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Stochastic Dominance with Nonadditive Probabilities

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Abstract: Choquet expected utility which uses capacities (i.e. nonadditive probability measures) in place of σ-additive probability measures has been introduced to decision making under uncertainty to cope with observed effects of ambiguity aversion like the Ellsberg paradox. In this paper we present necessary and sufficient conditions for stochastic dominance between capacities (i.e. the expected utility with respect to one capacity exceeds that with respect to the other one for a given class of utility functions). One wide class of conditions refers to probability inequalities on certain families of sets. To yield another general class of conditions we present sufficient conditions for the existence of a probability measure P with $\int f\,dC = \int f\,dP$ for all increasing functions f when C is a given capacity. Examples include n-th degree stochastic dominance on the reals and many cases of so-called set dominance. Finally, applications to decision making are given including anticipated utility with unknown distortion function.

Key words and Phrases: expected utility, Choquet integral, anticipated utility, rank dependent expected utility, n-th degree stochastic dominance, set dominance.

1 Introduction

A nonadditive probability measure (also called capacity) is a set function which is defined on a set algebra, ranges from 0 to 1, and is increasing. Every probability measure is a capacity. Integration with respect to a capacity has been investigated by Choquet (1953/54).

In recent years nonadditive probability measures have been used in decision theory to cope with observed violations of expected utility (EU). Schmeidler (1989), Gilboa (1987, 1989), Wakker (1989), and Nakamura (1990) provide sets of axioms under which a given preference between uncertain prospects can be represented by the Choquet integral of a utility function with respect to a capacity. These models were developed to explain observed effects of ambiguity aversion like the Ellsberg paradox. Closely related to this are the nonlinear expected utility models by Quiggin (1982), Segal (1984), and Chew, Karni, and

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Safra (1987) which are known as anticipated utility (AU), rank dependent expected utility (RDEU), or expected utility with rank dependent probabilities (EURDP), and the dual theory of choice under risk by Yaari (1987). These models employ different transforms or "distortions" of usual probabilities. Other applications of nonadditive probability measures arise in the computation of insurance premiums (Denneberg 1989) and in robust statistics (Huber and Strassen 1973).

In these models Choquet integrals of a utility function have to be evaluated to decide between two given decision alternatives. However, in practice, the exact form of the utility function may not be known. The question is how to decide between two given capacities when properties of the utility function such as increasingness and/or concavity have been assessed only.

Stochastic dominance between capacities with respect to a class $\mathscr{U}$ of utility functions is defined as follows: A capacity C_1 dominates another capacity C_2 with respect to $\mathscr{U}$ ($C_1 \succeq_{\mathscr{U}} C_2$) if for every utility function u in $\mathscr{U}$ the C_1-integral of u exceeds the C_2-integral of u.

In decision making the following standard framework is used: Let S be a set of possible states of nature, $\mathscr{C}$ a set of consequences. An act is a function $f: S \to \mathscr{C}$. The set of all acts is denoted by $\mathscr{F}(S, \mathscr{C})$. If an act $f \in \mathscr{F}(S, \mathscr{C})$ is chosen and $s \in S$ comes out to be the true state of nature the result is a consequence $f(s)$. This is the general framework of *decision making under uncertainty*. If, in addition, a probability measure P on S is given then we have the framework of *decision making under risk*.

In decision making under uncertainty a decision maker is called a *Choquet expected utility maximizer* (or short CEU-Maximizer) if there exists a capacity C on the states of nature and a utility function u on the space of consequences such that an act f is preferred to another act g if and only if

$$\int_S u \circ f \, dC \geq \int_S u \circ g \, dC \ .$$

See Gilboa (1987), Schmeidler (1989).

Let

$$C_f(A) = C(\{s \in S | f(s) \in A\}) \ , \qquad A \in \mathscr{C} \ ,$$

$$C_g(A) = C(\{s \in S | g(s) \in A\}) \ , \qquad A \in \mathscr{C}$$

and $\mathscr{U}$ be a class of utility functions. Then, $C_f \succeq_{\mathscr{U}} C_g$ if and only if every CEU maximizer with capacity C and utility function u in $\mathscr{U}$ prefers act f to act g. When $\mathscr{U}$ and C are known we can use stochastic dominance results as decision rules.

Apart from decision making under uncertainty, stochastic dominance between usual probability measures has been applied to many fields of operations research such as queueing, reliability, activity networks, simulation, stochastic programming, inventory, and financial management. See the collection of papers in Mosler and Scarsini (1991a).

In this paper we give necessary and sufficient conditions for stochastic dominance of capacities with respect to various classes of utility functions. We investigate which of the stochastic dominance results in EU can be extended to the nonadditive case. One can say that this extension is easy if the space of consequences is weakly ordered (complete and transitive). With a general space of consequences this extension is still possible if all considered utility functions represent the same preference over consequences.

One class of conditions for stochastic dominance refers to probability inequalities on certain families of sets. These dominance relations are referred to as *set dominance relations*. We demonstrate that many of the results on set dominance relations derived in Mosler and Scarsini (1991b) and Bergmann (1991) can be extended to the nonadditive case; see also Scarsini (1992).

We show this by using a weakened version of measurability which has been introduced by Greco (1981): If $\mathcal{T}$ is a class of sets then a function f is called quasi-$\mathcal{T}$-measurable if for every $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ there exists a set $T_{\alpha,\varepsilon}$ in $\mathcal{T}$ such that $\{x | f(x) \geq \alpha\} \subset T_{\alpha,\varepsilon} \subset \{x | f(x) \geq \alpha - \varepsilon\}$.

Now, if f is a quasi-$\mathcal{T}$-measurable function, it can be shown that the Choquet integral of f is fully determined by the values of C on $\mathcal{T}$. Therefore, if C_1, C_2 are two capacities and $C_1|_{\mathcal{T}} \leq C_2|_{\mathcal{T}}$ (where $C_1|_{\mathcal{T}}$ is the restriction of C_1 to $\mathcal{T}$ and the inequality is to be understood pointwise) then $\int f \, dC_1 \leq \int f \, dC_2$ for all f which are quasi-$\mathcal{T}$-measurable. Further, the set of quasi-$\mathcal{T}$-measurable functions is the largest set of functions for which the above implication holds. Characterizations of many stochastic dominance relations can be derived from these results.

Scarsini (1992) has shown that for every capacity C on a weakly ordered space there exists a finitely additive probability measure P with $\int f \, dC = \int f \, dP$ for all increasing functions. We investigate the problem under which circumstances this finitely additive measure can be extended to a σ-additive probability measure. We show that this is the case whenever the capacity is continuous on a certain class of subsets of the space and two additional technical assumptions apply which refer to the order structure of the space. The technical assumptions are fulfilled in most applications, in particular, if the weakly ordered space is the real line or a discrete space.

By this, well known results about classes of utility functions which represent the same preference order over outcomes can be extended to the nonadditive case. Examples are the results about n-th degree stochastic dominance on $\mathbb{R}$ (e.g. Mosler 1982) and other dominance relations of probability measures on the real line. Further examples are stochastic dominance relations with multivariate utility functions which have been investigated by Levy and Levy (1984).

Our results can be applied to decision making under Choquet expected utility as well as to other fields of operations research. Moreover, in the framework of anticipated utility (AU), we present rules which allow for dominance decision between risky alternatives when the distortion function is only known to belong to a given class of functions.

Section 2 reviews the definitions of upper and lower Choquet integrals and their main properties. Stochastic dominance with respect to capacities is introduced, and a first result concerning first order dominance on the reals is given. Section 3 includes an investigation of general set dominance relations and many examples. Section 4 addresses the problem of representing the Choquet integral as an integral with respect to a proper σ-additive measure and presents characterizations of higher degree dominance on the reals and of other dominance relations. Section 5 includes applications to decision making, in particular to anticipated utility.

2 Preliminaries

This section contains definitions and a preliminary result which is extended in subsequent sections.

Definition 1: Let Ω be a given set and $\mathcal{S}$ a set algebra on Ω. A set function $C: \mathcal{S} \to \mathbb{R}$ is called a nonadditive probability measure *if*

(i) $C(\varnothing) = 0$, $C(\Omega) = 1$,
(ii) $S, T \in \mathcal{S}$, $S \subset T$ *implies* $C(S) \leq C(T)$.

We also call C a *capacity* and $(\Omega, \mathcal{S}, C)$ a *capacity space.* (Parts of the literature use more restricted notions of capacity.) The set of capacities on $(\Omega, \mathcal{S})$ is denoted by $\mathcal{C}(\mathcal{S})$. We say that a function $f: \Omega \to \mathbb{R}$ is $\mathcal{S}$*-measurable* if $\{\omega \in \Omega | f(\omega) \geq t\}$ is in $\mathcal{S}$ for every t in $\mathbb{R}$, and we notate by $\mathcal{F}_0(\mathcal{S})$ the set of $\mathcal{S}$-measurable functions. For $C \in \mathcal{C}(\mathcal{S})$ and $f \in \mathcal{F}_0(\mathcal{S})$, the *upper Choquet integral* is defined as follows (Choquet 1953/54),

$$\int f \, dC = \int_0^\infty C(\{\omega | f(\omega) \geq t\}) \, dt + \int_{-\infty}^0 [C(\{\omega | f(\omega) \geq t\}) - 1] \, dt \tag{2.1}$$

whenever the right hand side integrals exist in the Riemann sense and their sum is defined. If both right hand side integrals are finite f is called *upper C-integrable.*

It can be shown that the upper Choquet integral satisfies

(i) *positive homogeneity*

$$\int \lambda f \, dC = \lambda \int f \, dC \ , \qquad \text{if } \lambda \geq 0 \ , \tag{2.2}$$

(ii) *comonotonic additivity*

$$\int (f + g) \, dC = \int f \, dC + \int g \, dC \ , \qquad \text{if } f \text{ and } g \text{ are comonotonic} \ , \tag{2.3}$$

i.e., if $(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0$ for all ω, ω' in Ω.

For $C \in \mathscr{C}(\mathscr{S})$, the *dual capacity* C^D is given by $C^D(S) = 1 - C(S^C)$, $S \in \mathscr{S}$. The integral of f with respect to C^D is called *lower Choquet integral* of f (Gilboa 1989),

$$\int^D f \, dC = \int f \, dC^D \ .$$

Upper and lower Choquet integral are related:

$$\int (-f) \, dC = -\int^D f \, dC \ , \tag{2.4}$$

$$\int 1_S \, dC = C(S) \ , \tag{2.5}$$

$$\int^D 1_S \, dC = 1 - C(S^C) \ , \tag{2.6}$$

where 1_S denotes the indicator function of S.

Note that the lower Choquet integral is a positive homogeneous and comonotonic additive functional, too. Upper and lower Choquet integral coincide if and only if $C^D = C$, i.e., $C(S^C) = 1 - C(S)$ for all S. If C is a probability measure ($\mathscr{S}$ σ-algebra, C σ-additive capacity), then both Choquet integrals are equal to the integral in the Lebesgue sense.

Stochastic Dominance of capacities is defined as follows.

Definition 2: (Stochastic dominance of capacities) *Let* $C_1, C_2 \in \mathscr{C}(\mathscr{S})$ *and* $\mathscr{F} \subset \mathscr{F}_0(\mathscr{S})$. *Then,* C_1 *dominates* C_2 *with respect to* $\mathscr{F}$ ($C_1 \succeq_{\mathscr{F}} C_2$), *if*

$$\int f\, dC_1 \geq \int f\, dC_2 \qquad \textit{for all } f \in \mathscr{F} \textit{ for which both integrals exist} \ . \tag{2.7}$$

Let Ω be a linear space and assume that for every set S in $\mathscr{S}$ the set $-S = \{-\omega | \omega \in S\}$ is in $\mathscr{S}$. Let

$$\mathscr{F}^D = \{g\colon \Omega \to \mathbb{R} | g(\omega) = -f(-\omega) \text{ for some } f \in \mathscr{F} \text{ and all } \omega \in \Omega\} \ .$$

Then, $\geq_{\mathscr{F}^D}$ is called the *dual dominance relation* of $\geq_{\mathscr{F}}$. Denote $\overline{C}(S) = C(-S)$, $S \in \mathscr{S}$. It is easily derived from (2.1) and (2.4) that

$$C_1 \geq_{\mathscr{F}^D} C_2 \quad \text{if and only if} \quad \overline{C}_1^D \geq_{\mathscr{F}} \overline{C}_2^D \ . \tag{2.8}$$

E.g., if $\mathscr{F}$ is the class of convex functions, $\mathscr{F}^D$ is the class of concave functions. Similarly, if $\mathscr{F}$ is the class of increasing[1] convex functions on some partially ordered set, then $\mathscr{F}^D$ is the class of increasing concave functions.

As a simple but important special case consider first degree stochastic dominance on the real line: Let $\Omega = \mathbb{R}$, $\mathscr{S} = \mathscr{B}$ (the Borel sets), and $\mathscr{F}_1 = \{f \in \mathscr{F}_0(\mathscr{B}) | f \textit{ increasing}\}$. The following proposition on dominance of capacities is well known in the context of probability measures.

Proposition 1: Let $C_1, C_2 \in \mathscr{C}(\mathscr{B})$ *and* $\mathscr{F}_1 = \{f \in \mathscr{F}_0(\mathscr{B}) | f \textit{ increasing}\}$. *Then* $C_1 \geq_{\mathscr{F}_1} C_2$ *if and only if*

$$C_1([t, \infty[) \geq C_2([t, \infty[) \qquad \textit{for all } t \in \mathbb{R}$$

and (2.9)

$$C_1(]t, \infty[) \geq C_2(]t, \infty[) \qquad \textit{for all } t \in \mathbb{R} \ .$$

Proof: Necessity: Let $C_1 \geq_{\mathscr{F}_1} C_2$, i.e., $\int f\, dC_1 \geq \int f\, dC_2$ for all $f \in \mathscr{F}_1$. As $1_{[t,\infty[}$ is in $\mathscr{F}_1$ for every $t \in \mathbb{R}$ and from (2.5) we get

$$C_1([t, \infty[) = \int 1_{[t,\infty[}\, dC_1 \geq \int 1_{[t,\infty[}\, dC_2 = C_2([t, \infty[) \ .$$

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[1] Here and in the whole paper by "increasing" we mean "nondecreasing" and by "decreasing" we mean "nonincreasing" with respect to the given ordering.

The same holds for indicator functions $1_{]t,\infty[}$, $t \in \mathbb{R}$. To show sufficiency let $f \in \mathscr{F}_1$ and assume (2.9). For $t \in \mathbb{R}$, define $f^{-1}(t) = \inf\{x | f(x) \geq t\}$; then $\{x | f(x) \geq t\} = U_{f^{-1}(t)}$, where $U_{f^{-1}(t)}$ equals $[f^{-1}(t), \infty[$ or $]f^{-1}(t), \infty[$. Hence, by (2.1) and (2.9),

$$\begin{aligned}\int f\, dC_1 &= \int_0^\infty C_1(U_{f^{-1}(t)})\, dt + \int_{-\infty}^0 [C_1(U_{f^{-1}(t)}) - 1]\, dt \\ &\geq \int_0^\infty C_2(U_{f^{-1}(t)})\, dt + \int_{-\infty}^0 [C_2(U_{f^{-1}(t)}) - 1]\, dt \\ &= \int f\, dC_2 \ . \end{aligned}$$

□

3 Set Dominance of Nonadditive Probability Measures

Let $\mathscr{C}(\mathscr{S})$ be the set of nonadditive probability measures on $(\Omega, \mathscr{S})$ as above, and let $\mathscr{F} \subset \mathscr{F}_0(\mathscr{S})$. The stochastic dominance relation is called a *set dominance relation* if there exists some $\mathscr{T} \subset \mathscr{S}$ such that $C_1 \succeq_{\mathscr{F}} C_2$ is equivalent to

$$C_1(T) \geq C_2(T) \qquad \text{for every } T \in \mathscr{T} \ . \tag{3.1}$$

From Proposition 1 we know that the usual stochastic order (or first degree stochastic dominance) of capacities on the real line is a set dominance relation, where $\mathscr{T} = \{[t, \infty[| t \in \mathbb{R}\} \cup \{]t, \infty[| t \in \mathbb{R}\}$.

With usual σ-additive probability measures, many other stochastic orderings can be shown to be set dominance relations (Mosler and Scarsini 1991b, Bergmann 1991). Most of these results are extended here to the nonadditive case. Further, for these orderings minimal and maximal families of functions are determined which are monotone with respect to the given stochastic ordering.

In search of families of functions $\mathscr{F}$ which are minimal or maximal (in a sense to be specified) to determine a set dominance relation, we use a weakened notion of measurability which was introduced by Greco (1981).

Definition 3: (Quasi-$\mathscr{T}$-measurability) *Let $\mathscr{T}$ be a subset of $\mathscr{S}$ such that $\varnothing$, $\Omega \in \mathscr{T}$. A function $f: \Omega \to \mathbb{R}$ is called* quasi-$\mathscr{T}$-measurable *if for all $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ there exists a set $T_{\alpha,\varepsilon}$ in $\mathscr{T}$ with*

$$\{x \in \Omega | f(x) \geq \alpha\} \subset T_{\alpha,\varepsilon} \subset \{x \in \Omega | f(x) \geq \alpha - \varepsilon\} \ . \tag{3.2}$$

Let $\mathscr{Q}(\mathscr{T})$ denote the set of quasi-$\mathscr{T}$-measurable functions. In particular, if the upper level sets $\{x \in \Omega | f(x) \geq \alpha\}$ are in $\mathscr{T}$ for every α then f is in $\mathscr{Q}(\mathscr{T})$ and thus $\mathscr{F}_0(\mathscr{S}) \subset \mathscr{Q}(\mathscr{S})$. If $\mathscr{T}$ is a monotone class[2] then a quasi-$\mathscr{T}$-measurable function is also $\mathscr{T}$-measurable. Moreover, if $\mathscr{T}$ is a σ-algebra, then every quasi-$\mathscr{T}$-measurable function is measurable in the usual sense. Let $\mathscr{I}(\mathscr{T})$ denote the set of indicator functions 1_T with $T \in \mathscr{T}$. Obviously, $\mathscr{I}(\mathscr{T}) \subset \mathscr{Q}(\mathscr{T})$, and for every indicator function 1_T that is in $\mathscr{Q}(\mathscr{T})$ we have $T \in \mathscr{T}$.

Theorem 1: Assume $\mathscr{F} \subset \mathscr{F}_0(\mathscr{S})$, $\mathscr{I}(\mathscr{T}) \subset \mathscr{F} \subset \mathscr{Q}(\mathscr{T})$, $C_1, C_2 \in \mathscr{C}(\mathscr{S})$. *Then*

$$\int f \, dC_1 \geq \int f \, dC_2 \qquad \text{for all } f \in \mathscr{F}$$

if and only if

$$C_1(T) \geq C_2(T) \qquad \text{for every } T \in \mathscr{T} \ .$$

Proof: For $T \in \mathscr{T}$ and $f = 1_T$, we have $f \in \mathscr{F}$ and $\int f \, dC_i = C_i(T)$, $i = 1, 2$. So, the necessity of (3.1) is obvious.

To prove sufficiency, let $f \in \mathscr{F}$, hence $f \in \mathscr{Q}(\mathscr{T})$. For $\alpha \in \mathbb{R}$ and $\varepsilon > 0$, there exists some $T_{\alpha,\varepsilon}$ with

$$\{f \geq \alpha\} \subset T_{\alpha,\varepsilon} \subset \{f \geq \alpha - \varepsilon\} \ .$$

It follows from (3.1) that

$$C_2(f \geq \alpha) \leq C_2(T_{\alpha,\varepsilon}) \leq C_1(T_{\alpha,\varepsilon}) \leq C_1(f \geq \alpha - \varepsilon)$$

and thus

$$\int_0^\infty C_2(f \geq \alpha) \, d\alpha \leq \int_0^\infty C_1(f \geq \alpha - \varepsilon) \, d\alpha = \int_{-\varepsilon}^\infty C_1(f \geq \alpha) \, d\alpha \ .$$

Similarly,

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[2] A set system is called monotone class if the union of an increasing sequence and the intersection of a decreasing sequence of sets $T_i \in \mathscr{T}$ are again in $\mathscr{T}$.

$$\int_{-\infty}^{0} [C_2(f \geq \alpha) - 1]\, d\alpha \leq \int_{-\infty}^{-\varepsilon} [C_1(f \geq \alpha) - 1]\, d\alpha \ .$$

Adding the two inequalities and letting ε approach zero yields

$$\int f\, dC_2 \leq \int f\, dC_1 \ . \qquad \square$$

Corollary 1: Assume $\mathscr{F} \subset \mathscr{F}_0(\mathscr{S})$, $\mathscr{I}(\mathscr{T}) \subset \mathscr{F} \subset \mathscr{Q}(\mathscr{T})$, $C_1, C_2 \in \mathscr{C}(\mathscr{S})$. *Then*

$$C_1 \succeq_{\mathscr{F}} C_2 \quad \textit{and} \quad C_2 \succeq_{\mathscr{F}} C_1$$

if and only if

$$C_1(T) = C_2(T) \qquad \textit{for every } T \in \mathscr{T} \ .$$

The corollary is immediate from Theorem 1. It follows from the corollary that a set dominance relation $\succeq_{\mathscr{F}}$ is in general not antisymmetric. The relation comes out to be antisymmetric if $\mathscr{T}$ is dense enough in $\mathscr{S}$, viz., if equality of the restrictions $C_1|_{\mathscr{F}} = C_2|_{\mathscr{F}}$ on $\mathscr{T}$ implies equality $C_1 = C_2$ on $\mathscr{S}$.

Remark: Corollary 1 tells further that for $f \in \mathscr{Q}(\mathscr{T})$, $C \in \mathscr{C}(\mathscr{S})$, the value of the integral $\int f\, dC$ does not depend on $C(T)$, $T \in \mathscr{S} \setminus \mathscr{T}$. Thus, an extended integral $\int f\, dC$ for $f \in \mathscr{Q}(\mathscr{T})$ can be defined (Greco 1981). Observe that by this, the assumption $f \in \mathscr{F}_0(\mathscr{S})$ can be dropped in Theorem 1 and Corollary 1.

$\mathscr{F} = \mathscr{I}(\mathscr{T})$ is not the smallest family of functions for which the equivalence statement of the theorem holds. However, when this statement holds with some $\mathscr{F}$ and $\mathscr{T}$, then necessarily $\succeq_{\mathscr{F}} = \succeq_{\mathscr{I}(\mathscr{T})}$. On the other hand, $\mathscr{F} = \mathscr{Q}(\mathscr{T})$ is the largest family of functions for which Theorem 1 holds when arbitrary $C_i \in \mathscr{C}$ are allowed. The reason is that for every $f \notin \mathscr{Q}(\mathscr{T})$ there exists $C_1, C_2 \in \mathscr{C}$ such that $C_1(T) \geq C_2(T)$ for all $T \in \mathscr{T}$, but $\int f\, dC_1 < \int f\, dC_2$ as Greco (1981) has shown. In other words:

Corollary 2: If $\mathscr{I}(\mathscr{T}) \subset \mathscr{F} \subset \mathscr{Q}(\mathscr{T})$, $C_1, C_2 \in \mathscr{C}$ *then* $\mathscr{Q}(\mathscr{T})$ *is the set of* $\preceq_{\mathscr{F}}$*-monotone functions, i.e., the set of functions f for which* $C_1 \preceq_{\mathscr{F}} C_2$ *implies*

$$\int f\, dC_1 \leq \int f\, dC_2 \ .$$

We continue with a series of examples. In each example, a family $\mathscr{T}$ of sets is considered and $\mathscr{Q}(\mathscr{T})$ or at least some $\mathscr{F}$, $\mathscr{I}(\mathscr{T}) \subset \mathscr{F} \subset \mathscr{Q}(\mathscr{T})$, is determined so that Theorem 1 applies.

Example 1: Let $(\Omega, \mathscr{S})$ be arbitrary and $h: \Omega \to \mathbb{R}$ be a given $\mathscr{S}$-measurable function. Consider the family $\mathscr{T}_h$ of upper level sets of h,

$$\mathscr{T}_h = \{\{x | h(x) \geq \alpha\} | \alpha \in \overline{\mathbb{R}}\} \ ,$$

where $\overline{\mathbb{R}}$ denotes the extended real line, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, and the family

$$\mathscr{F}_h = \{f: \Omega \to \mathbb{R} | f = \rho \circ h, \rho: h(\Omega) \to \mathbb{R} \text{ increasing and right continuous}\} \ .$$

Then

$$\mathscr{F}_h = \mathscr{Q}(\mathscr{T}_h)$$

and Theorem 1 holds with $\mathscr{T}_h$ and $\mathscr{F}_h$.

Proof: Let $\alpha \in \mathbb{R}$, $f \in \mathscr{F}_h$, $f = \rho \circ h$ with some $\rho: h(\Omega) \to \mathbb{R}$ increasing and right continuous. Then $\{t | \rho(t) \geq \alpha\} = [\beta, \infty[$ with some $\beta \in \mathbb{R}$, and

$$\{x | f(x) \geq \alpha\} = \{x | h(x) \geq \beta\} \in \mathscr{T}_h \ ,$$

hence $f \in \mathscr{Q}(\mathscr{T}_h)$. Conversely, let f be in $\mathscr{Q}(\mathscr{T}_h)$. First we show that $h(x) \leq h(y)$ implies $f(x) \leq f(y)$. Assume that $h(x) \leq h(y)$ holds at some x and y, but $f(x) > f(y)$. With $\alpha = f(x)$ and $\varepsilon = ((f(x) - f(y))/2$ there should be some $\beta \in \mathbb{R}$ such that

$$\{f \geq f(x)\} \subset \{h \geq \beta\} \subset \{f \geq (f(x) + f(y))/2\} \ .$$

Therefore x is in $\{h \geq \beta\}$. By $h(x) \leq h(y)$, y is in $\{h \geq \beta\}$ as well, but y is not in $\{f \geq (f(x) + f(y))/2\}$; contradiction. We conclude that $h(x) = h(y)$ implies $f(x) = f(y)$ hence there exists a real function $\rho: h(\Omega) \to \mathbb{R}$ with $f = \rho \circ h$. Moreover, this function is increasing. Second, let us assume that ρ is not right continuous, i.e., there exists a sequence $(\beta_n)_{n \in \mathbb{N}}$ in $h(\Omega)$ decreasing to $\beta \in h(\Omega)$ such that $\alpha = \lim_{n \to \infty} \rho(\beta_n) > \rho(\beta)$. Then, whenever $0 < \varepsilon < \alpha - \rho(\beta)$, for every $n \in \mathbb{N}$ we have

$$\{h \geq \beta_n\} \subsetneqq \{f \geq \alpha\} = \{h > \beta\} = \{f \geq \alpha - \varepsilon\} \subsetneqq \{h \geq \beta\} \ .$$

Thus, there is no γ such that

$$\{f \geq \alpha\} \subset \{h \geq \gamma\} \subset \{f \geq \alpha - \varepsilon\}$$

which contradicts $f \in \mathcal{Q}(\mathcal{T}_h)$. □

Observe that right continuity of ρ is necessary for $\mathcal{F}_h \subset \mathcal{Q}(\mathcal{T}_h)$. $\mathcal{F}_h$ contains "almost all" h-increasing functions: every h-increasing function f is equal to some $\tilde{f} \in \mathcal{F}_h$ except at a countable number of values.

In particular, Ω may be a normed space with norm h. Then $\mathcal{T}_h$ includes the balls around the origin, and $\mathcal{F}_h$ "almost all" norm-increasing functions. A similar situation arises when $\Omega = \Omega' \times \Omega'$ and h is a distance in Ω'.

In this example, h induces a weak order[3] on Ω. On the other hand, for every weakly ordered space $(\Omega, \preceq)$, provided $\Omega/\sim$ contains a countable subset which is order dense, there exists some $h\colon \Omega \to \mathbb{R}$ which induces the ordering.

Example 2: We consider a preordered space $(\Omega, \mathcal{S}, \leq)$ and the family $\mathcal{U} = \{S | S \text{ upper}\}$ of upper[4] sets. Let $\mathcal{F}_1 = \{f\colon \Omega \to \mathbb{R} | f \text{ increasing}\}$. Then it can be shown that

$$\mathcal{F}_1 = \mathcal{Q}(\mathcal{U}) \ . \tag{3.3}$$

Therefore, Theorem 1 is true with $\mathcal{F} = \mathcal{F}_1$ and $\mathcal{T} = \mathcal{U}$. This extends Proposition 1 of Section 2. For probability measures the general result is due to Lehmann (1955); see also Kamae, Krengel, and O'Brien (1977).

Example 3: Stronger results are obtained under topological assumptions. Let us add a topological structure to Example 2. Let $(\Omega, \mathcal{B}, \leq)$ be a topological space with Borel sets $\mathcal{B}$ and a closed preorder $\leq$. We denote

$$\mathcal{U}_{op} = \{S | S \text{ open and upper}\} \ ,$$

and

$$\mathcal{U}_{cl} = \{S | S \text{ closed and upper}\} \ .$$

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[3] A weak order is a reflexive, transitive, and complete relation.

[4] A set S is called *upper* if $x \in S$, $x \leq y$ implies $y \in S$.

Then

$$\mathcal{Q}(\mathcal{U}_{cl}) = \mathcal{Q}(\mathcal{U}_{op}) = \{f: \Omega \to \mathbb{R} | f \text{ increasing and continuous}\}$$

holds, and the theorem applies with either $\mathcal{T} = \mathcal{U}_{cl}$ or $\mathcal{T} = \mathcal{U}_{op}$.

Example 4: Let $(\Omega, \mathcal{S}, \leq)$ be a preordered linear space, $\mathcal{U}_{cv} = \{S | S \text{ upper and convex}\}$. Then

$$\mathcal{Q}(\mathcal{U}_{cv}) = \{f: \Omega \to \mathbb{R} | f \text{ increasing and quasiconcave}\} \ .$$

For σ-additive probability measures, see Levhari et. al. (1975).

Example 5: Endow a preordered linear space Ω with a locally convex topology and consider the closed halfspaces $H(l, \beta) = \{x \in \Omega | l(x) \geq \beta, l \in \Omega', \beta \in \mathbb{R}\}$. Let $\mathcal{U}_{hs}$ be the set of upper halfspaces. Then

$$\mathcal{Q}(\mathcal{U}_{hs}) = \{f: \Omega \to \mathbb{R} | f = \rho \circ l, \rho: \mathbb{R} \to \mathbb{R} \text{ increasing and}$$

$$\text{right continuous, } \{x | l(x) \geq 0\} \text{ upper}\}$$

In the case of σ-additive probabilities this ordering has been investigated on an inner product space Ω by Muliere and Scarsini (1989). Their paper includes economic applications.

The proofs of the preceeding examples are left to the reader.

Example 6: Let $\Omega = \mathbb{R}^d$, and $\mathcal{G}$ a subgroup of the orthogonal group $\mathcal{O}$. Let $\mathcal{S}$ be invariant under transformations $g \in \mathcal{G}$. For $x, y \in \mathbb{R}^d$, $x \leq_{\mathcal{G}} y$ if x is in the convex hull of the orbit of y, i.e., $x \in \text{conv}\{z | z = gy, g \in \mathcal{G}\}$. A function $f: \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{G}$-increasing if $x \leq_{\mathcal{G}} y$ implies $f(x) \leq f(y)$. With $\mathcal{T}_{\mathcal{G}} = \{S | 1_S \ \mathcal{G}\text{-increasing}\}$ we get

$$\mathcal{Q}(\mathcal{T}_{\mathcal{G}}) = \{f: \Omega \to \mathbb{R} | f \ \mathcal{G}\text{-increasing}\} \tag{3.4}$$

A special case arises when $\mathcal{G} = \mathcal{G}_{Sch} = \{\pi \in \mathcal{O} | \pi \text{ is a permutation matrix}\}$. In this case $\mathcal{Q}(\mathcal{T}_{\mathcal{G}})$ is the set of Schur-convex functions; see Marshall and Olkin (1979). For other $\mathcal{G}$ and applications with σ-additive probability measures we refer to the books by Eaton (1987) and Dharmadhikari and Joag-Dev (1988). Observe that $\mathcal{T}_{\mathcal{G}}$ is the family of upper sets with respect to $\leq_{\mathcal{G}}$; so (3.4) follows from (3.3).

Example 7: Consider $\Omega = \mathbb{R}^d$, $\mathcal{G}$ subgroup of $\mathcal{O}$, $\mathcal{S}$ $\mathcal{G}$-invariant, and $\mathcal{T}_{\mathcal{G}} = \{T | T$ $\mathcal{G}$-invariant and convex$\}$, e.g., $\mathcal{G} = \mathcal{G}_{Sym} = \{id, -id\}$ where id denotes the identity mapping. Then

$$\mathcal{Q}(\mathcal{T}_{\mathcal{G}}) = \{f: \Omega \to \mathbb{R} | f \text{ } \mathcal{G}\text{-invariant and unimodal}\}$$

Here, f is called unimodal if $\{f \geq \alpha\}$ is a convex set for every α. For applications in statistics (with σ-additive probabilities), see, e.g., Mosler (1987) and Dharmadhikari and Joag-Dev (1988).

Proof: If f is $\mathcal{G}$-invariant and unimodal, then $\{f \geq \alpha\}$ is $\mathcal{G}$-invariant and convex, hence $f \in \mathcal{Q}(\mathcal{T}_{\mathcal{G}})$. Conversely, let $f \in \mathcal{Q}(\mathcal{T}_{\mathcal{G}})$. For all $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ there is some $\mathcal{G}$-invariant and convex set $T_{\alpha,\varepsilon}$ with (3.2): $\{f \geq \alpha\} \subset T_{\alpha,\varepsilon} \subset \{f \geq \alpha - \varepsilon\}$. Then $\{f \geq \alpha\} = \bigcap_{\varepsilon > 0} T_{\alpha,\varepsilon}$ which is a convex set again; hence f is unimodal. Assume that f is not $\mathcal{G}$-invariant: $f(y) < f(gy)$ for some $y \in \Omega$, $g \in \mathcal{G}$. From (3.2) with $\alpha = f(gy)$, $\varepsilon = (f(gy) - f(y))/2$ we see that $gy \in T_{\alpha,\varepsilon}$, and, since $T_{\alpha,\varepsilon}$ is $\mathcal{G}$-invariant, $y \in T_{\alpha,\varepsilon}$, hence $f(y) > \alpha - 2\varepsilon = f(y)$; contradiction. □

Example 8: With $\Omega = \mathbb{R}^d$ and some $\mathcal{S}$ we consider the family of upper orthants $\mathcal{T}_{uo} = \{\{x | x \geq z\} | z \in \overline{\mathbb{R}}^d\}$ where $\geq$ is the usual partial order in $\mathbb{R}^d$. Then

$$\mathcal{Q}(\mathcal{T}_{uo}) \supset \{f: \Omega \to \mathbb{R} | f(x) = \rho(\min_i\{\lambda_i x_i\}), \lambda_i \geq 0,$$
$$\rho: \mathbb{R} \to \mathbb{R} \text{ increasing and right continuous}\}$$

The inclusion is strict: e.g., $f(x_1, x_2) = \min\{x_1, x_2^2\}$. Set dominance of σ-additive probabilities with respect to $\mathcal{T}_{uo}$ has been investigated in Mosler (1982, 1984) and Scarsini (1985, 1988). See also Rüschendorf (1981).

Proof: Assume $f(x) = \rho(\min_i\{\lambda_i x_i\})$ with some $\lambda_i \geq 0$ and an increasing right continuous ρ. Then

$$\{f \geq \alpha\} = \{x | \min_i \{\lambda_i x_i\} \geq \rho^{-1}(\alpha)\} = \{x | x \geq z\}$$

where $z = (z_1, \ldots, z_d)$, $z_i = \rho^{-1}(\alpha)/\lambda_i$. Therefore $f \in \mathcal{Q}(\mathcal{T}_{uo})$. □

Remark: Let $h \in \mathcal{F}_0(\mathcal{S})$ be given. If $C_1(h \geq \alpha) \geq C_2(h \geq \alpha)$ for all α, then $C_1(\rho \circ h \geq \alpha) \geq C_2(\rho \circ h \geq \alpha)$ holds as well for all α and for every increasing

function $\rho\colon \mathbb{R} \to \mathbb{R}$. Therefore,

$$\int (\rho \circ h)\, dC_1 \geq \int (\rho \circ h)\, dC_2 \qquad \text{for every increasing } \rho\colon \mathbb{R} \to \mathbb{R}\ .$$

I.e., set dominance with respect to the level sets of some given function h yields a large class of functions which are monotone in that dominance relation, viz. all increasing transforms of h, or, in other words, all functions which are increasing with respect to the weak order on Ω induced by h. Other cases of dominance with weakly ordered spaces Ω are treated in Section 4.

4 Dominance of Capacities on Weakly Ordered Spaces

Let Ω be a set and $\leq$ a weak order, i.e., a transitive and complete relation, on Ω. Further, let $\mathcal{S}$ be a set algebra on Ω which contains the set $\{\omega | \omega \geq \alpha\}$ for every $\alpha \in \Omega$. It has been shown (Scarsini 1992) that there exists a finitely additive probability measure m on $\mathcal{S}$ with

$$\int f\, dC = \int f\, dm \qquad \text{for all increasing functions } f$$

where both integrals are Choquet-integrals.

In this section we consider the question under what conditions this finitely additive probability measure can be extended to a σ-additive probability measure P on the σ-algebra $\mathcal{A}$ generated by all upper sets, such that

$$\int f\, dC = \int f\, dP \qquad \text{for all increasing functions } f\ . \tag{4.1}$$

The main result is Theorem 2 which presents a general sufficient condition for that. In case of (4.1) it is easy to translate stochastic dominance results from EU to nonadditive EU, for if $\mathcal{F} \subset \mathcal{F}_1 = \{f\colon \Omega \to \mathbb{R} | f \text{ increasing}\}$ we have

$$\int f\, dC_1 \geq \int f\, dC_2 \qquad \text{for all } f \in \mathcal{F}$$

if and only if

$$\int f\,dP_1 \geq \int f\,dP_2 \qquad \text{for all } f \in \mathcal{F}$$

where P_i denotes the probability measure associated with C_i, $i = 1, 2$. A necessary and sufficient condition for (4.1) to hold is that C and P coincide on all upper sets:

Lemma 1: The following conditions are equivalent

(i) $\int f\,dC = \int f\,dP$ *for all* $f \in \mathcal{F}_1$

(ii) $C(U) = P(U)$ *for all* $U \in \mathcal{U}$.

Proof (i) $\Rightarrow$ (ii) is obvious by inserting indicator functions of upper sets. (ii) $\Rightarrow$ (i): If f is increasing then the level sets $\{f \geq t\}$ are upper sets for all $t \in \mathbb{R}$. From

$$\begin{aligned}\int f\,dC &= \int_0^\infty C(f \geq t)\,dt + \int_{-\infty}^0 [C(f \geq t) - 1]\,dt \\ &= \int_0^\infty P(f \geq t)\,dt + \int_{-\infty}^0 [P(f \geq t) - 1]\,dt \\ &= \int f\,dP\end{aligned}$$

the assertion follows. □

With this lemma in mind we only have to look for a probability measure P which coincides with C on all upper sets. Obviously, a necessary condition for such a probability measure to exist is that C be continuous on upper sets, i.e., for every increasing or decreasing sequence $(U_n)_{n \in \mathbb{N}}$ in $\mathcal{U}$

$$\lim_{n \to \infty} C(U_n) = C\left(\lim_{n \to \infty} U_n\right)$$

has to hold.

Unfortunately this condition is not sufficient as is shown by the following example due to Halmos (1950, p. 40):

Example 9: Let $\Omega = \mathbb{Q} \cap [0, 1]$, $\mathcal{S} = \mathcal{P}(\Omega)$, $\leq$ the usual order on $\mathbb{Q}$, and C some capacity with $C(]a, 1]) = C([a, 1] = 1 - a$ for all $a \in \Omega$. Then of course C is continuous on upper sets. On the other hand, every probability measure which coincides with C on $\mathcal{U}$ must satisfy $P(\{a\}) = 0$ for every $a \in \Omega$. Since Ω is countable we have

$$1 = P(\Omega) = P\left(\bigcup_{a \in \Omega} \{a\} \right) = \sum_{a \in \Omega} P(\{a\}) = 0$$

Thus, there is no σ-additive probability measure which coincides with C on all upper sets.

In order to get sufficient conditions we have to investigate the order structure of the space $(\Omega, \leq)$. A weakly ordered space is called *order-complete* if every non-void subset of Ω has a supremum and an infimum. We say that a weakly ordered, order-complete space $(\Omega, \leq)$ possesses the *countable supremum property* (CSP) if for every set $A \subset \Omega$ the supremum and infimum of A can be represented as the supremum and the infimum of a countable set, i.e., if $a = \inf A$, $b = \sup A$, a, $b \notin A$, there exists some countable set $M \subset \Omega$ with $a, b \notin M$, and $a = \inf M$, $b = \sup M$. An example of a space which is order-complete but does not possess the CSP is the space $[0, \omega_1]$ of ordinal numbers where ω_1 is the first uncountable ordinal. A topology can be introduced on Ω by taking the sets $\{\omega | \omega < \alpha\}$ and $\{\omega | \omega > \alpha\}$ as a subbasis of the topology. This topology is called the *order topology* on Ω. It is well known (see, e.g., Willard 1970) that Ω is compact with respect to the order topology if $(\Omega, \leq)$ is order complete.

Theorem 2: Let $(\Omega, \leq)$ be a weakly ordered space which is order-complete and possesses the CSP. Let $\mathcal{S}$ be a set algebra, $\mathcal{U} \subset \mathcal{S}$ and $\mathcal{A}$ the σ-algebra generated by $\mathcal{U}$. Consider a capacity C on $(\Omega, \mathcal{S})$ which is continuous on upper sets. Then there exists a unique σ-additive probability measure P on $(\Omega, \mathcal{A})$ which coincides with C on upper sets.

Proof: Appendix

Remark: Note that $\mathcal{A}$ is also the Borel algebra for the order topology. Furthermore $\mathcal{A}$ is generated by the sets $\{\omega | \omega \geq \alpha\}$ and is the smallest σ-algebra for which every increasing function is measurable.

Corollary 3: For P in Theorem 2 the following equalities hold:

$$\int f\,dC = \int f\,dP \qquad \textit{for all increasing } f$$

$$\int^{D} f\,dC = \int f\,dP \qquad \textit{for all decreasing } f$$

Proof: The first equality is an immediate consequence of Lemma 1. The second statement follows from the fact that for a decreasing function f the level sets $\{f \geq \alpha\}$, $\alpha \in \Omega$ are upper sets. □

Corollary 4: Let $(\Omega, \leq)$ be as in Theorem 2, $\mathscr{S}$ a set algebra which contains all lower sets, $\mathscr{A}$ the σ-algebra generated by the lower sets, and C a capacity which is continuous on lower sets. Then there exists a unique σ-additive probability measure P on $(\Omega, \mathscr{A})$ which coincides with C on lower sets. For this probability measure the following equalities hold:

$$\int f\,dC = \int f\,dP \qquad \textit{for all decreasing } f$$

$$\int^{D} f\,dC = \int f\,dP \qquad \textit{for all increasing } f$$

Proof: By applying Theorem 2 to the space $(\Omega, \leq^{D})$ where $\leq^{D}$ is defined by $x \leq^{D} y \Leftrightarrow y \leq x$ the corollary follows immediately. □

The reason why all this works is that the upper (lower) level sets of the functions of interest are linearly ordered by inclusion. In fact we can go the opposite way and start with a class of functions where the upper level sets are ordered by inclusion. This is equivalent to the condition that the functions are pairwise comonotonic functions. Thus, let us consider a capacity space $(\Omega, \mathscr{S}, C)$ and a set $\mathscr{F}$ of pairwise comonotonic functions $\mathscr{F} \subset \mathscr{F}_0(\mathscr{S})$. We will call such a set a *comonotonic set of functions.*

Lemma 2: If $\mathscr{F}$ is a comonotonic set of functions then the relation $\leq_{\mathscr{F}}$ defined by

$$x \leq_{\mathscr{F}} y \quad \Leftrightarrow \quad f(x) \leq f(y) \textit{ for all } f \in \mathscr{F}$$

is a weak order on Ω and $\mathscr{F}$ is a subset of the increasing functions with respect to $\leq_{\mathscr{F}}$.

Proof: Obvious

If we have some set of comonotonic functions we only have to check if the weak order induced by these functions satisfies the assumptions of Theorem 2. Next, we describe the technique of transferring stochastic dominance results from EU to models with nonadditive probabilities by means of Theorem 2.

Proposition 2: Let Ω be a compact subset of the extended real line $\overline{\mathbb{R}}$, $\mathscr{B} = \mathscr{B}(\Omega)$ be the Borel sets on Ω. Let further $\mathscr{F} \subset \mathscr{F}_1(\mathscr{B})$ and $C_1, C_2 \in \mathscr{C}(\mathscr{B})$ be two capacities which are continuous on upper sets. Suppose that we have the following stochastic dominance result[5] *for probability measures P_1, P_2:*

$$P_1 \preceq_{\mathscr{F}} P_2 \quad \textit{if and only if} \quad U(P_1, P_2)$$

where $U(P_1, P_2)$ is a statement whose validity depends on the values of the probability measure on upper sets only. Then the corresponding result for capacities is as follows:

$$C_1 \preceq_{\mathscr{F}} C_2 \quad \textit{if and only if} \quad U(C_1, C_2) \ .$$

Proof: Every compact subset of $\overline{\mathbb{R}}$ is order-complete and possesses the CSP. Thus, the assumptions of Theorem 2 are fulfilled. There exist probability measures P_1, P_2 which coincide on upper sets with C_1 and C_2, respectively. Furthermore, by Lemma 1 we have

$$\int f \, dP_i = \int f \, dC_i \ , \qquad i = 1, 2 \text{ for all } f \in \mathscr{F}_1(\mathscr{B}) \ .$$

Then the following equivalences hold:

$$C_1 \preceq_{\mathscr{F}} C_2 \ \Leftrightarrow \ P_1 \preceq_{\mathscr{F}} P_2 \ \Leftrightarrow \ U(P_1, P_2) \ \Leftrightarrow \ U(C_1, C_2) \ . \qquad \square$$

Stochastic dominance results with respect to the lower Choquet integral may be derived in the same way. The previous result holds true if we replace "upper" by "lower". We now give some examples to illustrate the results of this section.

[5] We apologize for the lack of formalism.

Example 10: Let $\mathscr{F}_2 = \{u: \overline{\mathbb{R}} \to \overline{\mathbb{R}} | u \text{ increasing and concave}\}$ and $C_1, C_2 \in \mathscr{C}(\overline{\mathscr{B}})$. If C_1, C_2 are continuous on upper sets then

$$C_1 \preceq_{\mathscr{F}_2} C_2 \quad \Leftrightarrow \quad \int_{-\infty}^{x} [\overline{F}_1(t) - \overline{F}_2(t)]\, dt \leq 0 \qquad \forall x \in \mathbb{R} \ ,$$

where $\overline{F}_i(t) = C_i([t, \infty])$, $i = 1, 2$, denotes the decumulative distribution function. An analogous result holds for capacities which are continuous on lower sets if the lower Choquet integral is used and if F_i and $\geq$ are put in place of $\overline{F}_i$ and $\leq$.

Proof: It is well known that for probability measures P_1, P_2 we have

$$P_1 \preceq_{\mathscr{F}_2} P_2 \quad \Leftrightarrow \quad \int_{-\infty}^{x} P_1([t, \infty]) - P_2([t, \infty])\, dt \leq 0 \qquad \forall x \in \mathbb{R} \ .$$

The statement on the right hand side depends on probabilities of upper sets only. The assertion follows immediately from Proposition 2. □

Example 11: For $k \geq 2$ let

$$\mathscr{F}_k' = \{u: \overline{\mathbb{R}} \to \overline{\mathbb{R}} | u(x) = \int_x^{\infty} \int_{t_{k-2}}^{\infty} \cdots \int_{t_1}^{\infty} v(t_0)\, dt_0 \ldots dt_{k-2} \ ,$$

$$v \text{ increasing, right continuous, } v \leq 0, v(\infty) = 0\} \ .$$

Let $C_1, C_2 \in \mathscr{C}(\overline{\mathscr{B}})$ be two capacities which are continuous on upper sets and let $\overline{F}_i$ denote the decumulative distribution function of C_i, $i = 1, 2$. Then

$$C_1 \succeq_{\mathscr{F}_k'} C_2 \quad \Leftrightarrow \quad \int_{-\infty}^{x} \int_{-\infty}^{t_{k-2}} \cdots \int_{-\infty}^{t_1} [\overline{F}_1(t_0) - \overline{F}_2(t_0)]\, dt_0 \ldots dt_{k-2} \geq 0$$

$$\forall x \in \mathbb{R} \ .$$

Again, this follows immediately from a well known result (Rolski 1976) and Proposition 2.

Example 12: For $k \geq 2$ and $b \geq 0$ let

$$\mathscr{F}_k^c = \{u\colon [0, b] \to \overline{\mathbb{R}} | u \text{ has derivatives of order } k \text{ and } (-1)^i u^{(i)} \leq 0, i = 1, \ldots, k\} .$$

Let C_1, C_2 be capacities which are continuous on upper sets and let $\mu_j(C) = \int x^j C(dx)$ denote the j-th upper moment of C. Then the following condition is sufficient for $C_1 \preceq_{\mathscr{F}_k^c} C_2$ to hold:

$$\int_0^x \int_0^{t_{k-2}} \cdots \int_0^{t_1} [\bar{F}_1(t_0) - \bar{F}_2(t_0)]\, dt_0 \ldots dt_{k-2} \geq 0 \qquad \forall x \in [0, b] ,$$

and

$$(-1)^j \mu_j(C_1) \leq (-1)^j \mu_j(C_2) , \qquad 1 \leq j \leq k - 2 .$$

This follows from Proposition 2 and a result in Mosler (1982, p. 70).

Next we will investigate utility functions which are defined on a product space. For example consider $\mathbb{R}^n$ with the usual partial order. Whereas all increasing functions on $\mathbb{R}$ are comonotonic this is not the case on $\mathbb{R}^n$. Thus, to apply our results we will restrict the analysis to subsets of increasing functions which are comonotonic. In other words, we consider classes of utility functions which represent the same preference-ordering over consequences.

Example 13: Let $\Omega = \mathbb{R}_+^n$, $\mathscr{S}$ be the Borel sets and $u\colon \mathbb{R}_+^n \to \mathbb{R}$ be a continuous increasing quasi-concave utility function. Define

$$\mathscr{U}_1 = \{w\colon \mathbb{R}_+^n \to \mathbb{R}_+ | w = \rho \circ u, \rho\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ right continuous and increasing}\}$$

and

$$\mathscr{U}_2 = \{w \in \mathscr{U}_1 | w \text{ concave}\} .$$

These classes were considered by Levy/Levy (1984) and, in the context of non-additive EU, by Scarsini (1992). Debreu (1976) has shown that there exists a function $u^* \in \mathscr{U}_1$ such that

$$\mathcal{U}_2 = \{w: \mathbb{R}_+^n \to \mathbb{R}_+ | w = \rho \circ u^*, \rho: \mathbb{R}_+ \to \mathbb{R}_+ \text{ right continuous and increasing}\} .$$

The function u^* is called the *least concave utility function.* Note that $\mathcal{U}_1$ is a special case of the class $\mathcal{F}_h$ considered in Example 1. A weak order $\leq_{u^*}$ on $\mathbb{R}_+^n$ is introduced by $x \leq y \Leftrightarrow u^*(x) \leq u^*(y)$. Let $C_1, C_2 \in \mathcal{C}(\mathcal{S})$ be two capacities which are continuous on upper sets (upper with respect to $\leq_{u^*}$). Then

$$C_1 \leq_{\mathcal{U}_2} C_2 \quad \Leftrightarrow \quad \int_0^x C_1(u^* \geq t)\, dt \leq \int_0^x C_2(u^* \geq t)\, dt \qquad \forall x \in \mathbb{R}_+ .$$

Proof: We get this result from Theorem 2 and a result in Levy/Levy (1984). □

5 Applications to Decision Making

In decision making under risk a decision maker is called an *anticipated utility maximizer* (or AU maximizer) if there exists a utility function $u: \mathcal{C} \to \mathbb{R}$ and an increasing function $q: [0, 1] \to [0, 1]$ with $q(0) = 0$ and $q(1) = 1$ (the so-called *distortion function*) such that she prefers an act f over another act g if and only if

$$\int_S (u \circ f)\, d(q \circ P) \geq \int_S (u \circ g)\, d(q \circ P) .$$

See Quiggin (1982), Segal (1984). If we know the exact form of the distortion function q we can apply the stochastic dominance results to the capacities $q \circ P_f$ and $q \circ P_g$. However, in practice, we may only know that q belongs to some class of functions. Interesting classes of distortion functions are the following ones (see Quiggin (1982), Segal (1985), Chew, Karni, and Safra (1987)):

$$Q_1 = \{q: [0, 1] \to [0, 1] | q(0) = 0, q(1) = 1, \\ q \text{ continuous and strictly increasing}\}$$

$$Q_S = \{q \in Q_1 | q(0.5) = 0.5\}$$

$$Q_{SX} = \{q \in Q_S | q \text{ convex in } [0, 0.5], q \text{ concave in } [0.5, 1]\}$$

$$Q_{SV} = \{q \in Q_S | q \text{ concave in } [0, 0.5], q \text{ convex in } [0.5, 1]\}$$

$$Q_X = \{q \in Q_1 | q \text{ convex}\}$$

$$Q_V = \{q \in Q_1 | q \text{ concave}\}$$

First, let us consider set dominance relations between capacities. The following proposition is obvious.

Proposition 3: Let $\mathcal{U}$ be a class of utility functions and let $\succeq_{\mathcal{U}}$ be a set dominance relation. Let further f and g be acts and P a probability measure on the states of nature. Then

$$P_f \succeq_{\mathcal{U}} P_g \quad \Leftrightarrow \quad q \circ P_f \succeq_{\mathcal{U}} q \circ P_g \qquad \forall q \in Q_1$$

Proposition 3 tells that if every EU maximizer with utility function u in $\mathcal{U}$ prefers f to g then every AU maximizer with distortion function in Q_1 and utility function u in $\mathcal{U}$ prefers f to g, too. (The reverse holds anyway.) Note also that the class Q_1 is a very general class of distortion functions and that it is very reasonable to assume q to be strictly increasing.

The situation is more complicated if the dominance relation is not a set dominance relation but a stochastic dominance relation like the ones we considered in Section 4. A result which parallels Proposition 3 holds if the dominance relation is second degree stochastic dominance and the class of distortion functions includes the convex and increasing functions. This assumption is closely related to risk aversion, see Chew, Karni, and Safra (1987).

Proposition 4: Let $\mathcal{U}_2 = \{u: [0, b] \to \mathbb{R}_+ | u$ twice differentiable, $u' \geq 0$, $u'' \leq 0\}$, and let $Q_2 = \{q \in Q_1 | q$ twice differentiable, $q' \geq 0, q'' \geq 0\}$. Let further f and g be acts and P be a probability measure. Then

$$q \circ P_f \succeq_{\mathcal{U}_2} q \circ P_g \qquad \forall q \in Q_2 \quad \Leftrightarrow \quad P_f \succeq_{\mathcal{U}_2} P_g$$

Proof: Let $\bar{F}(t) = P_f([t, b])$, $t \in [0, b]$, and $\bar{G}(t) = P_g([t, b])$, $t \in [0, b]$. We define the quasi-inverse h^{-1} of a decreasing function h as usual by $h^{-1}(t) = \inf\{x | h(x) \leq t\}$. It can be shown (e.g., Röell 1987) that

$$\int_0^x [\bar{F}(t) - \bar{G}(t)]\, dt \geq 0 \qquad \forall x \in [0, b]$$

$$\Leftrightarrow \quad \int_z^1 [\bar{F}^{-1}(s) - \bar{G}^{-1}(s)]\, ds \geq 0 \quad \forall z \in [0, 1] \ .$$

From Example 10 we know that

$$q \circ P_f \succeq_{\mathcal{U}_2} q \circ P_g \quad \Leftrightarrow \quad \int_0^x [q \circ \bar{F}(t) - q \circ \bar{G}(t)]\, dt \geq 0 \qquad \forall x \in [0, b]$$

Since $(q \circ \bar{F})^{-1} = \bar{F}^{-1} \circ q^{-1}$ we get

$$\int_0^x [q \circ \bar{F}(t) - q \circ \bar{G}(t)]\, dt \geq 0 \qquad \forall x \in [0, b]$$

$$\Leftrightarrow \quad \int_z^1 [\bar{F}^{-1}(q^{-1}(s)) - \bar{G}^{-1}(q^{-1}(s))]\, ds \geq 0 \quad \forall z \in [0, 1]$$

$$\Leftrightarrow \quad \int_x^1 [\bar{F}^{-1}(u) - \bar{G}^{-1}(u)] q'(u)\, du \geq 0 \quad \forall x \in [0, 1]$$

Integration by parts gives

$$\int_x^1 [\bar{F}^{-1}(u) - \bar{G}^{-1}(u)] q'(u)\, du$$

$$= \int_x^1 [\bar{F}^{-1}(s) - \bar{G}^{-1}(s)]\, ds\, q'(x) + \int_x^1 \int_t^1 [\bar{F}^{-1}(s) - \bar{G}^{-1}(s)]\, ds\, q''(t)\, dt$$

If $P_f \succeq_{\mathcal{U}_2} P_g$ we have

$$\int_x^1 [\bar{F}^{-1}(s) - \bar{G}^{-1}(s)]\, ds \geq 0 \quad \forall x \in [0, 1]$$

and thus, since $q'(x) \geq 0$ and $q''(x) \geq 0$ we conclude $q \circ P_f \succeq_{\mathcal{U}_2} q \circ P_g$. The reverse implication is obvious. □

Appendix

Proof of Theorem 2: Let $(\Omega, \leq)$ be a weakly ordered space which is order complete and possesses the CSP. First assume that $(\Omega, \leq)$ is linearly ordered. Then there exist elements $\sup \Omega$ and $\inf \Omega$ which we denote by ∞ and $-\infty$. We

use the following notation: $]a, b] = \{x \in \Omega | a < x \leq b\}$. The notations $[a, b]$, $[a, b[$, and $]a, b[$ are to be understood analogously. Since Ω is order-complete it is compact in the order topology. Thus, every closed interval $[a, b]$ is compact. Let $\mathscr{K} = \{]a, \infty] | a \in \Omega\}$ and

$$\mathscr{T}^+ = \left\{ \bigcup_{i=1}^{n}]a_i, b_i] | n \in \mathbb{N}, a_i, b_i \in \Omega,]a_i, b_i], 1 \leq i \leq n, \text{disjoint} \right\} .$$

Then $\mathscr{T}^+$ is the set ring generated by $\mathscr{K}$. We define P on $\mathscr{K}$ by $P(]a, \infty]) = C(]a, \infty])$, $a \in \Omega$. Then P can be uniquely extended to a finite additive measure on $\mathscr{T}^+$ in the standard way. Since $\mathscr{T}^+$ is a set algebra in $\Omega' = \Omega \setminus \{-\infty\}$ we have to show only that P is a premeasure on Ω'; then P can be extended uniquely to a σ-additive measure on the σ-algebra $\mathscr{A}'$ which is generated by $\mathscr{T}^+$ in Ω' and thus can be extended uniquely to the σ-algebra $\mathscr{A}$ which is generated by $\mathscr{T}^+$ in Ω.

For P to be a premeasure on $\mathscr{T}^+$ it is sufficient to show that for every $A \in \mathscr{T}^+$ and $\varepsilon > 0$ there exists a set $B \in \mathscr{T}^+$ with $P(A \setminus B) < \varepsilon$ and a set K, $A \supset K \supset B$, which is compact in the order topology. W.l.o.g.: Let $A =]a, b]$. We consider two cases:

Case 1: If $\inf]a, \infty] = a$, then it follows from the CSP that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n > a$ for all $n \in \mathbb{N}$ and $\inf\{x_n | n \in \mathbb{N}\} = a$. Thus

$$\lim_{n \to \infty} C(]x_n, \infty]) = C(]a, \infty]) ,$$

since C is continuous on upper sets. Then for every $\varepsilon > 0$ there is an x_n such that $C(]x_n, \infty]) > C(]a, \infty]) - \varepsilon$ and it follows that $]x_n, b] \subset [x_n, b] \subset]a, b]$ and $P(]a, b] \setminus]x_n, b]) < \varepsilon$. Furthermore $[x_n, b]$ is compact in the order topology since $(\Omega, \leq)$ is order complete.

Case 2: If $\inf]a, \infty] = \alpha > a$, then $]a, \infty] = [\alpha, \infty]$ and with $A = B = K =]a, b]$ we get the desired results.

We still have to show that P coincides with C on all upper sets. From the construction of P it is obvious that C and P coincide on $\mathscr{K}$. Let A be an upper set that is not in $\mathscr{K}$. Then $A = [a, \infty]$ for some $a \in \Omega$. Again we consider two cases:

Case 1: If $\sup[-\infty, a[= a$ then it follows from the CSP that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $\sup\{x_n | n \in \mathbb{N}\} = a$ and $x_n < a$ for all $n \in \mathbb{N}$. Thus

$$\lim_{n\to\infty} C(]x_n, \infty]) = C([a, \infty]) ,$$

and since P and C are continuous on upper sets we get $P([a, \infty]) = C([a, \infty])$.

Case 2: If $\sup[-\infty, a[= \alpha < a$, then $[-\infty, \alpha] = [-\infty, a[$ and $]\alpha, \infty] = [a, \infty]$. Thus, $[a, \infty]$ is in $\mathscr{K}$ and nothing has to be shown.

Finally, if $(\Omega, \leq)$ is not linearly ordered we can pass to the factor space $(\Omega/\sim, \leq/\sim)$ which is linearly ordered. By applying the above we construct a probability measure P' on $\Omega/\sim$ with the desired property. This probability measure is extended in a standard way to a probability measure P on Ω which possesses the desired property, too. □

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References

Bergmann R (1991) Stochastic orders and their applications to a unified approach to various concepts of dependence and association. In Mosler, K. and Scarsini, M. (eds.), *Stochastic Orders and Decision under Risk.* Institute of Mathematical Statistics, Hayward, CA, 48–73

Chew SH, Karni E, Safra Z (1987) Risk aversion in the theory of expected utility with rank dependent probabilities. *Journal of Economic Theory* 42:370–381

Choquet G (1953/54) Theory of capacities. *Annales de l'Institut Fourier* 5:131–295

Dharmadhikari S, Joag-Dev K (1988) *Unimodality, Convexity and Applications.* Academic Press, Boston, MA

Eaton ML (1987) *Lectures on Topics in Probability Inequalities.* Centrum voor Wiskunde en Informatica, Amsterdam

Gilboa I (1987) Expected utility with purely subjective non-additive probabilities. *Journal of Mathematical Economics* 16:65–88

Gilboa I (1989) Duality in non-additive expected utility. *Annals of Operations Research* 19:405–414

Greco GH (1981) Sur la mesurabilité d'une fonction numérique par rapport a une famille d'ensembles. *Rendiconti del Seminario Matematico dell'Università di Padova* 65:163–176

Halmos P (1974) *Measure Theory.* 2nd printing. Springer, New York

Huber PJ, Strassen V (1973) Minimax tests and the Neyman-Pearson lemma for capacities. *The Annals of Statistics* 1:251–263

Kamae T, Krengel U, O'Brien GL (1977) Stochastic inequalities on partially ordered spaces. *Annals of Probability* 5:899–912

Lehmann EL (1955) Ordered families of distributions. *Annals of Mathematical Statistics* 26:399–419

Levhari D, Paroush J, Peleg B (1975) Efficiency analysis for multivariate distributions. *Review of Economic Studies* 42:87–91

Levy H, Levy A (1984) Multivariate decision-making. *Journal of Economic Theory* 32:36–51

Marshall AW, Olkin I (1979) *Inequalities: Theory of Majorization and Its Applications.* Academic Press, New York

Mosler KC (1982) *Entscheidungsregeln bei Risiko: Multivariate stochastische Dominanz.* Springer, Berlin

Mosler KC (1984) Stochastic dominance decision rules when the attributes are utility independent. *Management Science* 30:1311–1322

Mosler KC (1987) Estimation under G-invariant quasi-convex loss. *Journal of Multivariate Analysis* 22:137–143

Mosler K, Scarsini M (Eds) (1991a) *Stochastic Orders and Decision under Risk.* Institute of Mathematical Statistics, Hayward, CA

Mosler K, Scarsini M (1991b) Some theory of stochastic dominance. In Mosler K and Scarsini M (eds.), *Stochastic Orders and Decision under Risk.* Institute of Mathematical Statistics, Hayward, CA, 261–284

Muliere P, Scarsini M (1989). Multivariate decisions with unknown price vector. *Economics Letters* 29:13–19

Nakamura Y (1990) Subjective expected utility with nonadditive probabilities on finite state spaces. *Journal of Economic Theory* 51:346–366

Quiggin J (1982) A theory of anticipated utility. *Journal of Economic Behavior and Organization* 3:323–343

Röell A (1987) Risk aversion in Quiggin and Yaari's rank-order model of choice under uncertainty. *The Economic Journal* 97:143–159

Rolski T (1976) Order Relations in the Set of Probability Distribution Functions and their Applications in Queueing Theory. *Dissertationes Mathematicae* 132, Warszawa

Rüschendorf L (1981) Stochastically ordered distributions and monotonicity of the OC-function of sequential probability ratio tests. *Mathematische Operationsforschung und Statistik. Series Statistics* 12:327–338

Scarsini M (1985) Stochastic dominance with pair-wise risk aversion. *Journal of Mathematical Economics* 14:187–201

Scarsini M (1988) Dominance conditions for multivariate utility functions. *Management Science* 34:454–460

Scarsini M (1992) Dominance conditions in non-additive expected utility theory. *Journal of Mathematical Economics* 21:173–184

Schmeidler D (1989) Subjective probability and expected utility without additivity. *Econometrica* 57:571–587

Segal U (1984) Nonlinear decision weights with the independence axiom. Working Paper 353, Department of Economics, University of California, Los Angeles

Wakker P (1989) Continuous subjective expected utility with non-additive probabilities. *Journal of Mathematical Economics* 18:1–27

Willard S (1970) *General Topology.* Addison-Wesley, Reading, MA

Yaari ME (1987) The dual theory of choice under risk. *Econometrica* 55:95–115

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