DISCUSSION PAPERS IN STATISTICS
AND ECONOMETRICS

SEMINAR OF ECONOMIC AND SOCIAL STATISTICS
UNIVERSITY OF COLOGNE

No. 9/07

Dependence of Stock Returns in Bull and Bear Markets

by

Jadran Dobrić
Gabriel Frahm
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1st version
December 12, 2007

DISKUSSIONSBEITRÄGE ZUR
STATISTIK UND ÖKONOMETRIE

SEMINAR FÜR WIRTSCHAFTS- UND SOZIALSTATISTIK
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Albertus-Magnus-Platz, D-50923 Köln, Deutschland

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Keywords: Bear market, bootstrapping, bull market, conditional Spearman's rho, copulas, Monte Carlo simulation, stock returns.

JEL Subject Classification: Primary C14, Secondary C12.

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Dependence of Stock Returns in Bull and Bear Markets

Jadran Dobrić[*] Gabriel Frahm[†] Friedrich Schmid[‡]
Department of Economic and Social Statistics, University of Cologne
Albertus-Magnus-Platz, 50923 Cologne, Germany

December 5, 2007

Abstract

Pearson's correlation coefficient is typically used for measuring the dependence structure of stock returns. Nevertheless, it has many shortcomings often documented in the literature. We suggest to use a conditional version of Spearman's rho as an alternative dependence measure. Our approach is purely nonparametric and we avoid any kind of model misspecification. We derive hypothesis tests for the conditional Spearman's rho in bull and bear markets and verify the tests by Monte Carlo simulation. Further, we study the daily returns of stocks contained in the German stock index DAX 30. We find some significant differences in dependence of stock returns in bull and bear markets. On the other hand the differences are not so strong as one might expect.

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1 Introduction

The linear correlation coefficient according to Karl Pearson still seems to be the most commonly used measure of dependence of two random variables X and Y though its many shortcomings have been often documented (see, e.g., Embrechts et al., 2002). Pearson's correlation coefficient is strongly affected by the marginal distributions of X and Y and its estimates are sensitive to outliers (Lindskog, 2000). Further, the linear correlation coefficient quantifies only linear dependence though *monotone dependence* is often much more relevant. The random variables X and Y possess a strong monotone dependence if we can find two real-valued and strictly increasing functions f and g such that $|\mathrm{Corr}\{f(X), g(Y)\}|$ is large. It is easy to construct dependence structures

[*]Phone: +49 221 7212556, email: jaddob@web.de.
[†]Phone: +49 221 470-4267, email: frahm@statistik.uni-koeln.de.
[‡]Phone: +49 221 470-2813, email: schmid@wiso.uni-koeln.de.

where the linear correlation coefficient of X and Y is close to 0 but, however, corresponds to 1 after some monotone transformations of the random variables (McNeil et al., 2005, p. 205).

Copula theory and the dependence measures derived thereof are a convincing alternative. Due to Sklar's theorem (Sklar, 1959) it is known that a joint distribution function can be split up into its *copula* (i.e. its dependence structure) and its marginal distributions. A meaningful dependence measure should be invariant under monotone transformations of the components of the random vector. Examples of such measures are Spearman's rho, Kendall's tau, Gini's gamma, and Blomquist's beta. In this paper we confine ourselves to the *rank correlation coefficient*, i.e. Spearman's rho. For surveys on copulas and dependence measures see, e.g., Cherubini et al. (2004), Joe (1997), and Nelsen (2006).

We investigate the contemporaneous dependence of two stock returns X and Y. In particular, we concentrate on the question whether dependence is significantly different in bull and bear markets, i.e. in case of a joint upswing or downswing. This question and related problems have been already investigated in finance literature (see, e.g., Ang and Chen, 2002, Erb et al., 1994, Fortin and Kuzmics, 2002, Junker and May, 2005, Patton, 2004, Silvapulle and Granger, 2001, Vaz de Melo Mendes, 2005). But we think that the statistical methods, in particular the use of Pearson's correlation coefficient is unsatisfactory. Hence, there is space for further contributions.

Bear and bull markets are characterized as follows. There is a *bear market* if the two stock returns X and Y contemporaneously fall short of the $100p\%$ quantiles of their corresponding cumulative distribution functions. Analogously, a bull market is present whenever both $-X$ and $-Y$ fall short of the corresponding $100q\%$ quantiles. Here p and q have to be pre-determined. A quantile of the cumulative distribution function of a stock return is commonly known as the *value-at-risk* under the specified shortfall probability. This measure is frequently used in finance literature and risk management. So it seems to be a natural choice for characterizing bull and bear markets.

Our approach is purely nonparametric. Contrary to Patton (2004) and Vaz de Melo Mendes (2005) we do not fit specific copulas to the data. Specifying the copula by some parametric model can lead to erroneous conclusions if the chosen model is wrong. From our point of view it is not necessary to rely on the parametric approach if the sample size is large enough. We are interested in financial data analysis and in that context it is easy to access many thousands of observations. By following the nonparametric approach we avoid any kind of model misspecification.

In this work we develop conditional versions of Spearman's rho to assess the dependence structure of stock returns in bull and bear markets. In contrast, some authors analyze the dependence structure of outliers in financial data by using the so-called *tail-dependence coefficient* (Fortin and Kuzmics, 2002, Junker and May, 2005). After applying parametric methods these authors come to the conclusion that 'the empirical joint distribution of return pairs on stock indices displays high tail-dependence in the lower tail and low tail-dependence in the upper tail' (Fortin and Kuzmics, 2002). Dobrić and Schmid (2005) as well as Frahm et al. (2005) found that estimating the tail-dependence coefficient by *nonparametric methods* can lead to very large estimation errors even if there are many observations. Hence the tail-dependence coefficient is not an appropriate alternative.

Though we focus on computational statistics and the empirical analysis of stock returns we have to introduce some statistical theory in order to have a formal basis for our testing procedures. This is done in section 2.1, where some copula theory is presented. It allows a precise formulation of the null hypotheses to be tested. The testing procedures are described in section 2.2. A Monte Carlo (MC) simulation is presented in Section 2.3 which shows that the procedures work well for sample sizes which are typically available in practice. In particular the procedures keep the prescribed error probabilities of the first kind and have sufficient power to detect violations of the null hypothesis. In Section 3 we investigate the daily returns of stocks from the German stock index *DAX 30* between 1992-03-02 and 2002-03-01 and Section 4 concludes.

2 Testing Conditional Dependence

This section introduces some notions from copula theory which are required as a basis for the testing procedure to be described below. Comprehensive introductions to the theory of copulas are Joe (1997) and Nelsen (2006). The testing procedure is then introduced and its finite sample properties are investigated in the final part of the section.

2.1 Some Copula Theory

Let X and Y denote two random variables with joint distribution function $F(x,y) = \mathrm{P}(X \leq x, Y \leq y)$ and marginal distribution functions $G(x) = \mathrm{P}(X \leq x)$ and $H(y) = \mathrm{P}(Y \leq y)$ for all $x, y \in \mathbb{R}$. The corresponding quantile functions are given by $G^{-1}(p) = \inf\{x\colon G(x) \geq p\}$ and $H^{-1}(p) = \inf\{y\colon H(y) \geq p\}$ for $0 \leq p \leq 1$. In Section 3, X and Y will denote daily returns of two stocks.

Throughout this paper we assume that G and H are continuous functions. Therefore, according to Sklar's theorem (Sklar, 1959) there exists a unique copula $C\colon [0,1]^2 \to [0,1]$ such that

$$F(x,y) = C(G(x), H(y))\,, \qquad \forall\, x, y \in \mathbb{R}\,.$$

The function C is the joint distribution function of $U = G(X)$ and $V = H(Y)$. The rank correlation coefficient of X and Y is now given by

$$\rho := \mathrm{Corr}(G(X), H(Y)) = 12 \int_{[0,1]^2} uv \, dC(u,v) - 3\,.$$

See Nelsen (2006, p. 167) for the latter representation of Spearman's rho.

For every fixed p with $0 < p < 1$ we define

$$A_{\mathrm{L}} := \left\{(x,y)\colon x \leq G^{-1}(p), y \leq H^{-1}(p)\right\}.$$

In the following we assume that $\mathrm{P}\{(X,Y) \in A_{\mathrm{L}}\} = C(p,p) > 0$. Consider the *conditional* joint distribution function

$$\begin{aligned} F_{\mathrm{L}}(x,y) &:= \mathrm{P}(X \le x, Y \le y \,|\, (X,Y) \in A_{\mathrm{L}}) = \frac{F(x \wedge G^{-1}(p), y \wedge H^{-1}(p))}{F(G^{-1}(p), H^{-1}(p))} \\ &= \frac{C(G(x \wedge G^{-1}(p)), H(y \wedge H^{-1}(p)))}{C(p,p)}, \qquad \forall\, x, y \in \mathbb{R}\,. \end{aligned}$$

The corresponding conditional marginal distribution functions are given by

$$\begin{aligned} G_{\mathrm{L}}(x) &:= \mathrm{P}(X \le x \,|\, (X,Y) \in A_{\mathrm{L}}) = F_{\mathrm{L}}(x, H^{-1}(p)) \\ &= \frac{C(G(x \wedge G^{-1}(p)), p)}{C(p,p)}, \qquad \forall\, x \in \mathbb{R}\,, \end{aligned}$$

and $H_{\mathrm{L}}(y)$ respectively. As G_{L} and H_{L} are continuous distribution functions, according to Sklar's theorem there exists also a unique copula $C_{\mathrm{L}} : [0,1]^2 \to [0,1]$ such that

$$F_{\mathrm{L}}(x,y) = C_{\mathrm{L}}(G_{\mathrm{L}}(x), H_{\mathrm{L}}(y)), \qquad \forall\, x, y \in \mathbb{R}\,.$$

Indeed, Juri and Wüthrich (2002) call

$$C_{\mathrm{L}}(u,v) = F_{\mathrm{L}}(G_{\mathrm{L}}^{-1}(u), H_{\mathrm{L}}^{-1}(v)), \qquad \forall\, u, v \in [0,1]\,,$$

the *extreme tail dependence copula relative to C at the level p*. We call C_{L} *lower tail copula* and the phrase 'relative to C at the level p' will be usually dropped for convenience.

Using the lower tail copula we now can define the lower conditional Spearman's rho, viz

$$\rho_{\mathrm{L}} = 12 \int_{[0,1]^2} uv \, dC_{\mathrm{L}}(u,v) - 3\,.$$

Hence, ρ_{L} measures the rank correlation of stock returns conditional on $(X,Y) \in A_{\mathrm{L}}$.

An analogue definition can be found for the *upper tail copula* C_{U}. This is the lower tail copula relative to the *survival copula* according to C (Nelsen, 2006, Section 2.6), i.e.

$$\overline{C}(u,v) := u + v - 1 + C(1-u, 1-v), \qquad \forall\, u, v \in [0,1]\,,$$

at the level q ($0 < q < 1$). The survival copula corresponds to the copula of $(-X,-Y)$ and thus C_{U} is the copula of $(-X,-Y)$ under the condition that $(-X,-Y) \in A_{\mathrm{U}}$. Here the area A_{U} is calculated similarly to A_{L} just by using the quantile functions of $-X$ and $-Y$ at q rather than the quantile functions of X and Y at p. Hence, the upper conditional Spearman's rho ρ_{U} measures the rank correlation of stock returns in a bull market. In the following we will have to guarantee that $A_{\mathrm{L}} \cap A_{\mathrm{U}} = \emptyset$ and thus $p + q \le 1$.

In most cases it is not possible to derive the conditional copulas C_{L} or C_{U} in closed form. Therefore ρ_{L} and ρ_{U} cannot be calculated explicitly. However, MC simulation is a convenient tool for obtaining numerical approximations to ρ_{L} and ρ_{U} with sufficient precision. We apply this method to calculate the conditional rank correlation coefficients for the Gauss, t_3, Clayton, and Gumbel copula (see Table 1 and Table 2). The Gauss and t_3 copula are given by

$$C_{\text{Gauss}}(u, v\,;\theta) = \Phi_\theta(\Phi^{-1}(u), \Phi^{-1}(v))\,, \qquad \forall\, u, v \in [0,1]\,,$$

where

$$\Phi_\theta(x,y) := \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\theta^2}} \cdot \exp\left(-\frac{s^2 - 2\theta st + t^2}{2\,(1-\theta^2)}\right) dsdt$$

as well as

$$C_{t_3}(u, v\,;\theta) = t_{3,\theta}(t_3^{-1}(u), t_3^{-1}(v))\,, \qquad \forall\, u, v \in [0,1]\,,$$

with

$$t_{3,\theta}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-\theta^2}} \cdot \left(1 + \frac{s^2 - 2\theta st + t^2}{3\,(1-\theta^2)}\right)^{-\frac{5}{2}} dsdt\,,$$

where t_3 denotes Student's univariate t distribution function with 3 degrees of freedom and $-1 < \theta < 1$. Note that the linear correlation coefficient is symbolized by the parameter θ rather than ρ. This is because to avoid possible confusions with the unconditional rank correlation coefficient of C_{Gauss} or C_{t_3}. The unconditional Spearman's rho for the Gauss copula corresponds to $\rho = 6/\pi \cdot \arcsin(\theta/2)$ (Hult and Lindskog, 2002). For the t_3 copula to our knowledge there exists no closed-form expression.

The Clayton copula is given by

$$C_{\text{Clayton}}(u, v\,;\theta) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \qquad \forall\, u, v \in [0,1]\,,$$

where $\theta \geq 0$. In the limiting case $\theta = 0$ the Clayton copula corresponds to the *independence* or *product copula* $\Pi(u, v) := uv$ (Nelsen, 2006, p. 11).

The Gumbel copula can be written as

$$C_{\text{Gumbel}}(u, v\,;\theta) = \exp\Big[-\big\{(-\log u)^\theta + (-\log v)^\theta\big\}^{1/\theta}\Big]\,, \qquad \forall\, u, v \in [0,1]\,,$$

with $\theta \geq 1$. Note that for $\theta = 1$ once again the independence copula evolves. The values for θ in Table 2 are chosen such that the *unconditional* Spearman's rho corresponds to $\rho = 0.3, 0.5$, and 0.7. The relationship between θ and ρ can be obtained by numerical integration or MC simulation (cf. Joe, 1997, p. 147).

For our approximations of the conditional rank correlation coefficients (see Table 1 and Table 2) we use $N_{\text{MC}} = 1000$ MC replications, each one generating a sample from C with sample size $n = 10^6$. Both for the simulation study and for the empirical study following later on we set $p = q$. Note that only the Clayton copula allows for an explicit representation of C_{L}. If C is a Clayton copula then the lower tail copula C_{L} corresponds to C for any $0 < p < 1$ (Juri and Wüthrich, 2002). That means that ρ_{L} corresponds to the unconditional Spearman's rho of C.

Gauss copula						
	$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	lower	upper	lower	upper	lower	upper
0.05	.0404 (.0004)	.0407 (.0004)	.1109 (.0003)	.1114 (.0003)	.2622 (.0002)	.2624 (.0002)
0.20	.0601 (.0001)	.0601 (.0001)	.1595 (.0001)	.1593 (.0001)	.3485 (.0001)	.3483 (.0001)
0.35	.0775 (.0001)	.0774 (.0001)	.1972 (.0001)	.1973 (.0001)	.4090 (.0001)	.4091 (.0001)
0.50	.0962 (.0001)	.0962 (.0001)	.2354 (.0001)	.2356 (.0001)	.4655 (.0000)	.4656 (.0000)
t_3 copula						
	$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	lower	upper	lower	upper	lower	upper
0.05	.3373 (.0003)	.3369 (.0003)	.4043 (.0002)	.4044 (.0002)	.5264 (.0002)	.5265 (.0002)
0.20	.3186 (.0001)	.3183 (.0001)	.3968 (.0001)	.3967 (.0001)	.5361 (.0001)	.5361 (.0001)
0.35	.2984 (.0001)	.2984 (.0001)	.3913 (.0001)	.3913 (.0001)	.5484 (.0001)	.5485 (.0001)
0.50	.2756 (.0001)	.2756 (.0001)	.3882 (.0001)	.3882 (.0001)	.5651 (.0000)	.5652 (.0000)

Table 1: MC approximations to ρ_{L} and ρ_{U} for the Gauss and t_3 copula possessing different values for θ. We use $N_{\mathrm{MC}} = 1000$ MC replications, each one generating a sample from the corresponding copula with sample size $n = 10^6$. The standard errors of the approximations are given in parentheses.

The null hypothesis we are going to test can be formalized as

$$\begin{aligned} & H_0: \quad \rho_{\mathrm{L}} = \rho_{\mathrm{U}} \\ \text{vs.} \quad & H_1: \quad \rho_{\mathrm{L}} \neq \rho_{\mathrm{U}} \,, \end{aligned}$$

where some p and q with $p + q \leq 1$ are fixed. In our framework H_0 implies that the monotone dependence of stock returns in bear markets is the same as in bull markets. Here we consider the lower $100p\%$ and upper $100q\%$ of stock returns to characterize the bear and the bull market, respectively.

Instead of a two-sided hypothesis test, a one-sided test like

$$\begin{aligned} & H_0: \quad \rho_{\mathrm{L}} \leq \rho_{\mathrm{U}} \\ \text{vs.} \quad & H_1: \quad \rho_{\mathrm{L}} > \rho_{\mathrm{U}} \end{aligned}$$

is of general interest.

The null hypothesis $H_0\colon \rho_{\mathrm{L}} = \rho_{\mathrm{U}}$ stated above might be also of importance in another context. Both in theory and application of copulas it is sometimes of interest whether the random vector (X, Y) is *radially symmetric* or not (Nelsen, 2006, Section 2.7). Radial symmetry is a useful property which guarantees that $\rho_{\mathrm{L}} = \rho_{\mathrm{U}}$ for all $0 < p < 1$ since C and the corresponding survival copula coincide. In order to test the null hypothesis H_0': 'The random vector (X, Y) is radially symmetric', one can apply the two-sided test and reject H_0' if H_0 is rejected.

Clayton copula						
	$\theta = 0.5112$		$\theta = 1.0759$		$\theta = 2.1326$	
$p = q$	lower	upper	lower	upper	lower	upper
0.05	.3004 (.0002)	.0025 (.0005)	.5001 (.0002)	.0018 (.0004)	.7002 (.0001)	.0035 (.0004)
0.20	.3003 (.0001)	.0040 (.0001)	.4999 (.0001)	.0113 (.0001)	.7000 (.0001)	.0318 (.0001)
0.35	.3001 (.0001)	.0130 (.0001)	.4999 (.0001)	.0356 (.0001)	.7000 (.0000)	.0906 (.0001)
0.50	.3001 (.0001)	.0298 (.0001)	.5000 (.0000)	.0764 (.0001)	.7000 (.0000)	.1783 (.0001)
Gumbel copula						
	$\theta = 1.26$		$\theta = 1.54$		$\theta = 2.07$	
$p = q$	lower	upper	lower	upper	lower	upper
0.05	.0319 (.0004)	.3499 (.0002)	.0697 (.0003)	.4504 (.0002)	.1431 (.0003)	.5849 (.0001)
0.20	.0515 (.0001)	.3158 (.0001)	.1106 (.0001)	.4392 (.0001)	.2206 (.0001)	.5871 (.0001)
0.35	.0697 (.0001)	.2906 (.0001)	.1476 (.0001)	.4314 (.0001)	.2843 (.0001)	.5916 (.0000)
0.50	.0912 (.0001)	.2744 (.0001)	.1885 (.0001)	.4276 (.0001)	.3507 (.0000)	.5990 (.0000)

Table 2: MC approximations to ρ_{L} and ρ_{U} for the Clayton and Gumbel copula possessing different values for θ. We use $N_{\mathrm{MC}} = 1000$ MC replications, each one generating a sample from the corresponding copula with sample size $n = 10^6$. The standard errors of the approximations are given in parentheses.

2.2 The Testing Procedures

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. as (X, Y). As we do not assume that the marginal distribution functions G and H are known, we have to estimate them by

$$\widehat{G}_n(x) = \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbf{1}_{\{X_i \leq x\}} \qquad \text{and} \qquad \widehat{H}_n(y) = \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbf{1}_{\{Y_i \leq y\}} .$$

The corresponding estimates for $\widehat{G}_n^{-1}(p)$ and $\widehat{H}_n^{-1}(p)$ can be derived thereof. For some fixed p and q with $p + q \leq 1$ we can define

$$\widehat{A}_{\mathrm{L}} := \left\{ (x, y) \colon x \leq \widehat{G}_n^{-1}(p), y \leq \widehat{H}_n^{-1}(p) \right\}$$

and $\widehat{A}_{\mathrm{U}}$ respectively. Further, let $n_{\mathrm{L}} := |\widehat{A}_{\mathrm{L}}|$ and $n_{\mathrm{U}} := |\widehat{A}_{\mathrm{U}}|$, where $|\cdot|$ denotes the cardinality of a set. The observations in $\widehat{A}_{\mathrm{L}}$ and $\widehat{A}_{\mathrm{U}}$ can be used for estimating ρ_{L} and ρ_{U}. More precisely,

$$\hat{\rho}_{\mathrm{L},n} = \frac{12}{n_{\mathrm{L}}} \cdot \sum_{i \in \mathbf{I}_{\widehat{A}_{\mathrm{L}}}} \frac{\mathrm{r}_{\mathrm{L},n}(X_i)}{n_{\mathrm{L}}} \cdot \frac{\mathrm{r}_{\mathrm{L},n}(Y_i)}{n_{\mathrm{L}}} - 3 ,$$

where $\mathbf{I}_{\widehat{A}_{\mathrm{L}}}$ denotes the set of indices i where (X_i, Y_i) lies in $\widehat{A}_{\mathrm{L}}$.

Further, $\mathrm{r}_{\mathrm{L},n}(\cdot)$ is the rank of a marginal observation relative to all observations in $\widehat{A}_{\mathrm{L}}$, i.e. the lower left area of the empirical copula

$$\widehat{C}_n(u,v) := \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbf{1}_{\{\mathrm{r}_n(X_i)/n \leq u\}} \mathbf{1}_{\{\mathrm{r}_n(Y_i)/n \leq v\}} \,, \qquad \forall\, u,v \in [0,1] \,.$$

Note that $\mathrm{r}_{\mathrm{L},n}(X_i)/n_{\mathrm{L}} = \widehat{G}_{\mathrm{L},n}(X_i)$ and $\mathrm{r}_{\mathrm{L},n}(Y_i)/n_{\mathrm{L}} = \widehat{H}_{\mathrm{L},n}(Y_i)$, where $\widehat{G}_{\mathrm{L},n}$ is the empirical counterpart of G_{L}, i.e.

$$\widehat{G}_{\mathrm{L},n}(x) = \frac{\widehat{C}_n(\widehat{G}_n(x \wedge \widehat{G}_n^{-1}(p)),p)}{\widehat{C}_n(p,p)} \,, \qquad \forall\, x \in \mathbb{R} \,,$$

and $\widehat{H}_{\mathrm{L},n}$ is defined respectively. The estimator $\hat{\rho}_{\mathrm{U},n}$ for the upper conditional Spearman's rho can be similarly defined, just by using the survival copula according to $\widehat{C}_n$, i.e. the observations in the upper right area $\widehat{A}_{\mathrm{U}}$.

It has been already shown that Spearman's rho is consistent and asymptotically normally distributed (Schmid and Schmidt, 2006b). The same holds for the conditional versions of Spearman's rho described above, i.e.

$$\sqrt{n_{\mathrm{L}}} \cdot \big(\hat{\rho}_{\mathrm{L},n} - \rho_{\mathrm{L}}\big) \xrightarrow{\mathrm{d}} \mathcal{N}\big(0, \sigma_{\mathrm{L}}^2\big) \quad \text{and} \quad \sqrt{n_{\mathrm{U}}} \cdot \big(\hat{\rho}_{\mathrm{U},n} - \rho_{\mathrm{U}}\big) \xrightarrow{\mathrm{d}} \mathcal{N}\big(0, \sigma_{\mathrm{U}}^2\big)$$

as $n_{\mathrm{L}}, n_{\mathrm{U}} \to \infty$.

In practical situations p and q have to be sufficiently large such that n_{L} and n_{U} do not become too small. We have found $p, q \geq \log(n)/\sqrt{n}$ as an appropriate rule of thumb for analyzing daily stock returns. E.g. for the sample size $n = 1000$ (that means we have an observation period of approximately 4 years) p and q should be larger than 0.2184. In case of the product copula we would expect to meet $0.2184^2 \cdot 1000 \approx 48$ data points in the lower left or upper right corner of the empirical copula. Admittedly, financial data cannot be appropriately described by the product copula but we can assume that there is some sort of positive dependence between stock returns. So there are even *more* observations in the corresponding corners of the empirical copula. Thus our rule of thumb guarantees that there are always enough data for large sample inferences.

The asymptotic variances σ_{L}^2 and σ_{U}^2 depend on the tail copulas C_{L} and C_{U}. In general they cannot be calculated explicitly. However, they can be approximated by a simple bootstrap procedure (Schmid and Schmidt, 2006a). Note that the observations contained in A_{L} and A_{U} stem from two disjoint sets and thus are stochastically independent. The same holds for $\widehat{A}_{\mathrm{L}}$ and $\widehat{A}_{\mathrm{U}}$, asymptotically, and the following procedure for the two-sided hypothesis test becomes straightforward:

1. Compute $\hat{\rho}_{\mathrm{L},n}$ and $\hat{\rho}_{\mathrm{U},n}$ from the observations in $\widehat{A}_{\mathrm{L}}$ and $\widehat{A}_{\mathrm{U}}$, where p and q are fixed with $p + q \leq 1$.

2. Compute N_{B} bootstrap replications of $\hat{\rho}_{\mathrm{L},n}$ and $\hat{\rho}_{\mathrm{U},n}$ from the observations in $\widehat{A}_{\mathrm{L}}$ and $\widehat{A}_{\mathrm{U}}$ and calculate the corresponding estimates for the asymptotic variances σ_{L}^2 and σ_{U}^2, say $\widehat{\sigma}_{\mathrm{L}}^2$ and $\widehat{\sigma}_{\mathrm{U}}^2$.

3a. Reject $H_0 \colon \rho_{\mathrm{L}} = \rho_{\mathrm{U}}$ if

$$\left| \frac{\hat{\rho}_{\mathrm{L},n} - \hat{\rho}_{\mathrm{U},n}}{\sqrt{\widehat{\sigma}^2_{\mathrm{L}}/n_{\mathrm{L}} + \widehat{\sigma}^2_{\mathrm{U}}/n_{\mathrm{U}}}} \right| \geq \Phi^{-1}\Big(1 - \frac{\alpha}{2}\Big)\,,$$

where $\alpha > 0$ is a small error probability of the first kind and Φ denotes the distribution function of the standard normal distribution.

The one-sided hypothesis tests differ only in the third step from the two-sided test, i.e.

3b. Reject $H_0 \colon \rho_{\mathrm{L}} \leq \rho_{\mathrm{U}}$ or $H_0 \colon \rho_{\mathrm{L}} \geq \rho_{\mathrm{U}}$ if

$$\frac{\hat{\rho}_{\mathrm{L},n} - \hat{\rho}_{\mathrm{U},n}}{\sqrt{\widehat{\sigma}^2_{\mathrm{L}}/n_{\mathrm{L}} + \widehat{\sigma}^2_{\mathrm{U}}/n_{\mathrm{U}}}} \geq \Phi^{-1}(1 - \alpha)\,,$$

or

$$\frac{\hat{\rho}_{\mathrm{L},n} - \hat{\rho}_{\mathrm{U},n}}{\sqrt{\widehat{\sigma}^2_{\mathrm{L}}/n_{\mathrm{L}} + \widehat{\sigma}^2_{\mathrm{U}}/n_{\mathrm{U}}}} \leq \Phi^{-1}(\alpha)\,.$$

2.3 Finite Sample Properties

This section investigates the statistical properties of the testing procedures described at the end of the last section. The results are obtained by MC simulations for various special cases. These are essentially defined by the copula under study. First we are interested in the rejection probability of the procedure if $H_0 : \rho_{\mathrm{L}} = \rho_{\mathrm{U}}$ is true and α is the prescribed error probability of the first kind. We consider the Gauss and t_3 copula which belong to the class of elliptical copulas (Frahm et al., 2003). These are radially symmetric for every $-1 < \theta < 1$ and thus the null hypotheses holds. The selected values for the copula parameter are $\theta = 0.25, 0.5, 0.75$, the values for p are given by $p = 0.2, 0.35, 0.5$, and we validate the error probabilities $\alpha = 0.01, 0.05$, and $\alpha = 0.1$. The simulated sample size is $n = 2500$, the number of bootstrap replications corresponds to $N_{\mathrm{B}} = 1000$, and the number of MC replications is $N_{\mathrm{MC}} = 1000$. The results of the simulations are summarized in Panel 1 of Table 3. We can see that the approximated rejection probabilities satisfactorily agree with the prescribed error probabilities.

We are also interested in the *power* of the testing procedure, i.e. the probability of rejection provided H_0 is wrong. For that purpose we consider the Clayton and the Gumbel copula. It is well-known that these copulas are not radially symmetric and thus $\rho_{\mathrm{L}} \neq \rho_{\mathrm{U}}$ holds in general. Remember that the parameter θ of both copula families (see p. 5) has been selected in such a way that the unconditional Spearman's rho is equal to $\rho = 0.3, 0.5$ and $\rho = 0.7$. The results of the MC simulations are given in Panel 2 of Table 3. It can be seen that for every fixed p and α the power is an increasing function of θ. This is because the asymmetry of the Archimedean copulas C_{Clayton} and C_{Gumbel} increases with θ (cf. Nelsen, 2006, Ch. 4).

Similar results are obtained for the two one-sided tests which can be taken from Table 4 and Table 5. The rejection probabilities become very large whenever H_1 is true. In

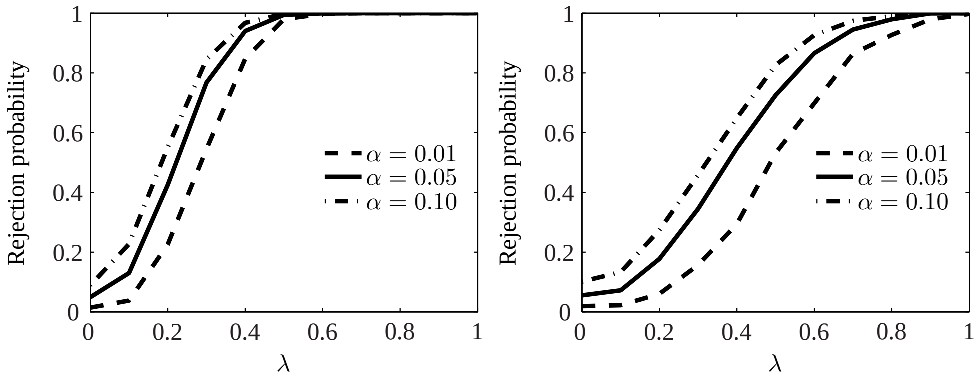

Figure 1: Power functions of the two-sided hypothesis test for the mixed copulas C_{Mix1} (left hand side) and C_{Mix2} (right hand side) as a function of λ. The results are obtained by MC simulation for the sample size $n = 2500$, $N_{\text{B}} = 1000$ bootstrap replications, and $N_{\text{MC}} = 1000$ MC replications using the threshold probability $p = q = 0.5$.

contrast, if H_0 is true our simulations produce no false rejection. For instance, consider the right-sided test $H_0 : \rho_{\text{L}} \leq \rho_{\text{U}}$ vs. $H_1 : \rho_{\text{L}} > \rho_{\text{U}}$. In that case the null hypothesis is fulfilled for the Gumbel copula. Panel 2 of Table 4 shows that there is no rejection for any given unconditional rank correlation coefficient ρ, threshold probability p, and error probability α. In contrast, for the Clayton copula the alternative hypothesis is true and consequently the rejection probabilities are very high (e.g. roughly 90% for $\rho = 0.3$, $p = 0.2$, and $\alpha = 0.1$). Moreover, for $\rho = 0.5$ and $\rho = 0.7$, H_0 is rejected for the Clayton copula in almost every simulated case.

Now we want to investigate the relationship between asymmetry and power. For that purpose we consider the mixed copula

$$C_{\text{Mix1}}(u, v\,; \lambda, \theta_0, \theta_1) := \lambda C_{\text{Clayton}}(u, v\,; \theta_1) + (1 - \lambda) C_{\text{Gauss}}(u, v\,; \theta_0)\,,$$

where $0 \leq \lambda \leq 1$. Further, the copula parameters θ_0, θ_1 are such that the *unconditional* Spearman's rho of $C_{\text{Clayton}}(u, v\,; \theta_1)$ and $C_{\text{Gauss}}(u, v\,; \theta_0)$ corresponds to $\rho = 0.5$. Hence, the mixed copula possesses the same unconditional rank correlation coefficient for every λ (see the formula for ρ on p. 3). Since $\rho_{\text{L}} = \rho_{\text{U}}$ is true for the Gauss copula but for the Gumbel copula it holds that $\rho_{\text{L}} < \rho_{\text{U}}$, the mixing parameter λ determines the degree of asymmetry given by $C_{\text{Mix1}}(u, v\,; \lambda, \theta_0, \theta_1)$. If we consider the two-sided hypothesis test, $\lambda = 0$ means that the null hypothesis is true whereas the alternative hypothesis holds for every $\lambda > 0$. The larger λ the more we shall expect to reject H_0.

A similar result is obtained for the mixed copula

$$C_{\text{Mix2}}(u, v\,; \lambda, \theta_0, \theta_2) := \lambda C_{\text{Gumbel}}(u, v\,; \theta_2) + (1 - \lambda) C_{\text{Gauss}}(u, v\,; \theta_0)\,,$$

where θ_2 is such that Spearman's rho of $C_{\text{Gumbel}}(u, v\,; \theta_2)$ once again corresponds to $\rho = 0.5$. The corresponding power functions are given in Figure 1. The power functions illustrated in Figure 1 are simulated on the basis of $n = 2500$, $N_{\text{B}} = 1000$, and $N_{\text{MC}} = 1000$ using the threshold probability $p = 0.5$. We can see that the two-sided hypothesis test exhibits more power in case of the Clayton/Gauss copula

$H_0: \rho_L = \rho_U$ vs. $H_1: \rho_L \neq \rho_U$							
Panel 1		$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	α	Gauss	t_3	Gauss	t_3	Gauss	t_3
0.20	0.10	.107 (.0098)	.082 (.0087)	.112 (.0100)	.107 (.0098)	.101 (.0095)	.086 (.0089)
	0.05	.056 (.0073)	.039 (.0061)	.054 (.0071)	.052 (.0070)	.045 (.0066)	.041 (.0063)
	0.01	.011 (.0033)	.005 (.0022)	.013 (.0036)	.008 (.0028)	.009 (.0030)	.009 (.0030)
0.35	0.10	.088 (.0090)	.102 (.0096)	.085 (.0088)	.100 (.0095)	.101 (.0095)	.094 (.0092)
	0.05	.057 (.0073)	.047 (.0067)	.041 (.0063)	.048 (.0068)	.041 (.0063)	.040 (.0062)
	0.01	.011 (.0033)	.013 (.0036)	.007 (.0026)	.009 (.0030)	.011 (.0033)	.009 (.0030)
0.50	0.10	.115 (.0101)	.094 (.0092)	.100 (.0095)	.115 (.0101)	.120 (.0103)	.088 (.0090)
	0.05	.051 (.0070)	.038 (.0060)	.049 (.0068)	.059 (.0075)	.063 (.0077)	.036 (.0059)
	0.01	.010 (.0031)	.005 (.0022)	.015 (.0038)	.016 (.0040)	.016 (.0040)	.006 (.0024)
Panel 2		$\rho = 0.30$		$\rho = 0.50$		$\rho = 0.70$	
$p = q$	α	Clayton	Gumbel	Clayton	Gumbel	Clayton	Gumbel
0.20	0.10	.836 (.0117)	.738 (.0139)	1.000 (.0000)	.984 (.0040)	1.000 (.0000)	.999 (.0010)
	0.05	.728 (.0141)	.637 (.0152)	.998 (.0014)	.953 (.0067)	1.000 (.0000)	.999 (.0010)
	0.01	.505 (.0158)	.393 (.0154)	.993 (.0026)	.845 (.0114)	1.000 (.0000)	.989 (.0033)
0.35	0.10	.994 (.0024)	.934 (.0079)	1.000 (.0000)	.999 (.0010)	1.000 (.0000)	1.000 (.0000)
	0.05	.985 (.0038)	.890 (.0099)	1.000 (.0000)	.995 (.0022)	1.000 (.0000)	1.000 (.0000)
	0.01	.929 (.0081)	.763 (.0134)	1.000 (.0000)	.983 (.0041)	1.000 (.0000)	1.000 (.0000)
0.50	0.10	1.000 (.0000)	.977 (.0047)	1.000 (.0000)	.999 (.0010)	1.000 (.0000)	1.000 (.0000)
	0.05	1.000 (.0000)	.952 (.0068)	1.000 (.0000)	.998 (.0014)	1.000 (.0000)	1.000 (.0000)
	0.01	.994 (.0024)	.831 (.0119)	1.000 (.0000)	.995 (.0022)	1.000 (.0000)	1.000 (.0000)

Table 3: MC approximations of the rejection probabilities for the Gauss and t_3 copula (Panel 1) and for the Clayton and Gumbel copula (Panel 2) given $H_0: \rho_L = \rho_U$. The simulated sample size is $n = 2500$, the number of bootstrap replications corresponds to $N_B = 1000$, and the number of MC replications is $N_{MC} = 1000$. The standard errors for the approximated rejection probabilities are given in parentheses.

$H_0: \rho_L \leq \rho_U$ vs. $H_1: \rho_L > \rho_U$							
Panel 1		$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	α	Gauss	t_3	Gauss	t_3	Gauss	t_3
0.20	0.10	.107 (.0098)	.089 (.0090)	.104 (.0097)	.099 (.0094)	.104 (.0097)	.103 (.0096)
	0.05	.050 (.0069)	.040 (.0062)	.053 (.0071)	.056 (.0073)	.049 (.0068)	.045 (.0066)
	0.01	.007 (.0026)	.007 (.0026)	.011 (.0033)	.012 (.0034)	.009 (.0030)	.007 (.0026)
0.35	0.10	.101 (.0095)	.114 (.0101)	.085 (.0088)	.094 (.0092)	.097 (.0094)	.089 (.0090)
	0.05	.047 (.0067)	.057 (.0073)	.038 (.0060)	.045 (.0066)	.055 (.0072)	.044 (.0065)
	0.01	.010 (.0031)	.011 (.0033)	.009 (.0030)	.009 (.0030)	.014 (.0037)	.010 (.0031)
0.50	0.10	.108 (.0098)	.093 (.0092)	.093 (.0092)	.110 (.0099)	.123 (.0104)	.089 (.0090)
	0.05	.052 (.0070)	.045 (.0066)	.051 (.0070)	.066 (.0079)	.063 (.0077)	.045 (.0066)
	0.01	.009 (.0030)	.007 (.0026)	.012 (.0034)	.017 (.0041)	.018 (.0042)	.005 (.0022)
Panel 2		$\rho = 0.30$		$\rho = 0.50$		$\rho = 0.70$	
$p = q$	α	Clayton	Gumbel	Clayton	Gumbel	Clayton	Gumbel
0.20	0.10	.903 (.0094)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.05	.836 (.0117)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.01	.600 (.0155)	.000 (.0000)	.997 (.0017)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
0.35	0.10	.999 (.0010)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.05	.994 (.0024)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.01	.962 (.0060)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
0.50	0.10	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.05	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)
	0.01	.995 (.0022)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)	.000 (.0000)

Table 4: MC approximations of the rejection probabilities for the Gauss and t_3 copula (Panel 1) and for the Clayton and Gumbel copula (Panel 2) given $H_0: \rho_L \leq \rho_U$. The simulated sample size is $n = 2500$, the number of bootstrap replications corresponds to $N_B = 1000$, and the number of MC replications is $N_{MC} = 1000$. The standard errors for the approximated rejection probabilities are given in parentheses.

$H_0: \rho_L \geq \rho_U$ vs. $H_1: \rho_L < \rho_U$							
Panel 1		$\theta = 0.25$		$\theta = 0.50$		$\theta = 0.75$	
$p = q$	α	Gauss	t_3	Gauss	t_3	Gauss	t_3
0.20	0.10	.101 (.0095)	.082 (.0087)	.096 (.0093)	.100 (.0095)	.084 (.0088)	.091 (.0091)
	0.05	.057 (.0073)	.042 (.0063)	.059 (.0075)	.051 (.0070)	.052 (.0070)	.041 (.0063)
	0.01	.015 (.0038)	.004 (.0020)	.014 (.0037)	.009 (.0030)	.006 (.0024)	.010 (.0031)
0.35	0.10	.098 (.0094)	.104 (.0097)	.099 (.0094)	.100 (.0095)	.098 (.0094)	.100 (.0095)
	0.05	.041 (.0063)	.045 (.0066)	.047 (.0067)	.055 (.0072)	.046 (.0066)	.050 (.0069)
	0.01	.014 (.0037)	.010 (.0031)	.010 (.0031)	.010 (.0031)	.005 (.0022)	.010 (.0031)
0.50	0.10	.118 (.0102)	.109 (.0099)	.103 (.0096)	.102 (.0096)	.110 (.0099)	.083 (.0087)
	0.05	.063 (.0077)	.049 (.0068)	.049 (.0068)	.049 (.0068)	.057 (.0073)	.043 (.0064)
	0.01	.015 (.0038)	.006 (.0024)	.014 (.0037)	.014 (.0037)	.010 (.0031)	.009 (.0030)
Panel 2		$\rho = 0.30$		$\rho = 0.50$		$\rho = 0.70$	
$p = q$	α	Clayton	Gumbel	Clayton	Gumbel	Clayton	Gumbel
0.20	0.10	.000 (.0000)	.838 (.0117)	.000 (.0000)	.996 (.0020)	.000 (.0000)	1.000 (.0000)
	0.05	.000 (.0000)	.738 (.0139)	.000 (.0000)	.984 (.0040)	.000 (.0000)	.999 (.0010)
	0.01	.000 (.0000)	.485 (.0158)	.000 (.0000)	.901 (.0094)	.000 (.0000)	.997 (.0017)
0.35	0.10	.000 (.0000)	.975 (.0049)	.000 (.0000)	.999 (.0010)	.000 (.0000)	1.000 (.000)
	0.05	.000 (.0000)	.934 (.0079)	.000 (.0000)	.999 (.0010)	.000 (.0000)	1.000 (.0000)
	0.01	.000 (.0000)	.829 (.0119)	.000 (.0000)	.987 (.0036)	.000 (.0000)	1.000 (.0000)
0.50	0.10	.000 (.0000)	.993 (.0026)	.000 (.0000)	1.000 (.0000)	.000 (.0000)	1.000 (.0000)
	0.05	.000 (.0000)	.977 (.0047)	.000 (.0000)	.999 (.0010)	.000 (.0000)	1.000 (.0000)
	0.01	.000 (.0000)	.886 (.0101)	.000 (.0000)	.997 (.0017)	.000 (.0000)	1.000 (.0000)

Table 5: MC approximations of the rejection probabilities for the Gauss and t_3 copula (Panel 1) and for the Clayton and Gumbel copula (Panel 2) given $H_0: \rho_L \geq \rho_U$. The simulated sample size is $n = 2500$, the number of bootstrap replications corresponds to $N_B = 1000$, and the number of MC replications is $N_{MC} = 1000$. The standard errors for the approximated rejection probabilities are given in parentheses.

	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
$\bar{\hat{\rho}}_{\mathrm{L}}$	.3609	.3009	.3149	.3387	.3479
$\bar{\hat{\rho}}_{\mathrm{U}}$	.2685	.2409	.2400	.2571	.2749
$\overline{\hat{\rho}_{\mathrm{L}} - \hat{\rho}_{\mathrm{U}}}$	.0924	.0600	.0749	.0816	.0730
$\overline{\lvert\hat{\rho}_{\mathrm{L}} - \hat{\rho}_{\mathrm{U}}\rvert}$	.1528	.0891	.0829	.0840	.0754

Table 6: Average conditional rank correlation coefficients, differences, and absolute differences of all 231 asset combinations for different threshold probabilities $p = q$.

C_{Mix1}. However, both figures demonstrate that the hypothesis test always keeps the prescribed error probability of the first kind and the rejection probability indeed is an increasing function of the mixing parameter λ. Similar results can be obtained for other constellations of ρ and p.

3 Empirical Results for German Stock Returns

Now we consider daily returns of 21 stocks of the German stock index DAX 30 from 1992-03-02 to 2002-03-01 and the stock index itself. More precisely, the considered stock prices were adjusted by dividends, splits, etc., and our analyzes are based on the daily log-returns of the stocks. The number of observations is $n = 2523$. Table 6 contains the sample means of the upper and lower conditional Spearman's rho for all asset combinations given the threshold probabilities or, say, value-at-risk levels $p = 0.1, 0.2, 0.3, 0.4$, and $p = 0.5$. Here $\bar{\hat{\rho}}_{\mathrm{L}}$ symbolizes the mean lower and $\bar{\hat{\rho}}_{\mathrm{U}}$ the mean upper conditional Spearman's rho, $\overline{\hat{\rho}_{\mathrm{L}} - \hat{\rho}_{\mathrm{U}}}$ is the mean difference, whereas $\overline{\lvert\hat{\rho}_{\mathrm{L}} - \hat{\rho}_{\mathrm{U}}\rvert}$ denotes the mean absolute difference between $\hat{\rho}_{\mathrm{L}}$ and $\hat{\rho}_{\mathrm{U}}$. We can see that in average the lower conditional rank correlations are between 6 and 10 points larger than the upper conditional rank correlations. However, without some meaningful economical arguments it is not possible to judge whether this gap between bull and bear markets is rather 'large' or 'small' and we would like to avoid such kind of statements.

In contrast, we will discuss how much of the empirical evidence leads to significant results in our hypothesis tests. It is worth to point out that the outcomes of the test generally depend on the probability threshold p . Figure 2 shows the estimates of the lower and upper conditional Spearman's rho as a function of p for Allianz vs. BASF and Allianz vs. Munich Re. We can see that the difference between $\hat{\rho}_{\mathrm{L}}$ and $\hat{\rho}_{\mathrm{U}}$ for Allianz vs. Munich Re essentially depends on the chosen threshold whereas for Allianz vs. BASF the difference is roughly stable. However, in both cases $\hat{\rho}_{\mathrm{L}} - \hat{\rho}_{\mathrm{U}}$ increases as p approaches to zero. That means the rank correlation coefficients of stock returns seem to be substantially different between situations of panic and elation. From this arguments it should be clear that the hypothesis test works only if p is chosen *before* examining different estimates for ρ_{L} and ρ_{U}. Otherwise the test would suffer from a selection bias.

Panel 1	$H_0: \rho_L = \rho_U$ vs. $H_1: \rho_L \neq \rho_U$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
0.10	37	34	67	89	107
0.05	25	16	51	67	72
0.01	6	3	19	31	30
Panel 2	$H_0: \rho_L \leq \rho_U$ vs. $H_1: \rho_L > \rho_U$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
0.10	55	61	94	136	148
0.05	35	32	67	89	107
0.01	11	4	30	42	50
$\hat{\rho}_L > \hat{\rho}_U$	170	173	196	218	217
Panel 3	$H_0: \rho_L \geq \rho_U$ vs. $H_1: \rho_L < \rho_U$				
α	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$	$p = 0.5$
0.10	9	7	3	0	0
0.05	2	2	0	0	0
0.01	1	0	0	0	0
$\hat{\rho}_L < \hat{\rho}_U$	61	58	35	13	14

Table 7: Numbers of rejections for the different hypothesis tests, various threshold probabilities $p = q$, and error probabilities α given 231 asset combinations. Further, the numbers of asset combinations where $\hat{\rho}_L$ is larger or smaller than $\hat{\rho}_U$ (in the last rows of Panel 2 and Panel 3).

3.1 Two-Sided Hypothesis Test

There are $\binom{22}{2} = 231$ combinations of the 21 stocks and the stock index. It is clear that the estimates $\hat{\rho}_L$ and $\hat{\rho}_U$ are different from each other for every combination and we want to see whether the differences are significant. That means we test $H_0: \rho_L = \rho_U$ against $H_1 : \rho_L \neq \rho_U$ by using the procedure described in Section 2.2. The first panel of Table 7 contains the number of rejections for all 231 asset combinations. For $p = 0.1$ only 25 of 231 asset combinations (i.e. roughly 11%) and for $p = 0.3$ only 51 (that means about 22%) are significantly different on the 5% level, etc. However, we see that for all p taken into consideration the proportions of rejection exceed the corresponding error probability of the first kind. More precisely, since the number of rejections is always larger than 231α we can conclude that in general ρ_L does not correspond to ρ_U for daily asset returns.

3.2 One-Sided Hypothesis Tests

Panel 2 and 3 of Table 7 contain the number of asset combinations where $\hat{\rho}_L > \hat{\rho}_U$ and $\hat{\rho}_L < \hat{\rho}_U$. For $p = 0.1$ there are 170 asset combinations with $\hat{\rho}_L > \hat{\rho}_U$ and only 35 ($\approx$ 21%) of these combinations are significant. For $p = 0.3$ there are 67 of 196 ($\approx$ 34%) significant asset combinations with $\hat{\rho}_L > \hat{\rho}_U$ on the 5% level, etc. It is clear that not every combination with $\hat{\rho}_L > \hat{\rho}_U$ or $\hat{\rho}_L < \hat{\rho}_U$ can be significant. This holds especially if the number of observations in the lower left and upper right part of the empirical

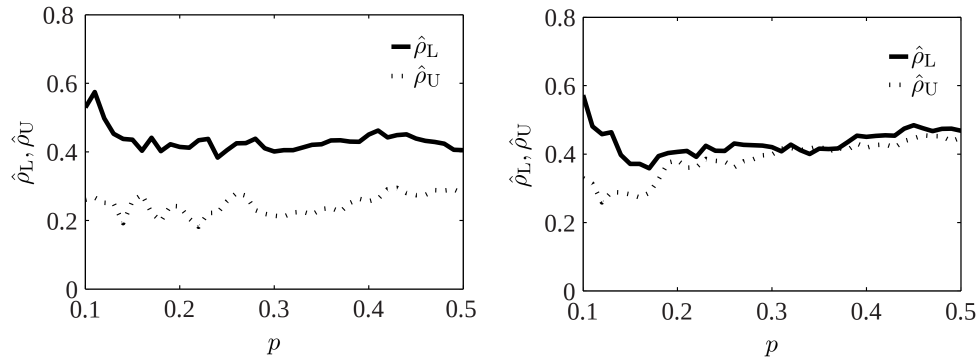

Figure 2: Estimates of the lower and upper conditional Spearman's rho as a function of $p = q$ for Allianz vs. BASF (left hand) and Allianz vs. Munich Re (right hand).

copula is small. However, the fact that the proportion of significant combinations is relatively small of course neither implicates that the corresponding null hypotheses are true nor that the distances between the true rank correlation coefficients ρ_{L} and ρ_{U} are small (cf. the last row of Table 6).

First consider the second panel of Table 7 which contains the number of rejections for various levels of p and α. The numbers of rejection exceed 231α in all cases. This indicates that $\rho_{\mathrm{L}} > \rho_{\mathrm{U}}$ is a typical constellation for daily asset returns. In contrast, for the opposite test $H_0 : \rho_{\mathrm{L}} \geq \rho_{\mathrm{U}}$ vs. $H_1 : \rho_{\mathrm{L}} < \rho_{\mathrm{U}}$ the number of rejections given in Table 7 (Panel 3) are always smaller than 231α. Thus we can assume that most of the documented rejections are errors of the first kind.

Many empirical studies suggest that the *linear* dependence of stock returns is larger in bear markets than in bull markets (see, e.g., Ang and Chen, 2002, Erb et al., 1994). Our results of the one-sided hypothesis tests confirm findings in the finance literature where Pearson's linear correlation coefficient is used as a dependence measure. That means in bear markets stock returns depend more on each other than in bull markets where the notion of 'dependence' is represented by the rank correlation coefficient.

4 Conclusion

Several authors have investigated the dependencies of stock returns in bull and bear markets. Pearson's correlation coefficient has been typically used as a canonical dependence measure. Unfortunately, it essentially depends on the marginal distributions of the random variables which are taken into consideration and quantifies only the degree of linear dependence. However, often we are interested in the degree of monotone rather than linear dependence. This holds especially if the marginal distributions are highly non-standard which is definitely the case if we concentrate on the tails of stock return distributions. So it is crucial to find a reasonable dependence measure for the degree of monotone dependence under the condition that stock returns go up or down, contemporaneously. We believe that copula theory can serve as an appropriate toolbox and suggest Spearman's rho as a dependence measure. This is in contrast to the previous literature where e.g. conditional correlation coefficients are used for the same purpose. Moreover, our approach is purely nonparametric. Since we do not fit specific copulas to the data we can avoid a model misspecification. We successfully verified the proposed one- and two-sided hypothesis tests by several MC simulations. Further, we conducted an empirical study using daily returns of stocks contained in the DAX 30. Of course, everybody can draw his own conclusions from the empirical results. But we think that there is sufficient evidence to support the hypothesis of different dependence structures in bull and bear markets. On the other hand the deviations probably are not as strong as one might expect.

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