implement and suffers from numerical instabilities.

Appendix A presents some computational details, and Appendix C contains tables with further power results.

2 Power on Two-Component Mixtures

The power of the above tests has been evaluated on a broad range of two-component mixture alternatives having density

$$f(y) = (1-\epsilon)\exp(-y) + \frac{\epsilon}{v}\exp(-\frac{y}{v}), \quad y \geq 0, \tag{10}$$

for various values of ϵ and v, n between 10 and 10 000, $\alpha = 0.10, 0.05$ and 0.01. At each alternative, 20 000 replications of the tests have been calculated.

Here we report empirical power results where $\epsilon = 0.1, 0.5$ and 0.9. $\epsilon = 0.5$ indicates a fifty-fifty mixture of exponentials having different means. The case $\epsilon = 0.1$ may be interpreted as an exponential distribution (whose mean is scaled down here to 1) that is slightly contaminated (at a ten percent rate) by another exponential distribution that has mean larger than 1. We call this shortly an *upper contamination*. On the other hand, $\epsilon = 0.9$ characterizes a mixture of two exponentials having different means where the larger weight ($\epsilon = 0.9$) is put on the higher mean. Due to the scale invariance, this is tantamount to having an exponential distribution with mean 1 that is slightly contaminated by an exponential distribution with mean smaller than 1 with contamination rate $1 - \epsilon = 0.1$. Hence the case $\epsilon = 0.9$ is regarded as a *lower contamination.*

First we compare the three goodness-of-fit tests. It is rather obvious that Tiku's test can never be better than the Kolmogorov-Smirnov test. Figures 1, 2 and 3 show, in addition, that the one-sided Kolmogorov-Smirnov test usually outperforms the Anderson-Darling test. We refer to the results of Balakrishnan and Ambagaspitiya (1989) who found that Tiku's test on mixtures of exponentials in the whole is more powerful than Durbin's test and the test by Shapiro and Wilks. Therefore the latter two are also inferior to the Kolmogorov-Smirnov test.

We compare the power of the overdispersion test with the best goodness-of-fit-test. From Figure 4 we see that an upper contamination is better detected by the overdispersion test than by the Kolmogorov-Smirnov test. This holds similarly for small and large samples. Figure 6 demonstrates that the reverse holds for lower contaminations. Here the Kolmogorov-Smirnov test clearly outperforms the overdispersion test; with large samples it can have nearly double power. For fifty-fifty mixtures (see Figure 5) the situation is mixed: While at $n = 100$ the Kolmogorov-Smirnov test has usually more power than the overdispersion test, at $n = 1000$ the latter is more powerful.

Next the power of the moment likelihood test is investigated. See Figures 7 to 9. While for upper contaminations the moment likelihood test does fairly well and better than the best goodness-of-fit test (Kolmogorov-Smirnov), for fifty-fifty mixtures and lower contaminations its performance is very bad. In many cases, even for large samples, we get the paradox result that the power decreases when the difference of the means increases.

The above power results have shown that, depending on the alternatives, in the whole either the overdispersion test or the Kolmogorov-Smirnov test have the best power among the tests considered so far. Figures 4 to 6 exhibit the power of the combined test as well, and it is seen that the combined test yields power which is near to optimum over all alternatives. In Figures 7 to 9 the same is done for the combination of the moment likelihood test with the Kolmogorov-Smirnov test.

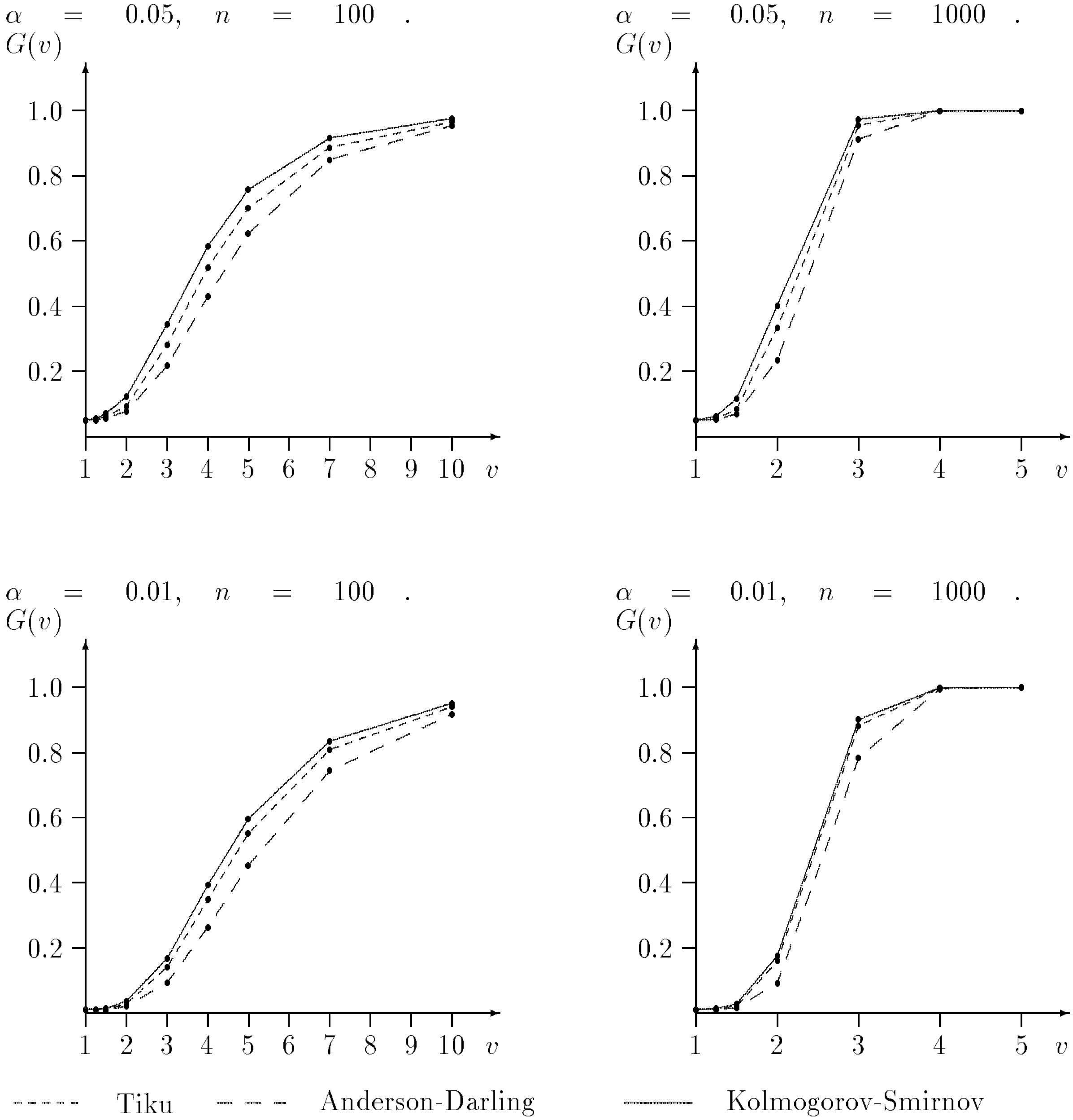

Figure 1: Power of three goodness-of-fit tests on alternative $f(y) = 0.9e^{-y} + 0.1\frac{1}{v}e^{-\frac{y}{v}}$ (upper contamination).

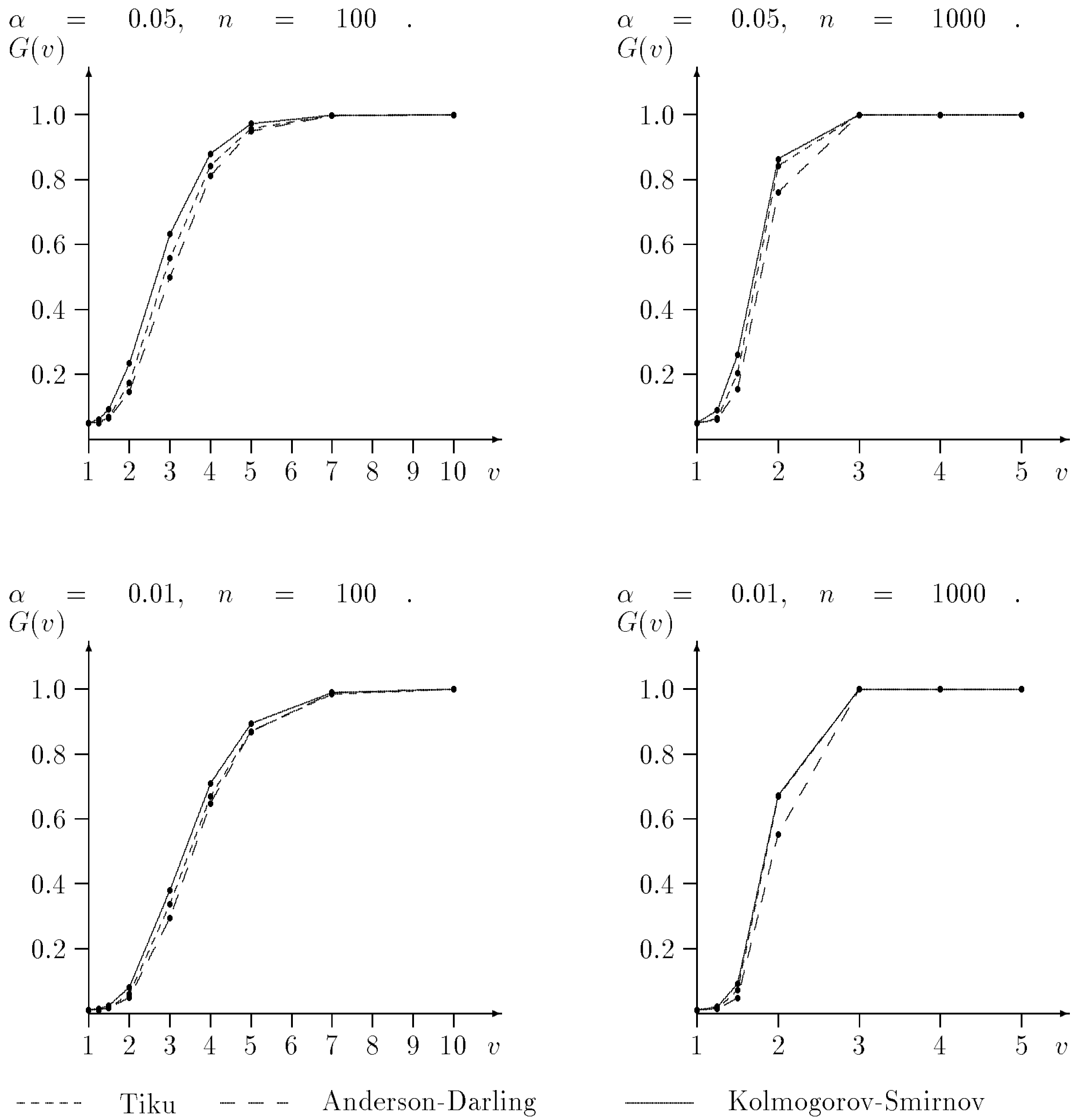

Figure 2: Power of three goodness-of-fit tests on alternative $f(y) = 0.5e^{-y} + 0.5\frac{1}{v}e^{-\frac{y}{v}}$ (fifty-fifty mixture).

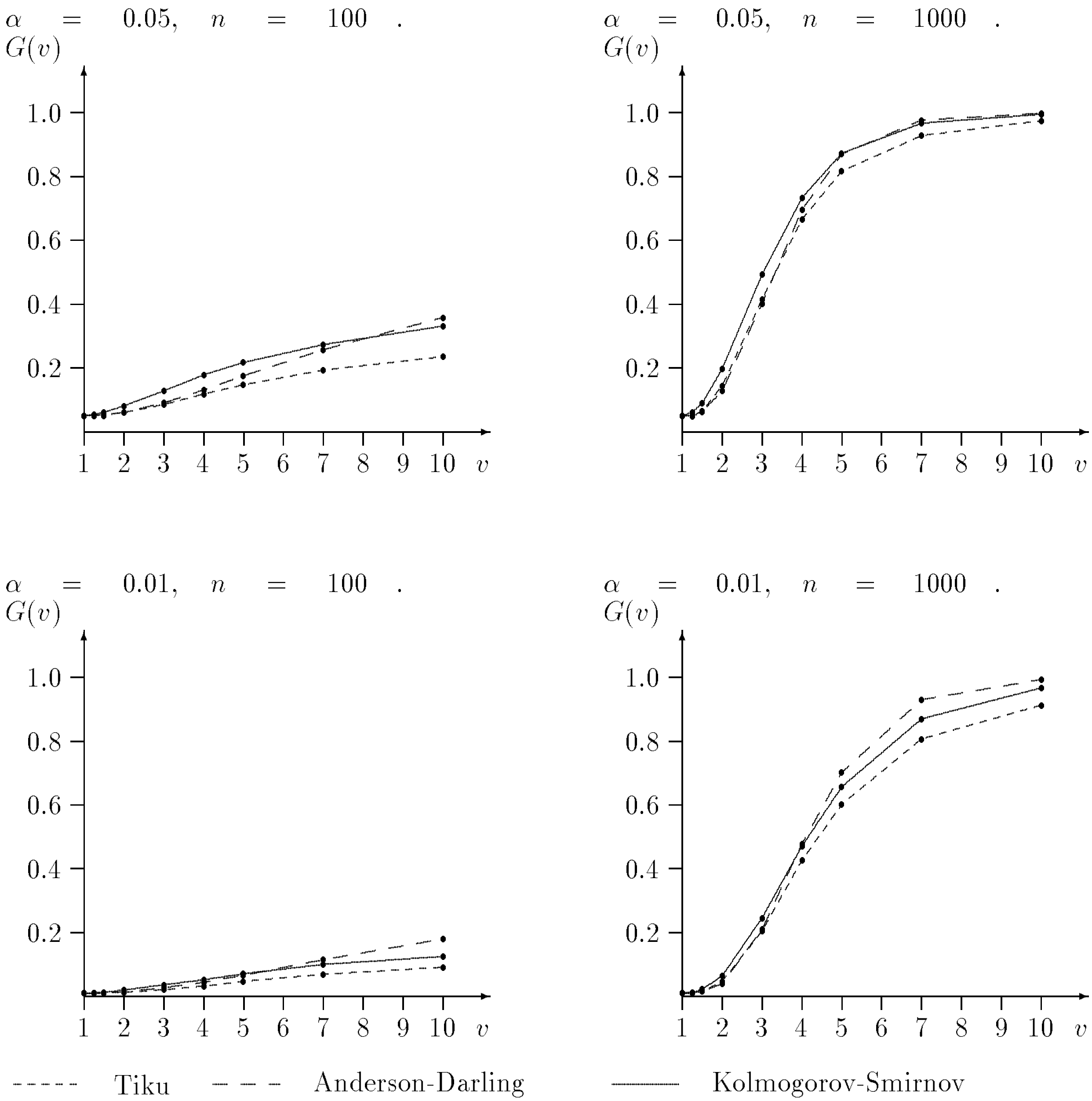

Figure 3: Power of three goodness-of-fit tests on alternative $f(y) = 0.1e^{-y} + 0.9\frac{1}{v}e^{-\frac{y}{v}}$ (lower contamination).

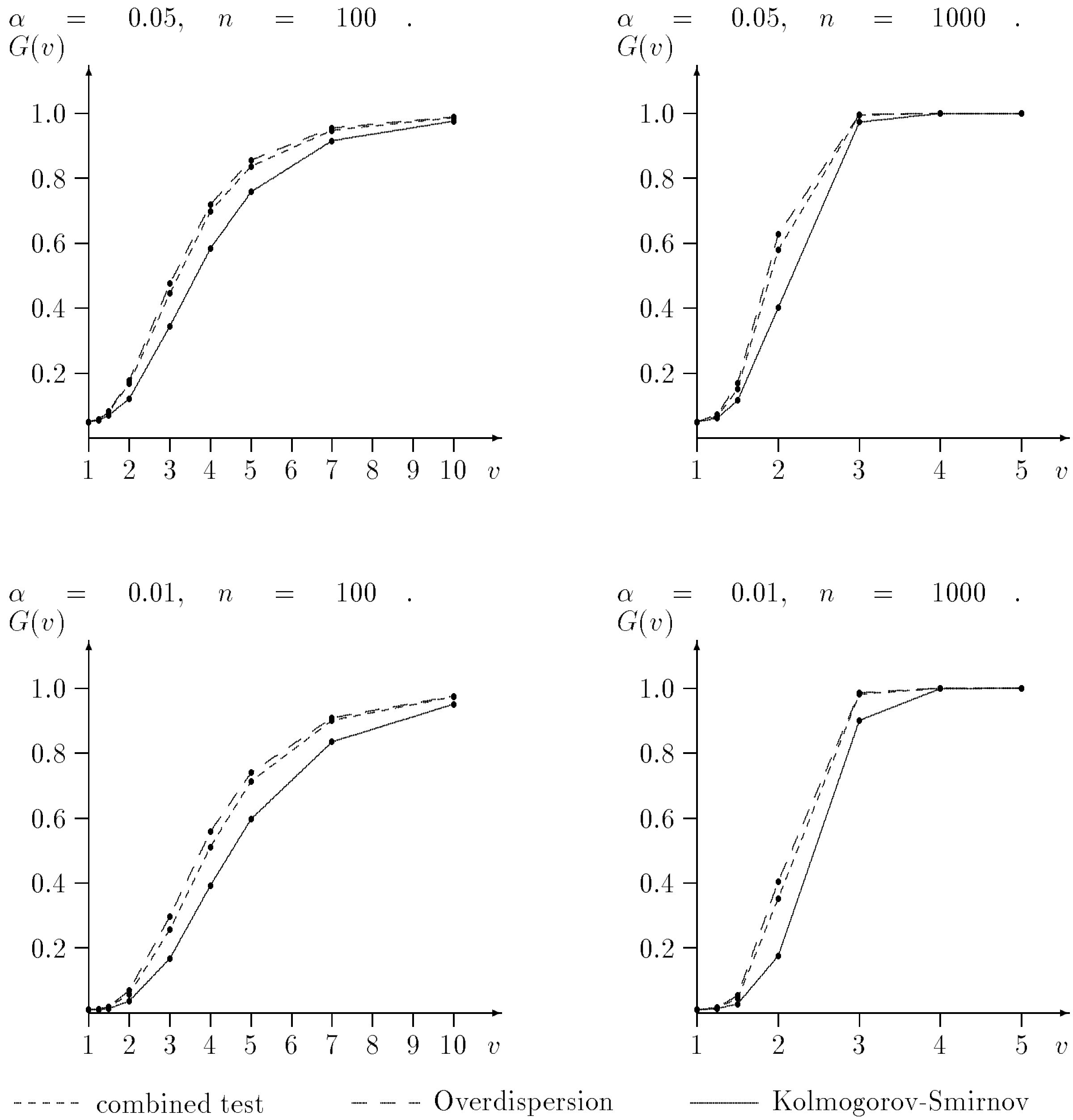

Figure 4: Power of overdispersion and Kolmogorov-Smirnov tests on alternative $f(y) = 0.9e^{-y} + 0.1\frac{1}{v}e^{-\frac{y}{v}}$ (upper contamination).

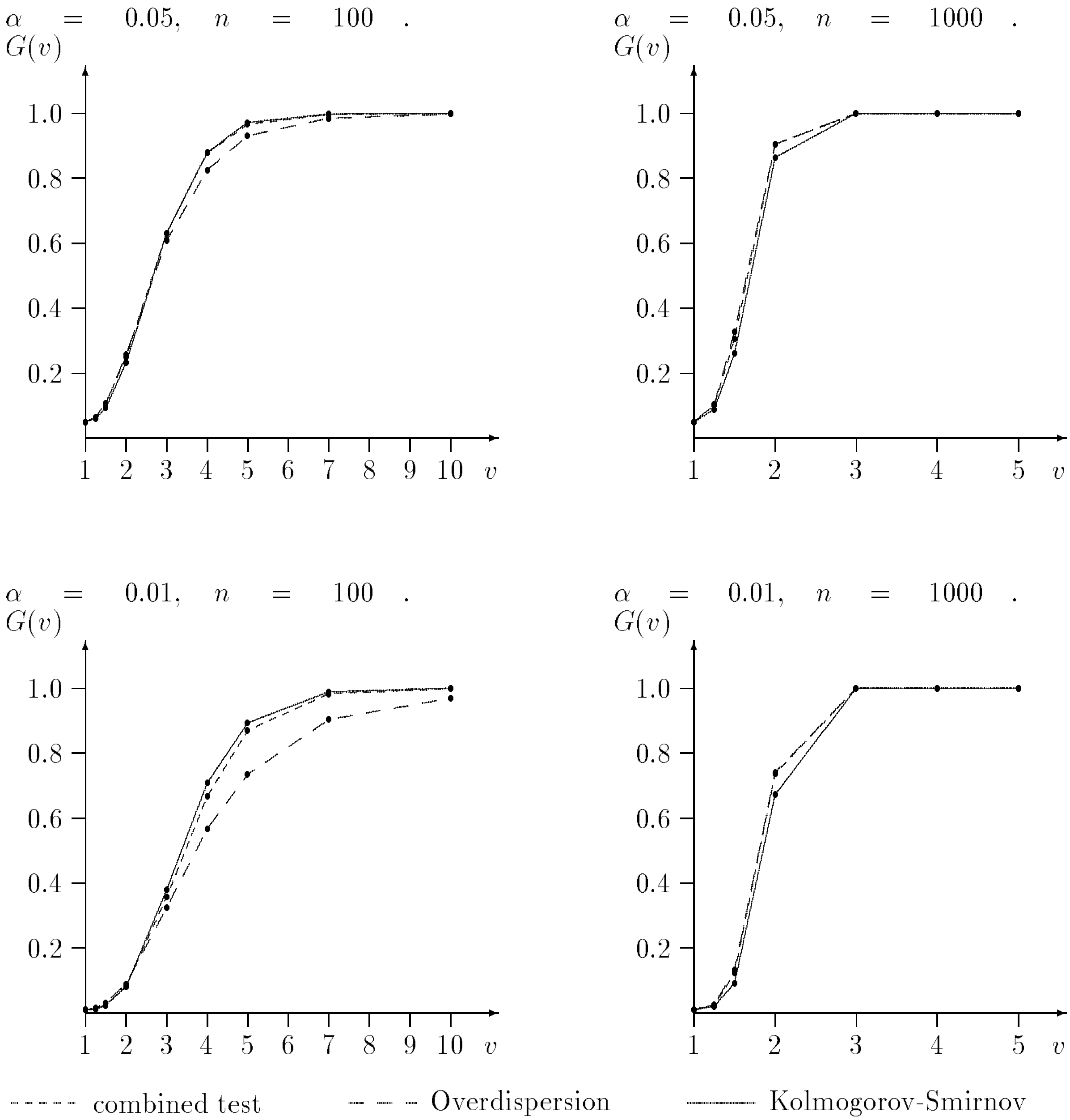

Figure 5: Power of overdispersion and Kolmogorov-Smirnov tests on alternative $f(y) = 0.5e^{-y} + 0.5\frac{1}{v}e^{-\frac{y}{v}}$ (fifty-fifty mixture).

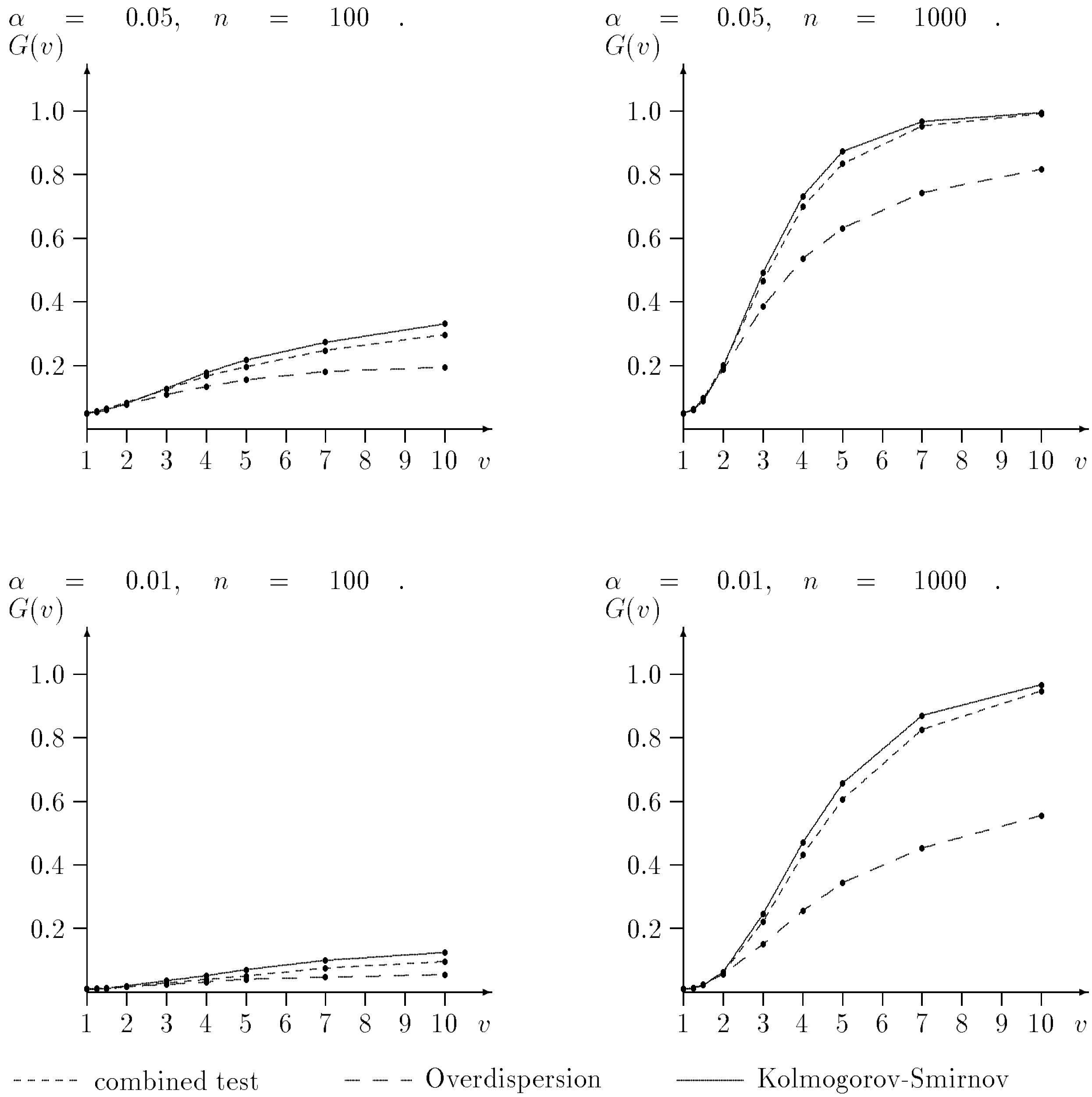

Figure 6: Power of overdispersion and Kolmogorov-Smirnov tests on alternative $f(y) = 0.1e^{-y} + 0.9\frac{1}{v}e^{-\frac{y}{v}}$ (lower contamination).

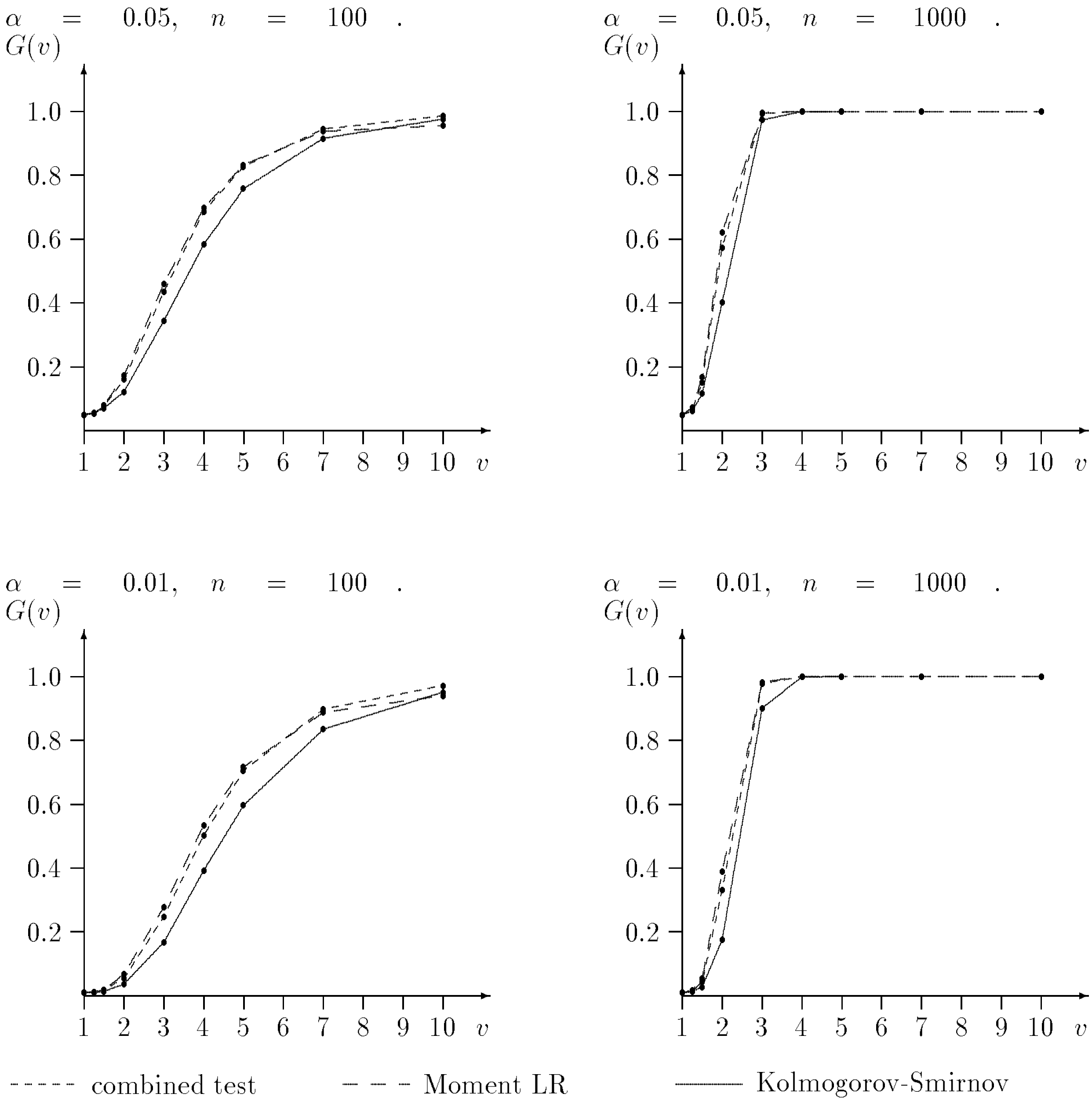

Figure 7: Power of moment likelihood and Kolmogorov-Smirnov tests on alternative $f(y) = 0.9e^{-y} + 0.1\frac{1}{v}e^{-\frac{y}{v}}$ (upper contamination).

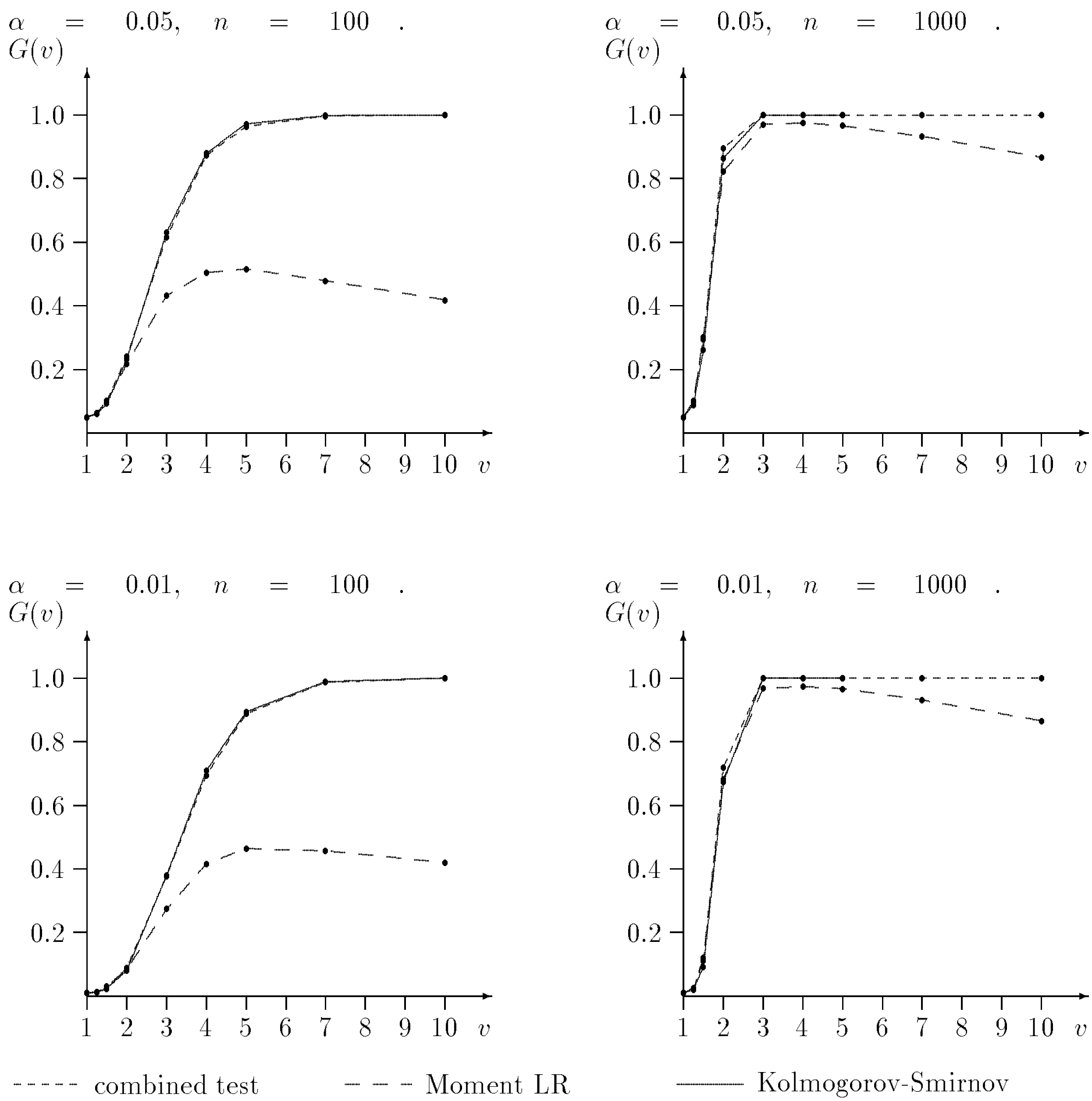

Figure 8: Power of moment likelihood and Kolmogorov-Smirnov tests on alternative $f(y) = 0.5e^{-y} + 0.5\frac{1}{v}e^{-\frac{y}{v}}$ (fifty-fifty mixture).

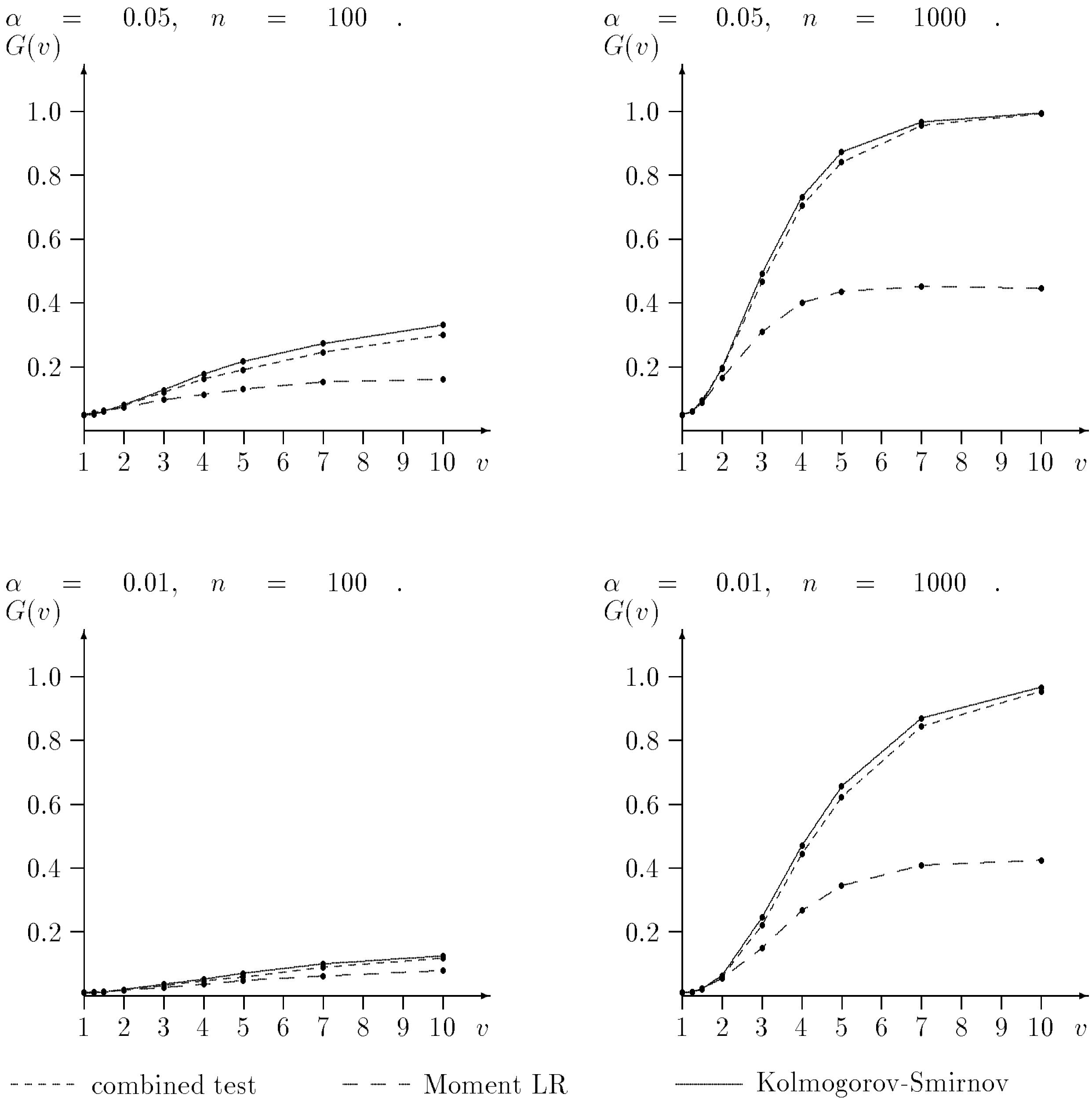

Figure 9: Power of moment likelihood and Kolmogorov-Smirnov tests on alternative $f(y) = 0.1e^{-y} + 0.9\frac{1}{v}e^{-\frac{y}{v}}$ (lower contamination).

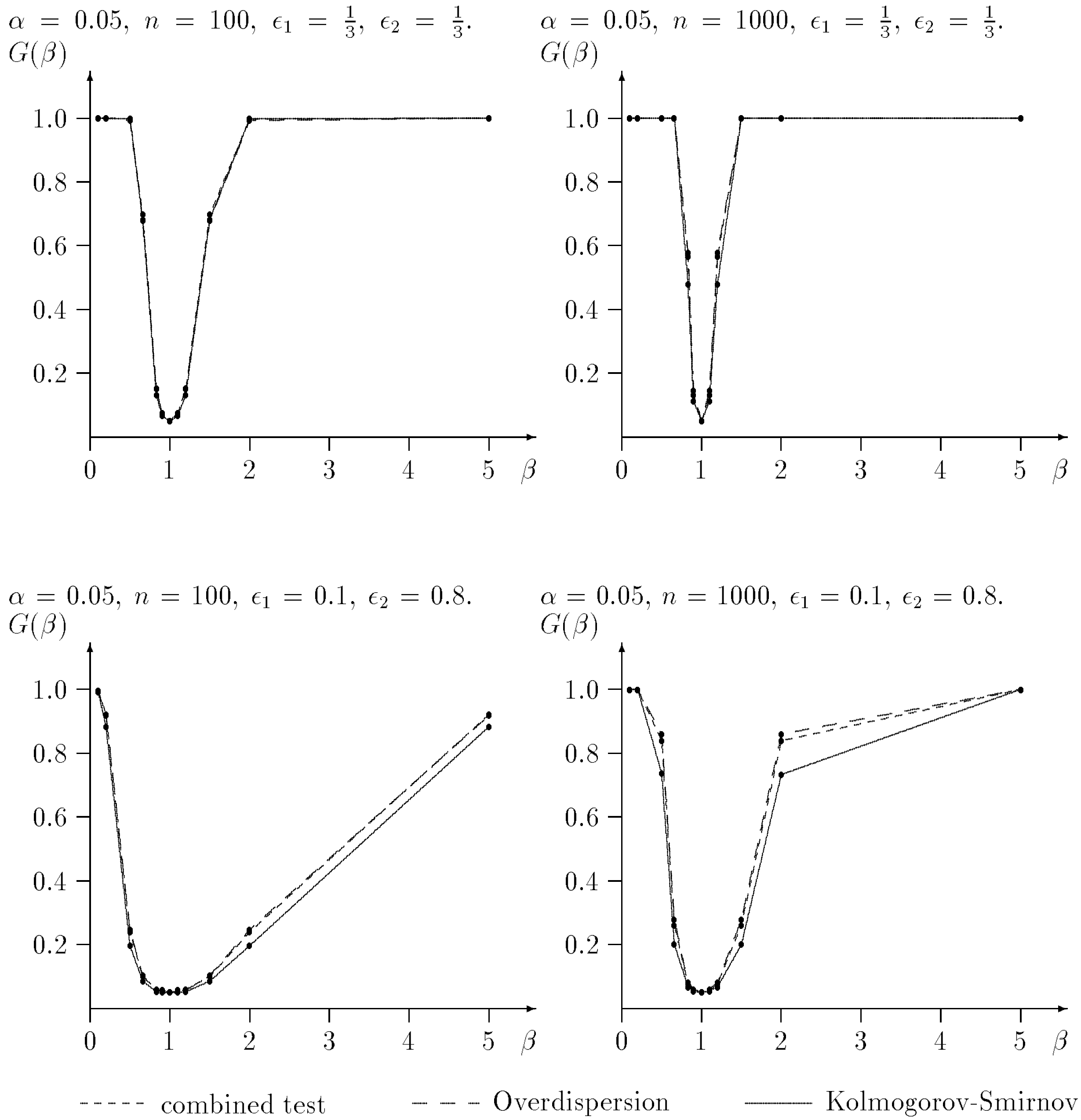

Figure 10: Power on three-component mixtures
$f(y) = \epsilon_1 \beta e^{-\beta y} + \epsilon_2 e^{-y} + (1 - \epsilon_1 - \epsilon_2)\frac{1}{\beta} e^{-\frac{y}{\beta}}$
.

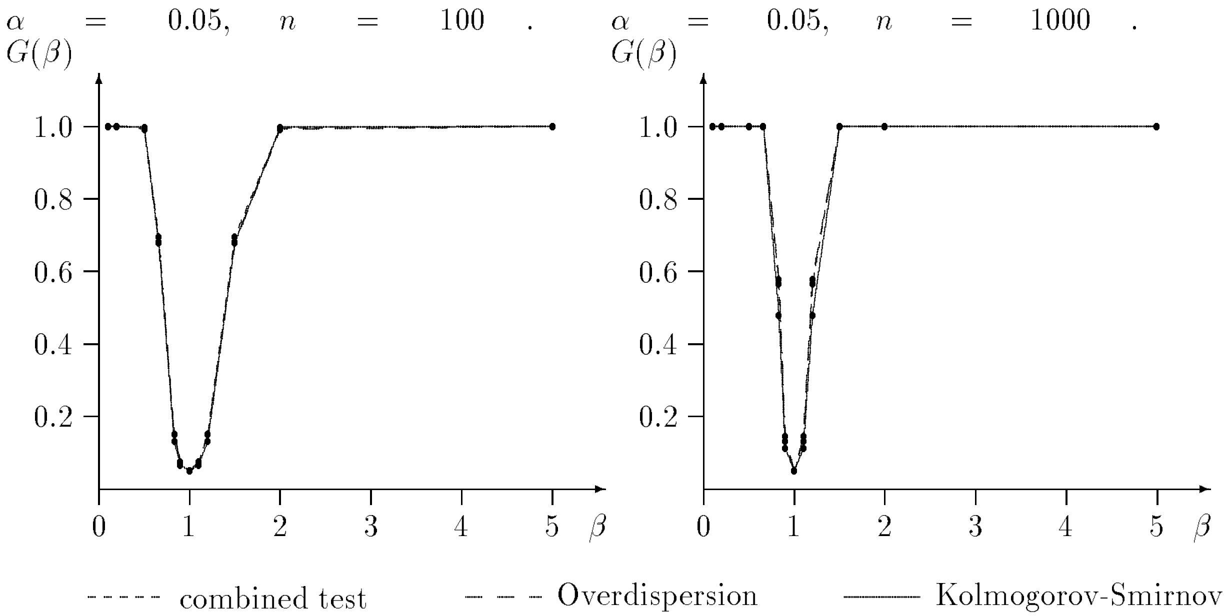

Figure 11: Power on five-component mixtures $f(y) = \frac{1}{5}\sum_{j=1}^{5} \beta^{3-j} \exp(-y\beta^{3-j})$.

3 Power on Higher Mixtures

For three- and five component mixtures, Figures 10 and 11 present the power of the overdispersion and the Kolmogorov-Smirnov tests and of their combination. It is seen that on equally weighted mixtures the differences in power may be neglected. Only the power of the moment likelihood test (not in the figures) is much lower. However, e.g., on three-component mixtures with unequal weights (10 percent contaminations on both sides) the overdispersion test does better than the Kolmogorov-Smirnov test. This example again suggests the use of the combined test.

The power on three different continuous mixture alternatives is presented in Tables 1 to 3. These results are in line with the above.

Test	n						
	10	20	50	100	200	500	1000
O_n	0.4920	0.7368	0.9653	0.9985	1.0000	1.0000	1.0000
R_n^{mom}	0.0366	0.1899	0.6083	0.8112	0.8724	0.9035	0.9176
A_n^2	0.4129	0.6474	0.9307	0.9955	1.0000	1.0000	1.0000
D_n^+	0.4758	0.7240	0.9595	0.9983	1.0000	1.0000	1.0000
T_n	0.4083	0.6738	0.9466	0.9977	1.0000	1.0000	1.0000
$O_n \mid D_n^+$	0.4905	0.7291	0.9620	0.9989	1.0000	1.0000	1.0000
$R_n^{\text{mom}} \mid D_n^+$	0.4258	0.7253	0.9598	0.9989	1.0000	1.0000	1.0000

Table 1: Continuous mixture with π inverse rectangular on $[0, 1]$, $\alpha = 0.05$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.2034	0.3042	0.5273	0.7694	0.9575	0.9999	1.0000
R_n^{mom}	0.0598	0.1336	0.3230	0.5097	0.7013	0.8966	0.9744
A_n^2	0.2090	0.3241	0.6153	0.8833	0.9914	1.0000	1.0000
D_n^+	0.2303	0.3480	0.6161	0.8703	0.9892	1.0000	1.0000
T_n	0.1252	0.2307	0.5378	0.8330	0.9856	1.0000	1.0000
$O_n \mid D_n^+$	0.2234	0.3421	0.5884	0.8622	0.9844	1.0000	1.0000
$R_n^{\text{mom}} \mid D_n^+$	0.1772	0.3242	0.6200	0.8574	0.9870	1.0000	1.0000

Table 2: Continuous mixture with π rectangular on $[0, 1]$, $\alpha = 0.05$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.3998	0.6171	0.9118	0.9941	1.0000	1.0000	1.0000
R_n^{mom}	0.0427	0.2022	0.5708	0.8046	0.9474	0.9987	1.0000
A_n^2	0.4127	0.6558	0.9512	0.9987	1.0000	1.0000	1.0000
D_n^+	0.4344	0.6659	0.9472	0.9985	1.0000	1.0000	1.0000
T_n	0.2898	0.5700	0.9249	0.9977	1.0000	1.0000	1.0000
$O_n \mid D_n^+$	0.4313	0.6698	0.9369	0.9980	1.0000	1.0000	1.0000
$R_n^{\text{mom}} \mid D_n^+$	0.3500	0.6390	0.9477	0.9977	1.0000	1.0000	1.0000

Table 3: Continuous mixture with π exponential, $\pi = Exp(1)$, $\alpha = 0.05$

4 Comparison with a maximum likelihood test

The likelihood ratio statistic is given by

$$R_n = \sup_{\theta \in H_1} l(y|\theta) - \sup_{\theta \in H_0} l(y|\theta), \tag{11}$$

where $l(y|\theta)$ is the log-likelihood given a parameter vector θ. While the likelihood in the second term is maximized (ML estimation under the null hypothesis) at θ equal to the sample mean, the first supremum (ML estimation under the alternative) has to be calculated numerically. Also the quantiles have to be assessed by simulation. This makes the likelihood ratio statistic more complicated and costly to apply than the statistics discussed so far. Things become additionally involved as the evaluation of the likelihood ratio test statistic (as well as the estimation of its quantiles) heavily depends on the numerical procedure used for calculating $\sup_{\theta \in H_1} l(y|\theta)$; see Seidel et al. (1997).

We studied the statistic (11) in the case that H_1 includes all two-component mixtures (3). For calculating the supremum we used an EM algorithm starting at $\epsilon = 0.5$, $u_1 = 0.5\bar{y}$ and $u_2 = 1.5\bar{y}$. For other starting strategies and computational details we refer to Seidel et al.(1997). Simulated quantiles (obtained from 100 000 replications) are presented in Table 9.

This numerical version of the likelihood ratio test has been compared with the above tests. The number of replications was 10 000. It comes out that for fifty-fifty mixtures its power is the same as that of the combined overdispersion-KS test. For lower contaminations the likelihood ratio test by far outperforms the combined test, while for upper contaminations the likelihood ratio test appears to have slightly less power. See Figures 12 and 13, which also contrast these two tests with the moment likelihood test.

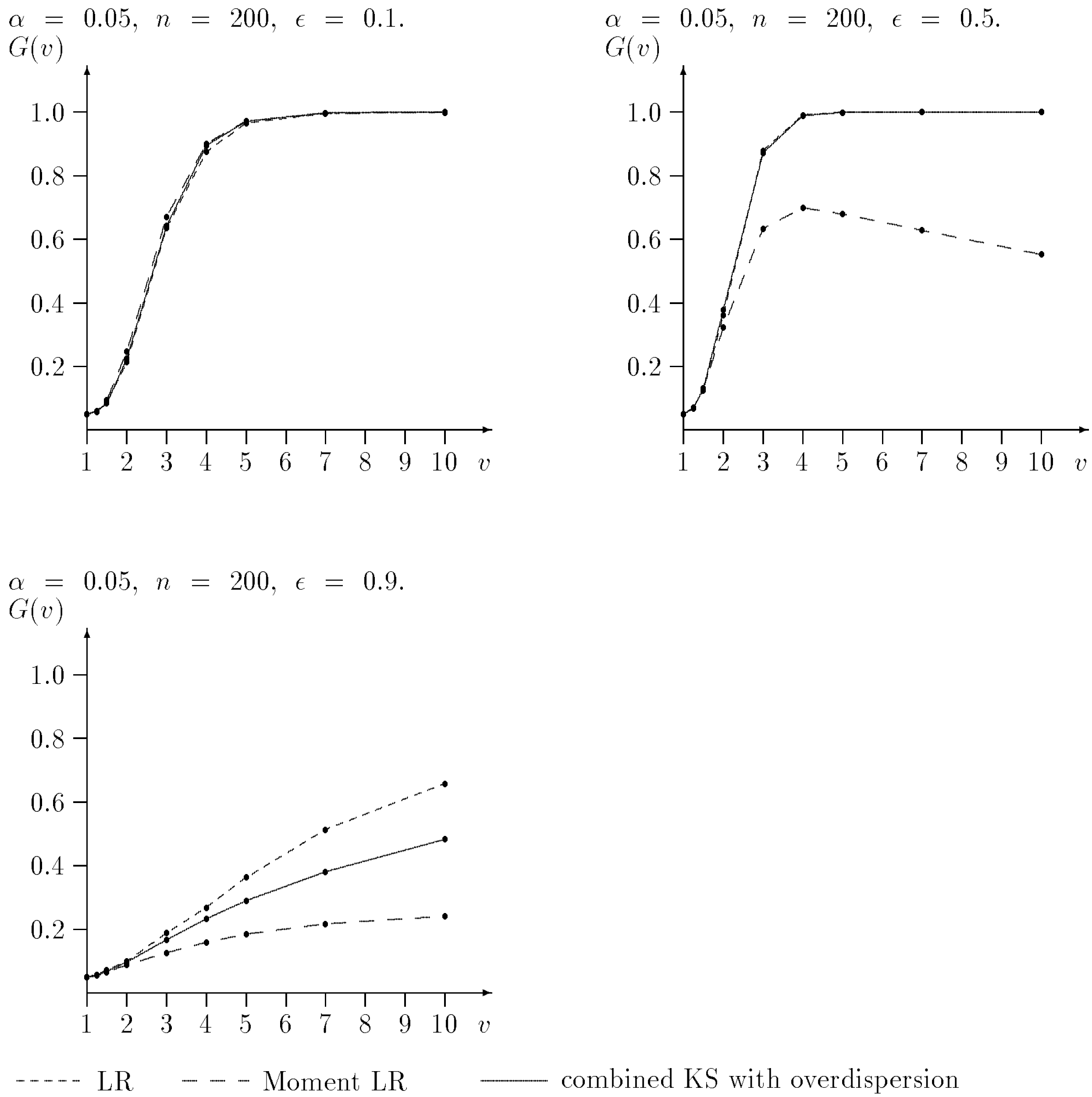

Figure 12: Power of likelihood ratio and other tests on alternative $f(y) = (1-\epsilon)e^{-y} + \frac{\epsilon}{v}e^{-\frac{y}{v}}$, upper contamination ($\epsilon = 0.1$), fifty-fifty mixture ($\epsilon = 0.5$), lower contamination ($\epsilon = 0.9$).

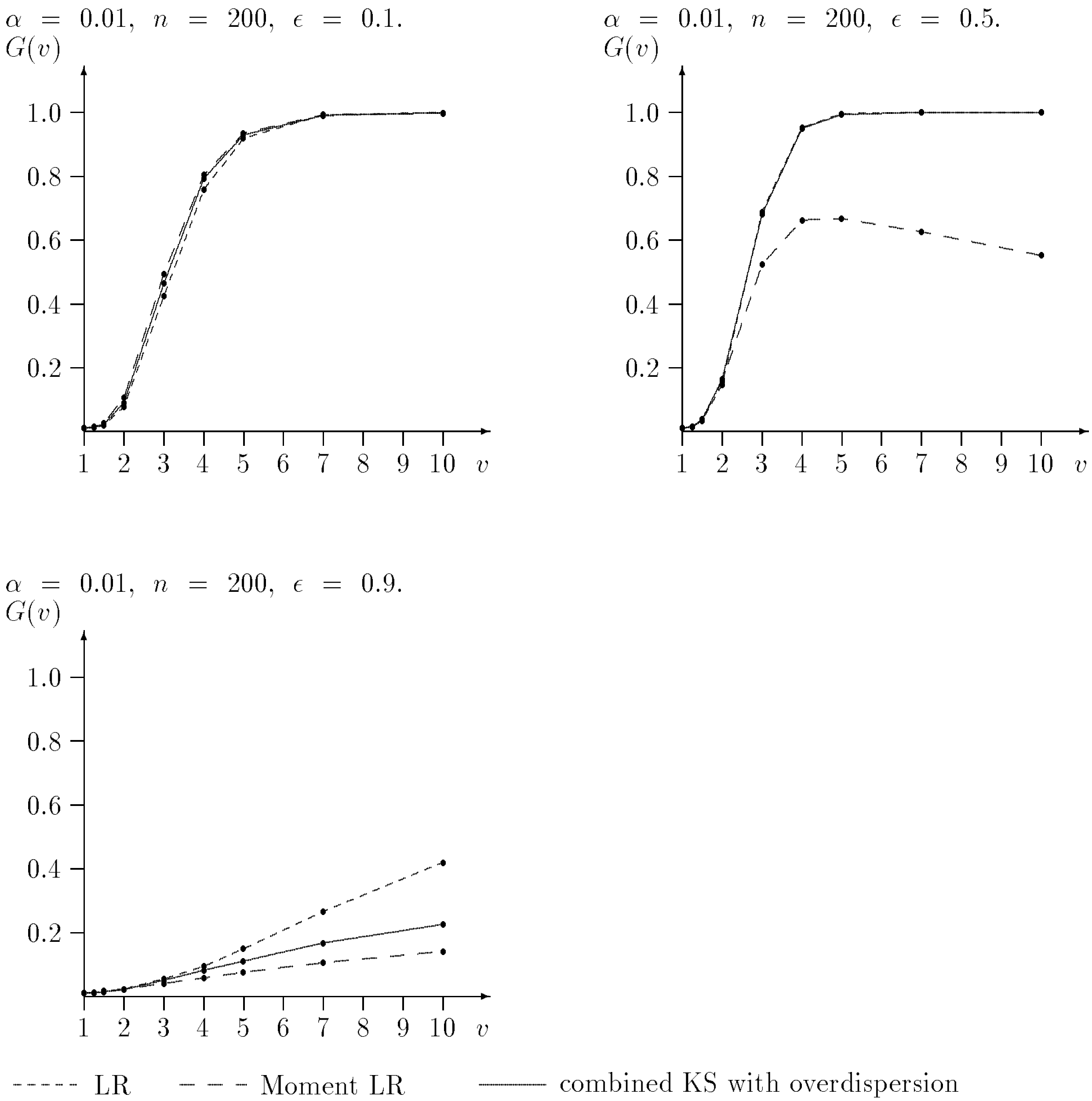

Figure 13: Power of likelihood ratio and other tests on alternative $f(y) = (1-\epsilon)e^{-y} + \frac{\epsilon}{v}e^{-\frac{y}{v}}$, upper contamination ($\epsilon = 0.1$), fifty-fifty mixture ($\epsilon = 0.5$), lower contamination ($\epsilon = 0.9$).

Appendix A: Computational Details

The number of replications was 1 000 000 for quantiles of the overdispersion test, the likelihood ratio and the moment likelihood test. 10 000 replications have been used in calculating the power of the likelihood ratio test, and 20 000 for the power of the other tests, on the various alternatives.

Quantiles for the combined tests have been determined in such a way that the individual tests have equal size (lower than the size of the combined test) and that the actual size of the combined test (calculated from 50 000 replications) is close to its nominal size. In fact, the given quantiles are preliminary ones, and in most cases they are conservative. The actual size of tests based on them differs from the nominal size by up to ten percent and, in a few cases, by more.

Appendix B: Quantiles

This appendix includes selected quantiles of the tests investigated.

α	n						
	10	20	50	100	200	500	1000
0.1	0.9612	1.0930	1.2046	1.2540	1.2840	1.2961	1.2976
0.05	1.4450	1.6078	1.7278	1.7500	1.7592	1.7312	1.7165
0.01	2.6400	2.9095	2.9635	2.8909	2.7777	2.6357	2.5617

Table 4: Quantiles of the overdispersion statistic O_n

α	n						
	10	20	50	100	200	500	1000
0.1	0.4303	0.8182	1.1494	1.4145	1.6633	1.9688	2.1544
0.05	0.6984	1.1018	1.8216	2.4454	3.0251	3.6787	4.0596
0.01	0.9545	2.2787	6.3414	9.5178	12.7995	16.6868	18.9016

Table 5: Quantiles of the moment likelihood statistic R_n^{mom}

α	Test	
	A_n^2	D_n^+
0.1	1.062	1.073
0.05	1.321	1.224
0.01	1.959	1.518

Table 6: Quantiles for the D_n^+ (Kolmogorov-Smirnov) and A_n^2 (Anderson-Darling) statistics

α	test component		n						
			10	20	50	100	200	500	1000
0.1	D_n^+	t_1	1.13	1.14	1.16	1.16	1.16	1.17	1.17
	O_n	t_2	1.15	1.33	1.49	1.55	1.57	1.57	1.57
0.05	D_n^+	t_1	1.29	1.30	1.36	1.31	1.32	1.33	1.32
	O_n	t_2	1.69	1.92	2.02	2.07	2.09	2.03	1.98
0.01	D_n^+	t_1	1.58	1.58	1.61	1.63	1.61	1.63	1.61
	O_n	t_2	2.97	3.41	3.41	3.37	3.13	3.00	2.81

Table 7: Quantiles for the combination of O_n and D_n^+ statistics

α	test component		n						
			10	20	50	100	200	500	1000
0.1	D_n^+	s_1	1.22	1.21	1.20	1.18	1.18	1.18	1.34
	R_n^{mom}	s_2	0.70	1.07	1.69	2.08	2.58	3.06	2.95
0.05	D_n^+	s_1	1.36	1.34	1.32	1.33	1.32	1.32	1.32
	R_n^{mom}	s_2	0.85	1.43	2.66	3.92	4.76	5.92	6.64
0.01	D_n^+	s_1	1.63	1.63	1.61	1.61	1.61	1.61	1.61
	R_n^{mom}	s_2	0.98	3.01	9.93	16.46	21.76	29.43	33.49

Table 8: Quantiles for the combination of R_n^{mom} and D_n^+ statistics

α	$n = 200$
0.1	2.59
0.05	3.97
0.01	6.99

Table 9: Quantiles for the maximum likelihood statistic R_n.

Appendix C: Further Power Results

In this appendix we present further power results. The first twelve tables present the power of the different tests on two-component mixtures dependent on the sample size n, for selected values of α, ϵ and v.

Test	n						
	10	20	50	100	200	500	1000
O_n	0.020	0.031	0.048	0.070	0.113	0.224	0.405
R_n^{mom}	0.010	0.026	0.044	0.068	0.107	0.211	0.389
A_n^2	0.017	0.018	0.020	0.021	0.027	0.047	0.092
D_n^+	0.018	0.024	0.029	0.037	0.052	0.098	0.176
T_n	0.012	0.018	0.022	0.029	0.040	0.081	0.160
$O_n \mid D_n^+$	0.018	0.026	0.042	0.058	0.091	0.185	0.353
$R_n^{\text{mom}} \mid D_n^+$	0.016	0.022	0.039	0.055	0.085	0.180	0.332

Table 10: Power on two-component mixture alternative. $\alpha = 0.01$, $v = 2.00$, $\epsilon = 0.1$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.024	0.033	0.058	0.090	0.172	0.412	0.741
R_n^{mom}	0.011	0.028	0.052	0.085	0.158	0.373	0.683
A_n^2	0.018	0.022	0.034	0.049	0.096	0.255	0.553
D_n^+	0.022	0.030	0.053	0.081	0.152	0.366	0.673
T_n	0.013	0.019	0.037	0.060	0.123	0.339	0.669
$O_n \mid D_n^+$	0.024	0.033	0.056	0.085	0.164	0.399	0.737
$R_n^{\text{mom}} \mid D_n^+$	0.020	0.030	0.056	0.090	0.160	0.399	0.719

Table 11: Power on two-component mixture alternative. $\alpha = 0.01$, $v = 2.00$, $\epsilon = 0.5$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.133	0.231	0.484	0.740	0.942	1.000	1.000
R_n^{mom}	0.010	0.117	0.425	0.717	0.935	1.000	1.000
A_n^2	0.075	0.126	0.262	0.454	0.745	0.987	1.000
D_n^+	0.099	0.186	0.369	0.597	0.853	0.997	1.000
T_n	0.066	0.146	0.328	0.552	0.825	0.995	1.000
$O_n \| D_n^+$	0.106	0.208	0.455	0.714	0.930	0.999	1.000
$R_n^{\text{mom}} \| D_n^+$	0.077	0.184	0.437	0.706	0.926	0.999	1.000

Table 12: Power on two-component mixture alternative. $\alpha = 0.01$, $v = 5.00$, $\epsilon = 0.1$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.102	0.177	0.421	0.735	0.972	1.000	1.000
R_n^{mom}	0.009	0.083	0.260	0.464	0.667	0.874	0.967
A_n^2	0.107	0.208	0.543	0.869	0.995	1.000	1.000
D_n^+	0.118	0.241	0.593	0.894	0.996	1.000	1.000
T_n	0.059	0.162	0.524	0.870	0.996	1.000	1.000
$O_n \| D_n^+$	0.113	0.232	0.548	0.871	0.994	1.000	1.000
$R_n^{\text{mom}} \| D_n^+$	0.079	0.215	0.564	0.888	0.995	1.000	1.000

Table 13: Power on two-component mixture alternative. $\alpha = 0.01$, $v = 5.00$, $\epsilon = 0.5$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.263	0.502	0.844	0.976	1.000	1.000	1.000
R_n^{mom}	0.009	0.146	0.676	0.940	0.998	1.000	1.000
A_n^2	0.209	0.378	0.693	0.917	0.995	1.000	1.000
D_n^+	0.252	0.456	0.778	0.951	0.999	1.000	1.000
T_n	0.189	0.403	0.744	0.940	0.998	1.000	1.000
$O_n \| D_n^+$	0.258	0.489	0.832	0.974	0.999	1.000	1.000
$R_n^{\text{mom}} \| D_n^+$	0.213	0.458	0.824	0.972	0.999	1.000	1.000

Table 14: Power on two-component mixture alternative. $\alpha = 0.01$, $v = 10.00$, $\epsilon = 0.1$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.201	0.353	0.738	0.969	1.000	1.000	1.000
R_n^{mom}	0.006	0.094	0.277	0.420	0.552	0.733	0.866
A_n^2	0.306	0.598	0.967	1.000	1.000	1.000	1.000
D_n^+	0.283	0.572	0.959	1.000	1.000	1.000	1.000
T_n	0.149	0.421	0.922	1.000	1.000	1.000	1.000
$O_n \vert D_n^+$	0.261	0.540	0.942	0.999	1.000	1.000	1.000
$R_n^{\mathrm{mom}} \vert D_n^+$	0.204	0.515	0.949	1.000	1.000	1.000	1.000

Table 15: Power on two-component mixture alternative. $\alpha = 0.01$, $v = 10.00$, $\epsilon = 0.5$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.074	0.092	0.129	0.178	0.250	0.422	0.628
R_n^{mom}	0.052	0.081	0.129	0.173	0.247	0.414	0.621
A_n^2	0.061	0.063	0.068	0.077	0.094	0.141	0.234
D_n^+	0.069	0.082	0.099	0.122	0.161	0.258	0.402
T_n	0.058	0.064	0.078	0.093	0.123	0.200	0.334
$O_n \vert D_n^+$	0.072	0.090	0.114	0.168	0.225	0.382	0.580
$R_n^{\mathrm{mom}} \vert D_n^+$	0.064	0.090	0.120	0.160	0.227	0.382	0.574

Table 16: Power on two-component mixture alternative. $\alpha = 0.05$, $v = 2.00$, $\epsilon = 0.1$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.095	0.122	0.174	0.258	0.379	0.676	0.905
R_n^{mom}	0.056	0.093	0.146	0.217	0.323	0.583	0.822
A_n^2	0.073	0.076	0.101	0.146	0.226	0.466	0.761
D_n^+	0.091	0.113	0.160	0.233	0.345	0.620	0.864
T_n	0.065	0.081	0.115	0.174	0.279	0.563	0.842
$O_n \vert D_n^+$	0.087	0.115	0.164	0.251	0.378	0.663	0.905
$R_n^{\mathrm{mom}} \vert D_n^+$	0.077	0.112	0.173	0.241	0.375	0.655	0.896

Table 17: Power on two-component mixture alternative. $\alpha = 0.05$, $v = 2.00$, $\epsilon = 0.5$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.222	0.367	0.639	0.856	0.976	1.000	1.000
R_n^{mom}	0.049	0.229	0.576	0.833	0.972	1.000	1.000
A_n^2	0.154	0.234	0.405	0.624	0.863	0.997	1.000
D_n^+	0.205	0.315	0.538	0.758	0.936	0.999	1.000
T_n	0.158	0.262	0.477	0.701	0.908	0.998	1.000
$O_n\|D_n^+$	0.216	0.356	0.609	0.837	0.971	1.000	1.000
$R_n^{\text{mom}}\|D_n^+$	0.177	0.349	0.611	0.828	0.971	1.000	1.000

Table 18: Power on two-component mixture alternative. $\alpha = 0.05$, $v = 5.00$, $\epsilon = 0.1$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.269	0.424	0.712	0.931	0.997	1.000	1.000
R_n^{mom}	0.061	0.144	0.331	0.516	0.679	0.884	0.966
A_n^2	0.222	0.381	0.726	0.949	1.000	1.000	1.000
D_n^+	0.286	0.479	0.803	0.972	1.000	1.000	1.000
T_n	0.177	0.356	0.735	0.957	1.000	1.000	1.000
$O_n\|D_n^+$	0.283	0.462	0.775	0.966	0.999	1.000	1.000
$R_n^{\text{mom}}\|D_n^+$	0.222	0.431	0.788	0.964	0.999	1.000	1.000

Table 19: Power on two-component mixture alternative. $\alpha = 0.05$, $v = 5.00$, $\epsilon = 0.5$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.388	0.616	0.899	0.989	1.000	1.000	1.000
R_n^{mom}	0.040	0.239	0.740	0.956	0.999	1.000	1.000
A_n^2	0.310	0.492	0.787	0.954	0.998	1.000	1.000
D_n^+	0.370	0.576	0.855	0.976	1.000	1.000	1.000
T_n	0.317	0.524	0.825	0.965	0.999	1.000	1.000
$O_n\|D_n^+$	0.383	0.606	0.894	0.987	1.000	1.000	1.000
$R_n^{\text{mom}}\|D_n^+$	0.332	0.599	0.894	0.986	1.000	1.000	1.000

Table 20: Power on two-component mixture alternative. $\alpha = 0.05$, $v = 10.00$, $\epsilon = 0.1$

Test	n						
	10	20	50	100	200	500	1000
O_n	0.442	0.660	0.928	0.998	1.000	1.000	1.000
R_n^{mom}	0.044	0.118	0.286	0.419	0.552	0.731	0.867
A_n^2	0.480	0.764	0.988	1.000	1.000	1.000	1.000
D_n^+	0.523	0.798	0.990	1.000	1.000	1.000	1.000
T_n	0.341	0.667	0.973	1.000	1.000	1.000	1.000
$O_n \vert D_n^+$	0.502	0.777	0.985	1.000	1.000	1.000	1.000
$R_n^{\mathrm{mom}} \vert D_n^+$	0.425	0.746	0.988	1.000	1.000	1.000	1.000

Table 21: Power on two-component mixture alternative. $\alpha = 0.05$, $v = 10.00$, $\epsilon = 0.5$

The Tables 22 to 25 exhibit the power on two-component mixtures depending on the mean v of the second component, for $\alpha = 0.10$, $n = 100$ and 1000, $\epsilon = 0.1$ and 0.5.

Test	v							
	1.25	1.50	2.00	3.00	4.00	5.00	7.00	10.00
O_n	0.110	0.148	0.269	0.577	0.789	0.893	0.969	0.991
R_n^{mom}	0.110	0.145	0.262	0.555	0.766	0.870	0.953	0.962
A_n^2	0.100	0.109	0.138	0.306	0.524	0.706	0.888	0.967
D_n^+	0.105	0.128	0.208	0.461	0.682	0.823	0.942	0.984
T_n	0.103	0.109	0.162	0.373	0.605	0.763	0.915	0.976
$O_n \vert D_n^+$	0.110	0.138	0.249	0.554	0.768	0.884	0.966	0.990
$R_n^{\mathrm{mom}} \vert D_n^+$	0.111	0.140	0.252	0.557	0.768	0.884	0.966	0.990

Table 22: Power on two-component mixture alternative. $\alpha = 0.10$, $n = 100$, $\epsilon = 0.1$

Test	v							
	1.25	1.50	2.00	3.00	4.00	5.00	7.00	10.00
O_n	0.122	0.190	0.380	0.736	0.911	0.973	0.996	0.999
R_n^{mom}	0.119	0.176	0.309	0.490	0.536	0.525	0.478	0.425
A_n^2	0.100	0.127	0.224	0.605	0.876	0.972	0.999	1.000
D_n^+	0.115	0.174	0.345	0.743	0.935	0.987	1.000	1.000
T_n	0.102	0.130	0.260	0.669	0.906	0.980	0.999	1.000
$O_n \vert D_n^+$	0.129	0.186	0.372	0.756	0.933	0.985	0.999	1.000
$R_n^{\mathrm{mom}} \vert D_n^+$	0.130	0.187	0.366	0.746	0.928	0.984	0.999	1.000

Table 23: Power on two-component mixture alternative. $\alpha = 0.10$, $n = 100$, $\epsilon = 0.5$

Test	v							
	1.25	1.50	2.00	3.00	4.00	5.00	7.00	10.00
O_n	0.136	0.266	0.737	0.998	1.000	1.000	1.000	1.000
$R_n^{\rm mom}$	0.134	0.265	0.728	0.998	1.000	1.000	1.000	1.000
A_n^2	0.103	0.127	0.340	0.949	1.000	1.000	1.000	1.000
D_n^+	0.123	0.195	0.534	0.987	1.000	1.000	1.000	1.000
T_n	0.107	0.144	0.442	0.970	1.000	1.000	1.000	1.000
$O_n\|D_n^+$	0.129	0.245	0.687	0.998	1.000	1.000	1.000	1.000
$R_n^{\rm mom}\|D_n^+$	0.116	0.236	0.687	0.998	1.000	1.000	1.000	1.000

Table 24: Power on two-component mixture alternative. $\alpha = 0.10$, $n = 1000$, $\epsilon = 0.1$

Test	v							
	1.25	1.50	2.00	3.00	4.00	5.00	7.00	10.00
O_n	0.192	0.463	0.951	1.000	1.000	1.000	1.000	1.000
$R_n^{\rm mom}$	0.183	0.419	0.859	0.969	0.977	0.967	0.939	0.865
A_n^2	0.114	0.241	0.844	1.000	1.000	1.000	1.000	1.000
D_n^+	0.161	0.393	0.930	1.000	1.000	1.000	1.000	1.000
T_n	0.123	0.298	0.906	1.000	1.000	1.000	1.000	1.000
$O_n\|D_n^+$	0.183	0.437	0.950	1.000	1.000	1.000	1.000	1.000
$R_n^{\rm mom}\|D_n^+$	0.169	0.403	0.929	1.000	1.000	1.000	1.000	1.000

Table 25: Power on two-component mixture alternative. $\alpha = 0.10$, $n = 1000$, $\epsilon = 0.5$

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