Nonparametric tests for constant tail dependence with an application to energy and finance

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September 22, 2014

Abstract

New tests for detecting structural breaks in the tail dependence of multivariate time series using the concept of tail copulas are presented. To obtain asymptotic properties, we derive a new limit result for the sequential empirical tail copula process. Moreover, consistency of both the tests and a break-point estimator are proven. We analyze the finite sample behavior of the tests by Monte Carlo simulations. Finally, and crucial from a risk management perspective, we apply the new findings to datasets from energy and financial markets.

Keywords: Break-point detection, Multiplier bootstrap, Tail dependence, Weak convergence

JEL classification: C12, C14, C32, C58, G32

We are grateful to comments from the co-editor Yacine Ait-Sahalia, an associate editor, four referees, Holger Dette and Walter Krämer. This work has been supported in part by the Collaborative Research Center *Statistical Modelling of Nonlinear Dynamic Processes* (SFB 823, projects A1 and A7) of the German Research Foundation (DFG) and by the IAP research network Grant P7/06 of the Belgian government (Belgian Science Policy). Parts of this paper were written when the first author was a post-doctoral researcher at Université catholique de Louvain, Belgium.

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1. Introduction

Modeling and estimating stochastic dependencies has attracted increasing attention over the last decades in various fields of applications, including mathematical finance, actuarial science or hydrology, among others. Of particular interest, especially in risk management, is a sensible quantitative description of the dependence between extreme events, commonly referred to as tail dependence; see for example Embrechts et al. (2003). A formal definition of this concept is given in Section 2 below.

In applications, tail dependence is often assessed by fitting a parametric copula family to the data and by subsequently extracting the tail behavior of that particular copula. Examples can be found in Breymann et al. (2003) and Malevergne and Sornette (2003), among others. Fitting the copula typically requires some sort of goodness-of-fit testing. Recent reviews on these methods are given by Genest et al. (2009) and Fermanian (2013). More robust methods to assess tail dependence are based on the assumption that the underlying copula is an extreme-value copula. The class of these copulas can be regarded as a nonparametric copula family indexed by a function on the unit simplex (Gudendorf and Segers, 2010). Since the copula is a rather general measure for stochastic dependence, the estimation techniques for both of the latter approaches are usually based on the entire available dataset (see, for instance, Genest et al. (1995); Chen and Fan (2006) for parametric families or Genest and Segers (2009) for extreme-value copulas). However, due to the fact that the center of a distribution does not contain any information about the tail behavior, these techniques might in general yield biased estimates for the tail dependence. We refer to Frahm et al. (2005) for a more elaborated discussion of this issue. In order to circumvent the problem and to obtain estimators that are robust with respect to deviations in the center of the distribution, there are basically two important approaches: either one could extract the tail dependence from subsamples of block maximal data, for which extreme-value copulas provide a natural model (McNeil et al., 2005, Section 7.5.4), or one could rely on extreme-value techniques some of which are presented in Section 2 below. Applications of these procedures can be found in Breymann et al. (2003); Caillault and Guégan (2005); Jäschke et al. (2012); Jäschke (2014), among others.

Most of the aforementioned applications to time series data are based on the implicit assumption that the tail dependence remains constant over time. Whereas nonparametric testing for constancy of the whole dependence structure, as for instance measured by the copula, has recently drawn some attention in the literature (Remillard, 2010; Busetti and Harvey, 2011; Krämer and van Kampen, 2011; Bücher and Ruppert, 2013; Bücher et al., 2014; Wied et al., 2014), there does not seem to exist a unified approach to testing for constancy of the tail dependence. It is the main purpose of the present paper to fill this gap. Our proposed testing procedures are genuine extreme-value methods depending only on the dependence between the tails of the data and are hence robust with respect to potential (non-)constancy of the dependence between the centers of the distributions. In particular, the presented tests do not rely on the assumption of a constant copula throughout the sample period.

Our procedures are based on new limit results for the sequential empirical tail copula process, formally defined in Section 3.1. We derive its asymptotic distribution under the null hypothesis and propose several variants to approximate the required critical values. When restricting to the case of testing for constancy of the simple tail dependence coefficient, the limiting process can be easily transformed into a Brownian bridge. In this case, the asymptotic critical values of the tests can be obtained by direct calculations or simulations. In the more complicated case of testing for constancy of the whole extremal dependence structure as measured by the tail copula, we propose a multiplier bootstrap procedure to obtain approximate asymptotic quantiles. The finite-sample performance of all proposals is assessed in a simulation study, which reveals accurate approximations of the nominal level and reasonable power properties.

We apply our methods to two real datasets. The first application revisits a recent investigation in Jäschke (2014) on the tail dependence between WTI and Brent crude oil spot log-returns, which is based on the implicit assumption that the tail dependence remains constant over time. Our testing procedures show that this assumption cannot be rejected. The second application concerns the tail dependence between Dow Jones Industrial Average and the Nasdaq Composite time series around Black Monday on 19th of October 1987, it reveals a significant break in the tail dependence. However, our results do not show clear evidence for the hypothesis that this break takes place at the particular date of Black Monday.

The structure of the paper is as follows: in Section 2, we briefly summarize the concept of tail dependence and corresponding nonparametric estimation techniques. The new testing procedures for constancy of the tail dependence are introduced in Section 3. In particular, we derive the asymptotic distribution of the sequential empirical tail copula process, propose a multiplier bootstrap approximation of the latter and show consistency of various asymptotic tests. Additionally, we deal with the estimation of break-points in case the null hypothesis is rejected and make use of a data-adaptive process for the necessary parameter choice, common to inference methods in extreme-value theory. A comprehensive simulation study is presented in Section 4, followed by the two elaborate empirical applications in Section 5. All proofs are deferred to an Appendix.

2. The concept of tail dependence and its nonparametric estimation

Let (X, Y) be a bivariate random vector with continuous marginal cumulative distribution functions (c.d.f.s) F and G. *Lower or upper tail dependence* concerns the tendency that extremely small or extremely large outcomes of X and Y occur simultaneously. Simple, widely used and intuitive scalar measures for these tendencies are provided by the well-established coefficients of tail dependence (TDC), defined as

$$\lambda_L = \lim_{t \searrow 0} \mathbb{P}\{F(X) \le t \mid G(Y) \le t\}, \qquad \lambda_U = \lim_{t \nearrow 1} \mathbb{P}\{F(X) \ge t \mid G(Y) \ge t\} \quad (1)$$

see for instance Joe (1997); Frahm et al. (2005), among others.

It is well-known that the joint c.d.f. H of (X, Y) can be written in a unique way as

$$H(x, y) = C\{F(x), G(y)\}, \qquad x, y \in \mathbb{R}, \quad (2)$$

where the copula C is a c.d.f. on $[0, 1]^2$ with uniform marginals. Elementary calculations show that the conditional probabilities in (1) can be written as

$$\lambda_L = \lim_{t \searrow 0} \frac{C(t,t)}{t}, \qquad \lambda_U = \lim_{t \searrow 0} \frac{\overline{C}(t,t)}{t},$$

where $\overline{C}$ denotes the survival copula of (X, Y). Therefore, the coefficients of tail dependence can be regarded as directional derivatives of C or $\overline{C}$ at the origin with direction $(1, 1)$. Considering different directions, we arrive at the so-called tail copulas, defined for any $(x, y) \in \mathbb{E} = [0, \infty]^2 \setminus \{(\infty, \infty)\}$ by

$$\Lambda_L(x,y) = \lim_{t \searrow 0} \frac{C(xt,yt)}{t}, \qquad \Lambda_U(x,y) = \lim_{t \searrow 0} \frac{\overline{C}(xt,yt)}{t}, \tag{3}$$

see Schmidt and Stadtmüller (2006). Note that the upper tail copula of (X, Y) is the lower tail copula of $(-X, -Y)$, whence there is no conceptual difference between upper and lower tail dependence.

Several variants of tail copulas have been proposed in the literature on multivariate extreme-value theory. For instance, $L(x, y) = x + y - \Lambda_U(x, y)$ denotes the stable tail dependence function, see, e.g., de Haan and Ferreira (2006). The function $A(t) = 1 - \Lambda_U(1 - t, t)$, which is simply the restriction of L to the unit sphere with respect to the $\|\cdot\|_1$-norm, is called Pickands dependence function, see Pickands (1981). All these variants are one-to-one and are known to characterize the extremal dependence of X and Y, see de Haan and Ferreira (2006). In the present paper we restrict ourselves to the case of tail copulas.

Nonparametric estimation of L and Λ has been addressed in Huang (1992); Drees and Huang (1998); Einmahl et al. (2006); de Haan and Ferreira (2006); Bücher and Dette (2013); Einmahl et al. (2012) for i.i.d. samples $(X_i, Y_i)_{i\in\{1,\dots,n\}}$. For instance, in the case of lower tail copulas, the considered estimators are slight variants, differing only up to a term of uniform order $O(1/k)$, of the function

$$(x,y) \mapsto \frac{1}{k} \sum_{i=1}^{n} \mathbb{1}\Big(R_i \le kx, S_i \le ky\Big) \tag{4}$$

where R_i (resp. S_i) denotes the rank of X_i (resp. Y_i) among $X_1, \dots, X_n$ (resp. $Y_1, \dots, Y_n$), and where $k = k_n \to \infty$ denotes an intermediate sequence to be chosen by the statistician. Under suitable assumptions on k_n and on the speed of convergence in (3) the estimators are known to be $\sqrt{k_n}$-consistent. Additionally, under certain smoothness conditions on Λ, the corresponding process $\sqrt{k_n}(\hat{\Lambda} - \Lambda)$ converges to a Gaussian limit process.

3. Testing for constant tail dependence

3.1. Setting and test statistics Let $(X_i, Y_i)_{i\in\{1,\dots,n\}}$ be an independent sequence of bivariate random vectors with joint c.d.f. H_i and identical continuous marginal c.d.f.s F and G, respectively. According to Sklar's Theorem, see (2), we can decompose

$$H_i(x, y) = C_i\{F(x), G(y)\}, \qquad x, y \in \mathbb{R},$$

where $C_i(u,v) = \mathbb{P}(U_i \le u, V_i \le v)$ with $U_i = F(X_i)$ and $V_i = G(Y_i)$. We assume that the corresponding lower tail copulas

$$\Lambda_i(x,y) = \lim_{t\to\infty} tC_i(x/t, y/t) \tag{5}$$

exist for all $(x,y) \in \mathbb{E} = [0,\infty]^2 \setminus \{(\infty,\infty)\}$ and all $i = 1,\ldots,n$.

At first sight, the assumption of serially independent time series may appear somewhat restrictive. However, the assumption does not seem to be too problematic because of the following argument. In Section 5, the role of (X_i, Y_i) will be played by the unobservable, serially independent innovations of common time series models such as AR or GARCH processes. We will apply the proposed tests to the observable, standardized residuals (obtained by univariate filtering) and consider these residuals as *marginally almost i.i.d.* Our extensive simulation study in Section 4 indicates that the additional estimation step does not influence the asymptotic behavior of our test statistics, i.e., the asymptotic distribution of the estimator based on residuals is the same as the one based on the unobservable, serially independent innovations. Note that this observation is supported by the results in Chen and Fan (2006); Remillard (2010); Chan et al. (2009), where it is shown that the asymptotic distributions of both semi- and nonparametric estimators in copula models are not influenced by marginal filtering.

Also, the assumption of strict stationarity of the marginal distributions may appear restrictive. Note that, in the literature on testing for constant copulas, it can be considered as a common practice hitherto, see for instance Busetti and Harvey (2011); Remillard (2010); Bücher and Ruppert (2013); Bücher et al. (2014). In Section 3.7, we adapt our methods to a more general setting that allows for potential breaks in the marginal distributions. Note that, as we are only interested in strict stationarity in the following (calculation of ranks, see (4), originating from different distributions is of doubtful validity), we drop the adjective strict.

Throughout this paper, it is our aim to develop tests for detecting breaks in the tail dependence, i.e., to test for

$$H_0^{\Lambda}: \text{ there exists } \Lambda > 0 \text{ such that } \Lambda_i \equiv \Lambda \text{ for all } i = 1,\ldots,n$$

against alternatives involving the non-constancy of Λ_i. A special case of this null hypothesis is given by considering the conventional lower tail dependence coefficient $\lambda_i = \Lambda_i(1,1)$. The corresponding null hypothesis reads as

$$H_0^{\lambda}: \text{ there exists } \lambda > 0 \text{ such that } \lambda_i = \lambda \text{ for all } i = 1,\ldots,n.$$

In order to motivate our test statistics, let us first recapitulate the *empirical tail copula* from Schmidt and Stadtmüller (2006) as the basic nonparametric estimator for Λ under H_0^{Λ}, see also (4) and the corresponding citations. Replacing the unknown copula in (5) by the empirical copula $\hat{C}_n$, it is defined as

$$\hat{\Lambda}_n(x,y) = \frac{n}{k}\hat{C}_n\left(\frac{kx}{n}, \frac{ky}{n}\right) = \frac{1}{k}\sum_{i=1}^{n} \mathbb{1}\left(\hat{U}_i \le kx/n, \hat{V}_i \le ky/n\right), \tag{6}$$

where $(\hat{U}_i, \hat{V}_i)$ denote *pseudo-observations* from the copula C, defined by

$$\hat{U}_i = \frac{n}{n+1} F_n(X_i), \qquad \hat{V}_i = \frac{n}{n+1} G_n(Y_i),$$

with F_n and G_n denoting the marginal empirical c.d.f.s. Additionally, $k = k_n \to \infty$, $k = o(n)$ as $n \to \infty$, represents a sequence of parameters discussed in detail below. The ratio k/n can be interpreted as the fraction of data that one considers as *being in the tail* and thus taken into account to estimate the tail dependence in Equation (6). Under suitable regularity conditions some of which are given in the subsequent Section 3.2, it is known that $\hat{\Lambda}_n$ is $\sqrt{k}$-consistent for Λ and that the corresponding *empirical tail copula process* $(x, y) \mapsto \sqrt{k}\{\hat{\Lambda}_n(x, y) - \Lambda(x, y)\}$ converges weakly to a Gaussian limit process.

Now, in order to test for H_0^{Λ}, it is natural to consider a suitable sequential version of $\hat{\Lambda}_n$. We define

$$\hat{\Lambda}_n^{\circ}(s, x, y) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \le kx/n, \hat{V}_i \le ky/n\right)$$

as the sequential empirical tail copula. Under H_0^{Λ}, $\hat{\Lambda}_n^{\circ}$ should be regarded as an estimator for $\Lambda^{\circ}(s, x, y) = s\Lambda(x, y)$. Note that $\hat{\Lambda}_n^{\circ}(1, x, y) = \hat{\Lambda}_n(x, y)$. The crucial quantity for all test procedures in this paper is now given by the *sequential empirical tail copula process* $\{\mathbb{G}_n(s, x, y), s \in [0, 1], (x, y) \in \mathbb{E}\}$ with

$$\mathbb{G}_n(s, x, y) = \sqrt{k}\left\{\hat{\Lambda}_n^{\circ}(s, x, y) - s\hat{\Lambda}_n^{\circ}(1, x, y)\right\}. \tag{7}$$

Note that, despite its name, the sequential empirical tail copula process is not completely sequential. More precisely, the unknown marginal distributions are estimated based on all the available marginal information, whereas only the quantity of interest, the dependence, is assessed sequentially.

Now, some simple calculations show that, for $s \in (0, 1)$, $\mathbb{G}_n$ can be written as

$$\mathbb{G}_n(s, x, y) = \sqrt{k}\{s(1-s)\} \Bigg\{ \frac{1}{ks} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \le kx/n, \hat{V}_i \le ky/n\right) - \frac{1}{k(1-s)} \sum_{i=\lfloor ns \rfloor + 1}^{n} \mathbb{1}\left(\hat{U}_i \le kx/n, \hat{V}_i \le ky/n\right) \Bigg\}.$$

Since $ks \approx \lfloor ks \rfloor$, $ns \approx \lfloor ns \rfloor$ and $k/n \approx \lfloor ks \rfloor / \lfloor ns \rfloor$ for any $s \in (0, 1)$, the two summands in the brackets on the right-hand side can be interpreted as (slightly adapted) empirical tail copulas of the subsamples $(X_1, Y_1), \ldots, (X_{\lfloor ns \rfloor}, Y_{\lfloor ns \rfloor})$ and $(X_{\lfloor ns \rfloor + 1}, Y_{\lfloor ns \rfloor + 1}), \ldots, (X_n, Y_n)$, respectively, with corresponding sequence of parameters $k' = \lfloor ks \rfloor$ and $k'' = \lfloor k(1-s) \rfloor$. Under H_0^{Λ}, one would expect that the difference between these two estimators converges to 0. Therefore, any statistic that can be interpreted as a distance between $\mathbb{G}_n$ and the function being constantly equal to 0 is a reasonable candidate for a test statistic for the null hypothesis. A simulation study similar to one presented in Section 4 showed that a Cramér-von Mises functional yields the best finite-sample performance, which is why we restrict ourselves to this case in the subsequent presentation. Consequently, in case of the simple null hypothesis H_0^λ, we propose the test statistic

$$\mathcal{S}_n := \hat{\lambda}_n^{-1} \int_0^1 \{\mathbb{G}_n(s,1,1)\}^2 \, ds, \tag{8}$$

where $\hat{\lambda}_n = \hat{\Lambda}_n^\circ(1,1,1)$, and to reject the null hypothesis whenever $\mathcal{S}_n$ is larger than an appropriate critical value to be determined later on.

For the construction of a test for the null hypothesis H_0^Λ, we make use of the fact that, by homogeneity, the lower tail copula is uniquely determined by its values on the sphere $S(c) = \{\mathbf{x} \in [0,\infty)^2 : \|\mathbf{x}\| = c\}$, where $\|\cdot\|$ denotes an arbitrary fixed norm on $\mathbb{R}^2$ and where $c > 0$ is an arbitrary fixed constant. The most popular choice in bivariate extreme value theory is $c = 1$ together with the $\|\cdot\|_1$-norm resulting in the function $\Lambda_{\|\cdot\|_1} : [0,1] \to [0,1/2] : t \mapsto \Lambda_{\|\cdot\|_1}(t) = \Lambda(1-t,t)$. Note that $\Lambda_{\|\cdot\|_1}(t) = 1 - A(t)$ with the Pickands dependence function A, see, e.g., Segers (2012).

In order to test for overall constancy of Λ_i it is sufficient to test for constancy of Λ_i on some sphere $S(c)$. In Section 3.5, we will propose a data-adaptive procedure for the choice of the parameter k, which will suggest to use a sphere that contains the point $(1,1)$. For that reason, we introduce the following test statistic

$$\mathcal{T}_n := \int_{[0,1]^2} \{\mathbb{G}_n(s, 2-2t, 2t)\}^2 \; d(s,t),$$

whose support corresponds to the $\|\cdot\|_1$-norm and $c = 2$, and let H_0^Λ again be rejected when $\mathcal{T}_n$ is larger than an appropriate critical value.

In order to determine the critical values, we will derive the asymptotic null distributions of the tests in the next subsection. For both statistics, they will rely on a limit result for the sequential empirical tail copula process.

3.2. Asymptotic null distributions Let $\mathcal{B}_\infty([0,1]\times\mathbb{E})$ denote the space of all functions $f : [0,1]\times\mathbb{E} \to \mathbb{R}$ which are uniformly bounded on every compact subset of $[0,1]\times\mathbb{E}$ (here and throughout, we understand $\mathbb{E} = [0,\infty]^2 \setminus \{(\infty,\infty)\}$ as the one-point uncompactification of the compact set $[0,\infty]^2$), equipped with the metric

$$d(f,g) := \sum_{m=1}^\infty 2^{-m}(\|f-g\|_{S_m} \wedge 1),$$

where $a \wedge b = \min(a,b)$, where the sets S_m are defined as $S_m = [0,1]\times T_m$ with

$$T_m := [0,m]^2 \cup (\{\infty\}\times[0,m)) \cup ([0,m]\times\{\infty\})$$

and where $\|\cdot\|_S$ denotes the sup-norm on a set S. Note that convergence with respect to d is equivalent to uniform convergence on each S_m.

In the following we are going to show weak convergence of $\mathbb{G}_n$ as an element of the metric space $(\mathcal{B}_\infty([0,1]\times\mathbb{E}), d)$. Similar as in related references on the estimation of tail copulas (see Section 2), we have to impose several regularity conditions. First, we need a second order condition quantifying the speed of convergence in (5) uniformly in i and (x,y).

Assumption 3.1. *We have* $\Lambda_i \not\equiv 0$ *and*

$$\Lambda_i(x, y) - tC_i(x/t, y/t) = O(S(t)), \quad t \to \infty, \tag{9}$$

uniformly on $\{(x, y) \in [0, 1]^2 : x + y = 1\}$ *(and hence uniformly on each* T_m*) and uniformly in* $i \in \mathbb{N}$*, where* $S : [0, \infty) \to [0, \infty)$ *denotes a function satisfying* $\lim_{t\to\infty} S(t) = 0$.

Second, the following conditions have to be imposed on the sequence $k = k_n$.

Assumption 3.2. *For some* $\alpha > 0$*, the non-decreasing sequence* $k = k_n \to \infty$ *satisfies the conditions*

$$(a)\ k_n/n \downarrow 0, \qquad (b)\ \sqrt{k}_n S(n/k_n) = o(1),$$

as n *tends to infinity.*

Condition (a) is needed anyway to define a meaningful estimator. Condition (b) allows to control appearing bias terms in the non-sequential empirical tail copula process, see also Schmidt and Stadtmüller (2006) and Bücher and Dette (2013).

With these assumptions we can now state the main result of our paper.

Proposition 3.3. *Suppose that Assumptions 3.1 and 3.2 hold. Then, under* H_0^Λ*,*

$$\mathbb{G}_n \rightsquigarrow \mathbb{G}_\Lambda \quad \text{in } (\mathcal{B}_\infty([0, 1] \times \mathbb{E}), d),$$

where $\mathbb{G}_\Lambda(s, x, y) = \mathbb{B}_\Lambda(s, x, y) - s\mathbb{B}_\Lambda(1, x, y)$*. Here,* $\mathbb{B}_\Lambda$ *is a tight centered Gaussian process with continuous sample paths and with covariance structure*

$$\mathbb{E}[\mathbb{B}_\Lambda(s_1, x_1, y_2)\mathbb{B}_\Lambda(s_2, x_2, y_2)] = (s_1 \wedge s_2)\Lambda(x_1 \wedge x_2, y_1 \wedge y_2).$$

As stated above, Assumption 3.2 (b) is needed to control bias terms occurring when estimating Λ by $\hat{\Lambda}_n$. As the process $\mathbb{G}_n$ does not involve the true tail copula Λ, the assertion of Proposition 3.3 actually holds if (b) is replaced by a quite technical, but less restrictive assumption, see Remark A.3 in the appendix. However, as an application of the proposed test procedures in this paper will usually be followed by the application of estimation techniques relying on (b), we do not feel that imposing this condition is too restrictive.

Proposition 3.3 immediately yields the asymptotic null distributions of $\mathcal{S}_n$ and $\mathcal{T}_n$.

Corollary 3.4. *Suppose that Assumptions 3.1 and 3.2 hold. Then, under* H_0^Λ*,*

$$\mathcal{S}_n \rightsquigarrow \mathcal{S} := \int_0^1 \{B(s)\}^2 ds,$$

where B *is a one-dimensional standard Brownian bridge, and*

$$\mathcal{T}_n \rightsquigarrow \mathcal{T} := \int_{[0,1]^2} \{\mathbb{G}_\Lambda(s, 2 - 2t, 2t)\}^2 \, d(s, t),$$

where $\mathbb{G}_\Lambda$ *is defined in Proposition 3.3.*

Note that, in fact, the weak convergence of $\mathcal{S}_n$ can be derived under a relaxation of H_0^Λ, as it suffices that $\Lambda_i(x, y) \not\equiv 0$ exists and is constant in time in a neighborhood of $(1, 1)$. This is, however, *a bit more* than assumed in H_0^λ.

Since the limiting distribution for $\mathcal{S}_n$ in Corollary 3.4 is pivotal, we directly obtain an asymptotic level α test for H_0^λ.

TDC-Test 1. Reject H_0^λ for $\mathcal{S}_n \geq q_{1-\alpha}^C$, where $q_{1-\alpha}^C$ denotes the $(1-\alpha)$-quantile of the Cramér-von Mises distribution, the latter being defined as the distribution of the random variable $\int_0^1 \{B(s)\}^2 \, ds$.

In order to derive critical values for the test based on $\mathcal{T}_n$, some more effort is needed. Its limiting distribution in Corollary 3.4 is not pivotal and cannot be easily transformed to a distribution which is independent of Λ. Therefore, we propose an appropriate bootstrap approximation for $\mathbb{G}_\Lambda$ which will also allow for the definition of an alternative test for H_0^λ.

Let $B \in \mathbb{N}$ be a large integer and let $\xi_1^{(1)}, \dots, \xi_n^{(1)}, \dots, \xi_1^{(B)}, \dots, \xi_n^{(B)}$ be an independent sequence of $n \times B$ i.i.d. random variables with mean 0 and variance 1 which are independent of the data $(X_1, Y_1), \dots, (X_n, Y_n)$ and possess finite moments of any order. We will refer to $\xi_i^{(b)}$ as a multiplier. Similar in spirit as in Remillard (2010); Bücher and Dette (2013) we define, for any $(s, x, y) \in [0,1] \times \mathbb{E}$ and $b \in \{1, \dots, B\}$,

$$\mathbb{G}_{n,\xi^{(b)}}(s, x, y) = \mathbb{B}_{n,\xi^{(b)}}(s, x, y) - s\mathbb{B}_{n,\xi^{(b)}}(1, x, y), \tag{10}$$

where

$$\begin{aligned}\mathbb{B}_{n,\xi^{(b)}}(s, x, y) &:= \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i^{(b)} \left\{ \mathbb{1}\left(\hat{U}_i \leq kx/n, \hat{V}_i \leq ky/n \right) - \hat{C}_n(kx/n, ky/n) \right\} \\ &= \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i^{(b)} \left\{ \mathbb{1}\left(\hat{U}_i \leq kx/n, \hat{V}_i \leq ky/n \right) - k/n \times \hat{\Lambda}_n^\circ(1, x, y) \right\}.\end{aligned}$$

The following proposition essentially states that, for large n, $\mathbb{G}_{n,\xi^{(1)}}, \dots, \mathbb{G}_{n,\xi^{(B)}}$ can be regarded as *almost* independent copies of $\mathbb{G}_n$. To prove the result, one additional technical assumption on the sequence k_n is required, which can be regarded as very light.

Assumption 3.5. *There exists some $p \in \mathbb{N}$ such that $n/k_n^p = o(1)$.*

Proposition 3.6. *Suppose that Assumptions 3.1, 3.2 and 3.5 hold. Then, under H_0^Λ,*

$$(\mathbb{G}_n, \mathbb{G}_{n,\xi^{(1)}}, \dots, \mathbb{G}_{n,\xi^{(B)}}) \rightsquigarrow (\mathbb{G}_\Lambda, \mathbb{G}_\Lambda^{(1)}, \dots, \mathbb{G}_\Lambda^{(B)})$$

in $(\mathcal{B}_\infty([0,1] \times \mathbb{E}), d)^{B+1}$, where $\mathbb{G}_\Lambda^{(1)}, \dots, \mathbb{G}_\Lambda^{(B)}$ are independent copies of $\mathbb{G}_\Lambda$.

For $b = 1, \dots, B$, define $\mathcal{S}_{n,\xi^{(b)}}$ and $\mathcal{T}_{n,\xi^{(b)}}$ by

$$\mathcal{S}_{n,\xi^{(b)}} = \hat{\lambda}_n^{-1} \int_0^1 \{\mathbb{G}_{n,\xi^{(b)}}(s, 1, 1)\}^2 ds, \quad \mathcal{T}_{n,\xi^{(b)}} = \int_{[0,1]^2} \{\mathbb{G}_{n,\xi^{(b)}}(s, 2-2t, 2t)\}^2 d(s,t).$$

We obtain the following tests for H_0^λ and H_0^Λ, respectively.

TDC-Test 2. Reject H_0^λ for $\mathcal{S}_n \geq \hat{q}_{\mathcal{S}_n, 1-\alpha}$, where $\hat{q}_{\mathcal{S}_n, 1-\alpha}$ denotes the $(1-\alpha)$-sample quantile of $\mathcal{S}_{n,\xi^{(1)}}, \dots, \mathcal{S}_{n,\xi^{(B)}}$.

TC-Test. Reject H_0^Λ for $\mathcal{T}_n \geq \hat{q}_{\mathcal{T}_n, 1-\alpha}$, where $\hat{q}_{\mathcal{T}_n, 1-\alpha}$ denotes the $(1-\alpha)$-sample quantile of $\mathcal{T}_{n,\xi^{(1)}}, \dots, \mathcal{T}_{n,\xi^{(B)}}$.

The final result of this subsection shows that all proposed tests in this paper asymptotically hold their level.

Corollary 3.7. *Suppose that Assumptions 3.1 and 3.2 hold and that H_0^Λ is valid. Then TDC-Test 1 is an asymptotic level α test for H_0^λ. If, additionally, Assumption 3.5 holds, then TDC-Test 2 and TC-Test are asymptotic level α tests for H_0^λ and H_0^Λ, respectively, in the sense that, for any $\alpha \in (0,1)$,*

$$\lim_{B\to\infty}\lim_{n\to\infty}\mathbb{P}(\mathcal{S}_n \geq \hat{q}_{\mathcal{S}_n,1-\alpha}) = \alpha, \qquad \lim_{B\to\infty}\lim_{n\to\infty}\mathbb{P}(\mathcal{T}_n \geq \hat{q}_{\mathcal{T}_n,1-\alpha}) = \alpha.$$

3.3. Asymptotics under fixed alternatives

In the present subsection we are going to show consistency of the proposed test statistics under fixed alternatives. We observe a triangular array of row-wise independent random vectors $(X_{i,n}, Y_{i,n})$, $i = 1, \ldots, n$, such that $X_{i,n} \sim F$ and $Y_{i,n} \sim G$ for all i and n and such that the copula $C_{i,n}$ of $(X_{i,n}, Y_{i,n})$ may vary over time. Slightly abusing notation, we omit the index n wherever it does not cause any ambiguity. For the sake of a clear exposition, we first consider the following two simple alternatives for H_0^λ and H_0^Λ. Later on, we provide a discussion on how to detect multiple break-points and how the test statistics behave in the presence of smooth changes.

$$\begin{aligned}
H_1^\lambda:\ & \text{there exists } \bar{s} \in (0,1), \lambda^{(1)} \neq \lambda^{(2)} \text{ such that} \\
& \qquad \lambda_i = \lambda^{(1)} \text{ for } i = 1, \ldots, \lfloor n\bar{s} \rfloor \text{ and } \lambda_i = \lambda^{(2)} \text{ for } i = \lfloor n\bar{s} \rfloor + 1, \ldots, n. \\
H_1^\Lambda:\ & \text{there exists } \bar{s} \in (0,1), \Lambda^{(1)} \not\equiv \Lambda^{(2)} \text{ such that} \\
& \qquad \Lambda_i = \Lambda^{(1)} \text{ for } i = 1, \ldots, \lfloor n\bar{s} \rfloor \text{ and } \Lambda_i = \Lambda^{(2)} \text{ for } i = \lfloor n\bar{s} \rfloor + 1, \ldots, n.
\end{aligned}$$

Proposition 3.8. *Suppose that Assumptions 3.1 and 3.2 hold.*

(i) If H_1^λ and H_1^Λ are true, then

$$\sup_{s\in[0,1]}\left|\frac{1}{\sqrt{k_n}}\mathbb{G}_n(s,1,1) - A_\lambda(s)\right| = o_P(1)$$

where $A_\lambda(s) = s(1-\bar{s})(\lambda^{(1)} - \lambda^{(2)})$ for $s \leq \bar{s}$ and $A_\lambda(s) = \bar{s}(1-s)(\lambda^{(1)} - \lambda^{(2)})$ for $s > \bar{s}$. Moreover, $\mathcal{S}_n$ converges to infinity in probability.

(ii) If H_1^Λ is true, then

$$\sup_{s\in[0,1],(x,y)\in T_m}\left|\frac{1}{\sqrt{k_n}}\mathbb{G}_n(s,x,y) - A_\Lambda(s,x,y)\right| = o_P(1)$$

for any $m \in \mathbb{N}$, where $A_\Lambda(s,x,y) = s(1-\bar{s})\{\Lambda^{(1)}(x,y) - \Lambda^{(2)}(x,y)\}$ for $s \leq \bar{s}$ and $A_\Lambda(s,x,y) = \bar{s}(1-s)\{\Lambda^{(1)}(x,y) - \Lambda^{(2)}(x,y)\}$ for $s > \bar{s}$. Moreover, $\mathcal{T}_n$ converges to infinity in probability.

As already mentioned after Corollary 3.4, it is not necessary to assume global constancy of the tail copulas in the respective subsamples in part (i) of Proposition 3.8, constancy in a neighborhood of $(1,1)$ is sufficient. Moreover, Proposition 3.8 implies consistency of the proposed tests.

Corollary 3.9. *Suppose that Assumptions 3.1 and 3.2 are satisfied. Then TDC-Test 1 is consistent for H_1^λ. If, additionally, Assumption 3.5 holds, then TDC-Test 2 and TC-Test are consistent for H_1^λ and H_1^Λ, respectively, in the sense that,* *for any $B \in \mathbb{N}$ and $\alpha \in (0,1)$,*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{S}_n \geq \hat{q}_{\mathcal{S}_n,1-\alpha}) = 1, \qquad \lim_{n\to\infty} \mathbb{P}(\mathcal{T}_n \geq \hat{q}_{\mathcal{T}_n,1-\alpha}) = 1.$$

Under H_1^λ and H_1^Λ, consistent estimators for the break-point $\bar{s}$ are given by $\hat{s}^\lambda := \mathrm{argmax}_{s\in[0,1]} |\mathbb{G}_n(s,1,1)|$ and $\hat{s}^\Lambda := \mathrm{argmax}_{s\in[0,1]} \sup_{t\in[0,1]} |\mathbb{G}_n(s, 2-2t, 2t)|$, respectively.

Proposition 3.10. *Suppose that Assumptions 3.1 and 3.2 hold.*

(i) If H_1^λ and H_1^Λ are true, $\hat{s}^\lambda \to_p \bar{s}$.

(ii) If H_1^Λ is true, $\hat{s}^\Lambda \to_p \bar{s}$.

Note that, if one of the alternatives H_1^λ or H_1^Λ holds, then the other one cannot hold with a different value for $\bar{s}$. Hence, the break-point $\bar{s}$ in Proposition 3.10 (i) is well-defined.

Up to now, we have assumed the existence of at most one single break-point. As is shown in the end of this subsection, an analog consistency result for the test can be obtained in the case of an arbitrary finite number of break-points between which the tail copula is constant, respectively. For example, a corresponding alternative for H_0^λ would then read as: there exists a finite number of points $0 = s_0 < s_1 < \ldots < s_\ell < \ldots < s_L = 1$ such that, for any $\ell \in \{1, \ldots, L\}$, the TDC of the sample $(X_{\lfloor ns_{\ell-1}\rfloor+1}, Y_{\lfloor ns_{\ell-1}\rfloor+1}), \ldots, (X_{\lfloor ns_\ell\rfloor}, Y_{\lfloor ns_\ell\rfloor})$ is given by $\lambda^{(\ell)}$, with $\lambda^{(\ell)} \neq \lambda^{(\ell+1)}$.

Estimating the break-points $s_1, s_2, \ldots, s_{L-1}$ is slightly more complicated than it is in the case of just one break-point. In principle, it is also possible to work with the argmax-estimator $\hat{s}^\lambda$ here, but, by construction, this estimator only estimates a single break-point. The number and the location of the other break-points can be estimated by a binary segmentation algorithm going back to Vostrikova (1981). This procedure is for instance applied in Galeano and Wied (2014) to the problem of detecting changing correlations. The basic principle is as follows: at first, the test is applied to the whole sample. If the null hypothesis gets rejected, the argmax-estimator $\hat{s}^\lambda$ can be shown to be a consistent estimator for the *dominating* break-point (see Galeano and Wied, 2014). In the next step, the sample is divided into two parts with the split point given by $\lfloor n\hat{s}^\lambda \rfloor$. The test is applied to both parts separately to decide whether one gets additional break-points in the corresponding subsamples. In that case, the respective subsample is further divided at the corresponding estimated break-point. This procedure is repeated until no further break-points are detected.

The setting with a fixed number of break-points as described above is a special case of a general class of alternatives in which Λ_i (and thus also λ_i) is described by a non-constant function g. More precisely, let $\mathcal{G}$ denote the class of all functions $g : [0,1] \times \mathbb{E} \to \mathbb{R}$ such that $g(s,\cdot,\cdot)$ is a tail copula for any $s \in [0,1]$ and such that, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \sup_{(s,x,y)\in S_m} \left| \frac{1}{n} \sum_{i=1}^{\lfloor sn \rfloor} g\left(\frac{i}{n}, x, y\right) - \int_0^s g(z,x,y)\,dz \right| = 0.$$

The class $\mathcal{G}$ allows to consider the following general class of alternatives, see also

Wied et al. (2012):

$$H_{1,g}^{\lambda}: \text{ there exists } g \in \mathcal{G} \text{ such that } \Lambda_i = g(i/n, \cdot, \cdot) \text{ and such that}$$
$$\int_0^s g(z,1,1)\,dz \neq s \int_0^1 g(z,1,1)\,dz \text{ for some } s \in [0,1],$$
$$H_{1,g}^{\Lambda}: \text{ there exists } g \in \mathcal{G} \text{ such that } \Lambda_i = g(i/n, \cdot, \cdot) \text{ and such that}$$
$$\int_0^s g(z,x,y)\,dz \neq s \int_0^1 g(z,x,y)\,dz \text{ for some } (s,x,y) \in [0,1] \times \mathbb{E}.$$

The former setting with a fixed number of break-points corresponds to a function g that is piecewise constant in s, but in the general case, continuous functions are explicitly allowed. The latter, for instance, may occur in models with time varying copula parameters (see, e.g., Hafner and Manner, 2012 or Patton, 2006).

In general, CUSUM-type procedures as those considered in Section 3.2 are not constructed for detecting smooth changes in the first place. Here, it would perhaps be more advisable to consider a setup based on locally stationary processes. Nevertheless, the test statistics converge to infinity in probability under smooth alternatives.

Proposition 3.11. *Suppose that Assumptions 3.1 and 3.2 hold.*

(i) If $H_{1,g}^{\lambda}$ is true, then

$$\sup_{s\in[0,1]} \left| \frac{1}{\sqrt{k_n}} \mathbb{G}_n(s,1,1) - A_{\lambda}^{g}(s) \right| = o_P(1),$$

where $A_{\lambda}^{g}(s) = \int_0^s g(z,1,1)dz - s\int_0^1 g(z,1,1)dz$. Moreover, $\mathcal{S}_n$ converges to infinity in probability.

(ii) If $H_{1,g}^{\Lambda}$ is true, then

$$\sup_{s\in[0,1],(x,y)\in T_m} \left| \frac{1}{\sqrt{k_n}} \mathbb{G}_n(s,x,y) - A_{\Lambda}^{g}(s,x,y) \right| = o_P(1),$$

for any $m \in \mathbb{N}$, where $A_{\Lambda}^{g}(s,x,y) = \int_0^s g(z,x,y)dz - s\int_0^1 g(z,x,y)dz$. Moreover, $\mathcal{T}_n$ converges to infinity in probability.

As a simple consequence, we obtain consistency of TDC-Test 1 under the setting of Proposition 3.11(i).

3.4. Testing for a break at a specific time point In certain applications, one might have a reasonable guess for a potential break-point in the tail dependence of a time series. Important econometric examples can be seen in Black Monday on 19th of October 1987, the introduction of the Euro on 1st of January 1999 or the bankruptcy of Lehman Brothers Inc. on 15th of September 2008. In that case, it might be beneficial to test for constancy against a break at that specific time point rather than testing against the existence of *some* unspecified break-point. The results in the previous sections easily allow to obtain simple tests in this setting.

Under the situation of Section 3.1, let $\bar{s} \in (0,1)$ be some fixed time point of interest. Suppose we know that the tail dependence is constant in the two subsamples before $\lfloor n\bar{s} \rfloor$ and after $\lfloor n\bar{s} \rfloor + 1$, which, in practice, can be verified by the tests in the preceding sections. Then, to test for H_0^λ against

$$H_1^\lambda(\bar{s}): \text{ there exists } \lambda^{(1)} \neq \lambda^{(2)} \text{ such that}$$
$$\lambda_i = \lambda^{(1)} \text{ for } i = 1, \dots, \lfloor n\bar{s} \rfloor \text{ and } \lambda_i = \lambda^{(2)} \text{ for } i = \lfloor n\bar{s} \rfloor + 1, \dots, n,$$

we propose to use the test statistic

$$\mathcal{S}_n(\bar{s}) := \left(\bar{s}\hat{\lambda}_n\right)^{-1} \mathbb{G}_n(\bar{s}, 1, 1)^2. \tag{11}$$

It easily follows from Proposition 3.3 that, under the null hypothesis, $\mathcal{S}_n(\bar{s})$ weakly converges to a chi-squared distribution with one degree of freedom. Under the alternative, it follows from Proposition 3.8 that $\mathcal{S}_n(\bar{s})$ converges to infinity, in probability. Hence, rejecting H_0^λ if $\mathcal{S}_n(\bar{s})$ exceeds a corresponding quantile of the chi-squared distribution, yields a consistent test for H_0^λ against $H_1^\lambda(\bar{s})$, which asymptotically holds its significance level. Similar results can be obtained for the bootstrap analog and for the test for constancy of the entire tail copula, the details are omitted for the sake of brevity.

3.5. Choice of the parameter k As usual in extreme-value theory, the choice of k_n plays a crucial role for statistical applications. The asymptotic properties of the tests proposed in this paper hold as long as the assumptions on the sequence k_n from Assumption 3.2 (and of course other assumptions) hold. This, of course, allows for a large number of possible choices of k_n. However, the results of the testing procedures may depend crucially on the specific choice of k_n.

The common approach in extreme-value theory to cope with this problem is to consider the outcome of statistical procedures, for instance of an estimator, for several different values of k. The set of all these outcomes should give a clearer picture of the underlying data-generating process. This, for instance, is the basic motivation for the Hill plot used in univariate extreme-value theory for estimating the extreme-value index, see, e.g., Embrechts et al. (1997). Additionally, in certain univariate settings some refined data-adaptive choices to estimate an *optimal* k have been developed, see for instance Drees and Kaufmann (1998) or Danielsson et al. (2001).

In the specific context of estimating tail dependence, Frahm et al. (2005) use plots of the function $k \mapsto \text{TDC}(k)$ to define a *plateau-finding* algorithm that provides a single data-adaptive choice of k. In most of the application in this paper, we closely follow their approach for which reason we briefly summarize this algorithm in the following.

The aim of the algorithm is to search for a value k^* such that the TDC, as a function of k, is *as constant as possible* in a suitable neighborhood of k^*. This is achieved by accomplishing the following steps: first, the function $k \mapsto \text{TDC}(k)$ is smoothed by a box kernel depending on a bandwidth b; we denote the smoothed plot by $k \mapsto \tilde{\lambda}_b(k), k = 1, \dots, n-2b$. In our simulation study, we use $b = \lfloor 0.005n \rfloor$. In a second step, we consider a rolling window of vectors or plateaus (having length $\ell = \lfloor \sqrt{n-2b} \rfloor$) with their entries consisting of successive values of the smoothed TDC-plot, formally defined as $P(k) = (\tilde{\lambda}_b(k), \tilde{\lambda}_b(k+1), \dots, \tilde{\lambda}_b(k+\ell-1)) \in \mathbb{R}^\ell$, where $k = 1, \dots, n-2b-\ell+1$. We calculate the sum of the absolute deviations between all entries and the first entry in each vector, i.e., $\text{MAD}(k) = \sum_{j=1}^{\ell} |(P(k))_1 -$

$(P(k))_j|$. The algorithm searches for the first vector such that MAD(k) is smaller than two times the sample standard deviation of all values of the smoothed TDC-plot $\tilde{\lambda}_b(1), \ldots, \tilde{\lambda}_b(n-2b)$. Finally, k^* is defined as the index which corresponds to the middle entry (the floor function if the length is even) of this vector. For further details, we refer to Frahm et al. (2005).

3.6. Higher dimensions Although we have focused on the case of two dimensions so far, it is basically straightforward (although notationally more involved) to deal with d-dimensional random vectors for a fixed number d. Consider a sequence of marginally i.i.d. random vectors $(X_{i1}, \ldots, X_{id})_{i \in \{1,\ldots,n\}}$ with continuous marginal c.d.f.s $F_1, \ldots, F_d$ and d-dimensional copulas C_i. We suppose that the corresponding lower tail copulas

$$\Lambda_i(x_1, \ldots, x_d) := \lim_{t \to \infty} tC_i(x_1/t, \ldots, x_d/t).$$

exist for all $x = (x_1, \ldots, x_d) \in \mathbb{E}_d = [0, \infty]^d \setminus \{(\infty, \ldots, \infty)\}$. Note that Λ_i is in one-to-one correspondence to the familiar d-dimensional stable tail dependence function of $(X_{i1}, \ldots, X_{id})$, see, e.g., Einmahl et al. (2012) for its definition. Define pseudo-observations $(\hat{U}_{i1}, \ldots, \hat{U}_{id})$ from the copula C_i by $\hat{U}_{ij} = \frac{n}{n+1} F_{nj}(X_{ij}),\ j = 1, \ldots, d$, where F_{nj} denote the marginal empirical c.d.f.s. The d-dimensional sequential empirical tail copula process is defined, for any $(s, x_1, \ldots, x_d) \in [0, 1] \times \mathbb{E}_d$, by

$$\mathbb{G}_n(s, x_1, \ldots, x_d) = \sqrt{k} \left\{ \hat{\Lambda}_n^{\circ}(s, x_1, \ldots, x_d) - s\hat{\Lambda}_n^{\circ}(1, x_1, \ldots, x_d) \right\},$$

where $\hat{\Lambda}_n^{\circ}(s, x_1, \ldots, x_d) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}(\hat{U}_{i1} \leq kx_1/n, \ldots, \hat{U}_{id} \leq kx_d/n)$. A test statistic only focussing on the d-dimensional TDC can be defined analog to the 2-dimensional case,

$$\mathcal{S}_n := \left\{ \hat{\Lambda}_n^{\circ}(1, 1, \ldots, 1) \right\}^{-1} \int_0^1 \{\mathbb{G}_n(s, 1, \ldots, 1)\}^2 \, ds,$$

while test statistics focussing on the entire tail copula look slightly more complicated. For instance, one might use

$$\mathcal{T}_n := \int_{[0,1] \times \Delta} \{\mathbb{G}_n(s, \boldsymbol{t})\}^2 \, d(s, \boldsymbol{t}),$$

where $\Delta := \{\boldsymbol{t} \in (t_1, \ldots, t_d) \in \mathbb{E} \mid \text{at least 2 of the } t_j \text{ are } \neq \infty, \sum_{j=1, t_j \neq \infty}^{d} t_j = 1\}$. Note that the restriction of a tail copula to Δ uniquely determines the whole tail copula by homogeneity. Bootstrap statistics can be defined analogously. For the asymptotic results, one has to modify the metric defined in the beginning of Section 3.2 such that

$$T_m := \bigcup_{j=0}^{d-1} \bigcup_{\ell=1}^{\binom{d}{j}} U_{m,j,\ell},$$

where, for each $m \in \mathbb{N}$ and $j = 0, \ldots, d-1$, the $U_{m,j,\ell}$ are the $\binom{d}{j}$ different d-fold cartesian products that contain j times $\{\infty\}$ and $d - j$ times $[0, m]$.

3.7. Testing for a break under non-stationarity of the marginals Throughout the previous sections, we made the assumption that the marginal laws of (X_i, Y_i) are constant over time. A less stringent assumption would be to allow for breaks in the marginal laws. In the present section, we outline how the proposed methods can be adapted to that setting.

For the sake of brevity, we restrict ourselves to the case of one known break in each marginal. Let (X_i, Y_i) be an independent sequence of random variables with copula C_i and continuous marginal c.d.f.s $F^{(i)}$ and $G^{(i)}$, respectively. Suppose that there exist $t_F, t_G \in (0,1)$ such that $F^{(1)} = \cdots = F^{(\lfloor nt_F \rfloor)} \neq F^{(\lfloor nt_F \rfloor+1)} = \cdots = F^{(n)}$ and $G^{(1)} = \cdots = G^{(\lfloor nt_G \rfloor)} \neq G^{(\lfloor nt_G \rfloor+1)} = \cdots = G^{(n)}$. Define pseudo-observations $(\hat{U}_i, \hat{V}_i)$ of C_i through

$$\hat{U}_i = \begin{cases} F_{1:\lfloor nt_F \rfloor}(X_i), & i \le \lfloor nt_F \rfloor, \\ F_{\lfloor nt_F \rfloor+1:n}(X_i), & i > \lfloor nt_F \rfloor, \end{cases} \quad \hat{V}_i = \begin{cases} G_{1:\lfloor nt_G \rfloor}(Y_i), & i \le \lfloor nt_G \rfloor, \\ G_{\lfloor nt_G \rfloor+1:n}(Y_i), & i > \lfloor nt_G \rfloor, \end{cases} \tag{12}$$

where $F_{(k+1):\ell}(x) := (\ell - k + 1)^{-1} \sum_{j=k+1}^{\ell} \mathbb{1}(X_j \le x)$, and similarly for the second coordinate. Define $\mathbb{G}_n$ exactly as in (7). For the derivation of asymptotic properties, we need an additional smoothness assumption on Λ.

Assumption 3.12. *The first order partial derivative $\dot{\Lambda}_x = \frac{\partial}{\partial x}\Lambda$ exists and is continuous on $\{(x,y) \in \mathbb{E} : 0 < x < \infty\}$. The first order partial derivative $\dot{\Lambda}_y = \frac{\partial}{\partial y}\Lambda$ exists and is continuous on $\{(x,y) \in \mathbb{E} : 0 < y < \infty\}$.*

Proposition 3.13. *Suppose that Assumptions 3.1, 3.2 and 3.12 are satisfied. Then, under H_0^{Λ}, we have $\mathbb{G}_n \rightsquigarrow \mathbb{G}_{\Lambda,t_F,t_G}$ in $(\mathcal{B}_\infty([0,1] \times \mathbb{E}), d)$, where*

$$\mathbb{G}_{\Lambda,t_F,t_G}(s,x,y) = \mathbb{G}_\Lambda(s,x,y) - \dot{\Lambda}_x(x,y) \frac{s \wedge t_F - st_F}{t_F(1-t_F)} \mathbb{G}_\Lambda(t_F, x, \infty) - \dot{\Lambda}_y(x,y) \frac{s \wedge t_G - st_G}{t_G(1-t_G)} \mathbb{G}_\Lambda(t_G, \infty, y).$$

The limiting distribution is different from the one under constant margins in Proposition 3.3. As a consequence, for approximating critical values of an appropriate test statistic, one needs to modify the methods described in the previous sections. In the following, we restrict ourselves to the case of testing for a constant coefficient of tail dependence. Let $\hat{\Lambda}_{x,n}(1,1)$ and $\hat{\Lambda}_{y,n}(1,1)$ denote estimators for the partial derivatives of Λ at $(1,1)$ which are consistent under the null hypothesis, for instance

$$\hat{\Lambda}_{x,n}(1,1) := \frac{k^{1/2}}{2} \left\{ \hat{\Lambda}_n(1 + k^{-1/2}, 1) - \hat{\Lambda}_n(1 - k^{-1/2}, 1) \right\},$$

and similar for the partial derivative with respect to y (Bücher and Dette, 2013). Furthermore, let $\mathcal{S}_n$ be defined as in (8) with pseudo-observations as in (12). Observing that $s \mapsto \lambda^{-1/2}\mathbb{G}_\Lambda(s,1,1)$ is a standard Brownian bridge, Proposition 3.13 suggests the following test procedure.

TDC$^{\mathbf{MB}}$-Test. Reject H_0^{λ} if the test statistic $\mathcal{S}_n$ is larger than the $(1-\alpha)$-quantile of $\int_0^1 \{\hat{B}_{t_F,t_G}(s)\}^2 \, ds$, where

$$\hat{B}_{t_F,t_G}(s) := B(s) - \hat{\Lambda}_{x,n}(1,1) \frac{s \wedge t_F - st_F}{t_F(1-t_F)} B(t_F) - \hat{\Lambda}_{y,n}(1,1) \frac{s \wedge t_G - st_G}{t_G(1-t_G)} B(t_G),$$

with marginal break points t_F and t_G, and a standard Brownian bridge B.

Analogs of the tests in Section 3.4 for the detection of breaks at a given time point $\bar{s}$ are straightforward.

In practice, the marginal break points t_F and t_G are rarely known. However, they can usually be estimated at rate n^{-1} which suggests that the previous results remain valid provided t_F and t_G are replaced by suitable estimators $\hat{t}_F$ and $\hat{t}_G$ (see, e.g., Dümbgen, 1991) both within the definition of the pseudo-observations in (12) and the approximation of the limit distribution stated in Proposition 3.13. For instance, in a model with a structural break in the (unconditional) mean for the first marginal, one might use

$$\hat{t}_F = \frac{1}{n} \operatorname*{argmax}_{j=1,\ldots,n} \left| \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{j} X_i - \frac{j}{n} \sum_{i=1}^{n} X_i \right) \right|, \tag{13}$$

(see, e.g., Bai, 1997; Aue and Horváth, 2013). The simulation results in Section 4 show that, indeed, the approximation of the nominal size is quite good.

4. Evidence in finite samples

This section investigates the finite sample properties of the proposed testing procedures by means of a simulation study. We observe that the tests are slightly conservative and that they have reasonable power properties. As a main conclusion, we obtain that the tests based on i.i.d. observations and on time series residuals show the same asymptotic behavior.

4.1. Setup As outlined in Jäschke (2014) (see also McNeil et al., 2005, Section 7.5), many commonly applied symmetric tail copulas exhibit a quite similar behavior. When comparing, for instance, the Gumbel model (Gumbel, 1961), the Galambos model (Galambos, 1975) or the Hüsler-Reiss model (Hüsler and Reiss, 1989), the plots of $t \mapsto \Lambda(1-t, t)$, which uniquely determine the tail copula by homogeneity, are nearly indistinguishable. We therefore stick to two cases of one common symmetric and one common asymmetric tail copula model as follows.

(Λ1) The **negative logistic** or Galambos model (Galambos, 1975), defined by

$$\Lambda(1-t, t) = \left\{ (1-t)^{-\theta} + t^{-\theta} \right\}^{-1/\theta}, \quad t \in [0, 1],$$

where we chose the parameter $\theta \in [1, \infty)$ such that $\lambda = \Lambda(1, 1) = 2^{-1/\theta}$ varies in the set $\{0.25, 0.50, 0.75\}$.

(Λ2) The **asymmetric negative logistic** model (Joe, 1990), defined by

$$\Lambda(1-t, t) = \left\{ (\psi_1(1-t))^{-\theta} + (\psi_2 t)^{-\theta} \right\}^{-1/\theta}, \quad t \in [0, 1],$$

with two fixed parameters $\psi_1 = 2/3$, $\psi_2 = 1$ and parameter $\theta \in [1, \infty)$ such that $\lambda = \Lambda(1, 1) = 2\left((\psi_1/2)^{-\theta} + (\psi_2/2)^{-\theta}\right)^{-1/\theta}$ varies in the set $\{0.2, 0.4, 0.6\}$.

Note that (Λ1) is a special case of (Λ2) with $\psi_1 = \psi_2 = 1$. Tail copulas being directional derivatives of copulas in the origin, there are of course many copulas that result in the same tail copula. In our simulation study, we basically stick to simulating from one of following two copula families.

(C1) The **Clayton copula**, defined by

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad u,v \in [0,1],$$

possesses the negative logistic tail copula as specified in (Λ1). The Clayton copula is widely used for modeling negative tail dependent data.

(C2) The **survival copula of the extreme-value copula**, defined by

$$C(u,v) = \exp\left\{\log(uv) A\left(\frac{\log(v)}{\log(uv)}\right)\right\} \quad u,v \in [0,1], \tag{14}$$

where $A(t) = 1 - \Lambda(1-t,t)$ with Λ as in (Λ2), see Segers (2012), possesses the asymmetric negative logistic tail copula specified in (Λ2).

In order to show that our methods have better power properties than tests for constancy of the whole copula, provided the change only takes place in the tail, we consider a third model.

(C3) Instead of giving a closed form expression for the copula, we state the simulation algorithm for generating a pair (U,V) from that copula.

(a) First, generate $(\tilde{U},\tilde{V}) \sim C$, with C being one of the aforementioned copulas (C1) or (C2).

(b) Then, if $(\tilde{U},\tilde{V}) \in [a,1]^2$, set $(X,Y) = (\tilde{U},\tilde{V})$. If $(\tilde{U},\tilde{V}) \in [0,1]^2 \setminus [a,1]^2$, toss a coin with success probability p. In case of success, define $(X,Y) = (a\bar{U}, a\bar{V})$ with $(\bar{U},\bar{V}) \sim C$, independent of $(\tilde{U},\tilde{V})$, otherwise set $(X,Y) = (\tilde{U},\tilde{V})$.

Note that, for $p=0$, (X,Y) is distributed according to the initial copula C. Some tedious calculations show that

$$\begin{aligned} H(x,y) &= \mathbb{P}(X \le x, Y \le y) \\ &= \begin{cases} \mu p C(x_{\min}, y_{\min}) + (1-p)C(x,y), & (x,y) \in [0,1]^2 \setminus [a,1]^2, \\ \mu p + C(x,y) - p\{C(x,a) + C(a,y) - C(a,a)\}, & (x,y) \in [a,1]^2, \end{cases} \end{aligned}$$

where $x_{\min} := \min(x/a,1)$, $y_{\min} := \min(y/a,1)$ and $\mu := C(a,1) + C(1,a) - C(a,a)$ denotes the C-measure of $[0,1]^2 \setminus [a,1]^2$.

(c) Finally, define (U,V) by $U = H(X,1)$ and $V = H(1,Y)$.

(Estimated) densities of the resulting copulas are depicted in Figure 5 in the supplementary material for the Clayton copula with $\theta = 0.5$ (or equivalently $\lambda = 0.25$), for $a = 0.1$ and for $p \in \{0, 0.3\}$. One can clearly see that the two densities are very close to each other on $[a,1]^2$, while they differ significantly in the tail.

Our simulation results will show that the distribution of the test statistic based on estimated *marginally almost i.i.d.* residuals is the same as the one of the test statistic based on the unobservable, marginally i.i.d. innovations. Regarding the marginal time series behavior, we consider three different cases. We begin with a consideration of i.i.d. marginals. Subsequently, the simulation results in this case will serve as a benchmark for the application of the tests to *marginally almost i.i.d.* residuals of AR and GARCH time series models. Note that, under the null hypothesis, the latter two models satisfy the assumptions that Remillard (2010) imposed in the context of related residual-based tests for constancy of the entire copula.

(T1) **Serial independence.** Here, we simply generate independent random vectors (U_i, V_i), $i = 1, \ldots, n$, of one of the aforementioned copulas (C1), (C2) or (C3). Note that, without loss of generality, the marginal distribution can be chosen as standard uniform in this case, since all estimators in this paper are rank-based and hence invariant with respect to monotone transformations.

(T2) **AR(1) residuals.** This setting considers the (under H_0 stationary) solution (Q_i, R_i) of the first order autoregressive process

$$\begin{cases} Q_i = \beta_1 Q_{i-1} + X_i, \\ R_i = \beta_2 R_{i-1} + Y_i, \end{cases} \quad i \in \mathbb{Z}, \tag{15}$$

where (X_i, Y_i), $i \in \mathbb{Z}$, are serially independent bivariate random vectors (innovations) whose copula is either from model (C1) or (C2). Here, the (stationary) marginals X_i, $i \in \mathbb{Z}$, are standard normally distributed and Y_i, $i \in \mathbb{Z}$, are t_3-distributed, respectively. The coefficients (β_1, β_2) of the lagged variables vary in the set $\{1/3, 1/2, 2/3\}$. We simulate a time series of length n of this model as follows:

(a) choose some reasonably large negative number M, e.g., $M = -100$;

(b) generate a serially independent sequence $(U_i, V_i) \sim C_i$, $i = M, \ldots, n$ of one of the aforementioned copulas C and apply the inverse of the marginal c.d.f.s F and G to the copula sample, vis. $(X_i, Y_i) = (F^{-1}(U_i), G^{-1}(V_i))$;

(c) calculate recursively the values (Q_i, R_i) according to (15) for all $i = M + 1, \ldots, n$, starting with $(Q_M, R_M) = (X_M, Y_M)$; the last n observations form the final sample.

Since we do not observe the innovations (X_i, Y_i), $i = 1, \ldots, n$, we estimate β_1 and β_2 by the Yule-Walker estimators and obtain an *marginally almost i.i.d.* sample (see Section 3.1) by considering the time series $(\hat{X}_i, \hat{Y}_i)$ of corresponding estimated residuals defined as

$$\hat{X}_i = Q_i - \hat{\beta}_1 Q_{i-1}, \quad \hat{Y}_i = R_i - \hat{\beta}_2 R_{i-1}, \quad i = 1, \ldots, n.$$

(T3) **GARCH(1,1) residuals.** The final setting analyses a two-dimensional time series model which is based on the frequently applied univariate GARCH(1,1) model. More precisely, for $i \in \mathbb{Z}$, we consider the (under H_0 stationary) solution (Q_i, R_i) of

$$\begin{cases} Q_i = \sigma_{i,1} X_i, & \sigma_{i,1}^2 = \omega_1 + \alpha_1 Q_{i-1}^2 + \beta_1 \sigma_{i-1,1}^2, \\ R_i = \sigma_{i,2} Y_i, & \sigma_{i,2}^2 = \omega_2 + \alpha_2 R_{i-1}^2 + \beta_2 \sigma_{i-1,2}^2, \end{cases} \tag{16}$$

where (X_i, Y_i), $i \in \mathbb{Z}$, are serially independent bivariate random vectors (innovations) whose copula is again either from model (C1) or (C2). This time, the (stationary) marginals X_i, $i \in \mathbb{Z}$, are standard normally distributed and Y_i, $i \in \mathbb{Z}$, are *normalized* t_3-distributed (i.e., $\sqrt{3} Y_i$, $i \in \mathbb{Z}$, are t_3-distributed), respectively. Following the empirical application of modeling volatility of S&P 500 and DAX daily log-returns in Jondeau et al. (2007) we set the coefficients $\omega_1 = 0.012$, $\omega_2 = 0.037$, $\alpha_1 = 0.072$, $\alpha_2 = 0.115$, $\beta_1 = 0.919$ and $\beta_2 = 0.868$. The long run average variances in this model are given by $\sigma_{M,j} = \sqrt{\omega_j/(1-\alpha_j-\beta_j)}$ which also serve as initial values for simulating a sample from (16). The simulation algorithm reads as follows:

(a) generate an independent sample (X_i, Y_i), $i = M, \ldots, n$, as described in steps (a) and (b) of the previous AR(1) setting;

(b) recursively calculate the values (Q_i, R_i) according to (16) for all $i = M+1, \ldots, n$, starting with $(Q_M, R_M) = (\sigma_{M,1}X_M, \sigma_{M,2}Y_M)$; again, the last n observations form the final sample.

A *marginally almost i.i.d.* sample $(\hat{X}_i, \hat{Y}_i)$, $i = 1, \ldots, n$, to which we apply the tests is obtained by estimating the standardized residuals

$$\hat{X}_i := \sigma_{i,1}^{-1}(\hat{\omega}_1, \hat{\alpha}_1, \hat{\beta}_1)Q_i, \quad \hat{Y}_i := \sigma_{i,2}^{-1}(\hat{\omega}_2, \hat{\alpha}_2, \hat{\beta}_2)R_i, \quad i = 1, \ldots, n,$$

where the estimates $\hat{\omega}_j, \hat{\alpha}_j$ and $\hat{\beta}_j$, $j = 1, 2$, are calculated by applying standard constraint non-linear optimization routines.

4.2. Results and discussion The target values of our finite sample study are the simulated rejection probabilities (s.r.p.s) of the Cramér-von Mises tests described in Sections 3.2 and 3.7 under the null hypothesis and under various alternatives. We calculate the s.r.p.s for three levels of significance $\alpha \in \{1\%, 5\%, 10\%\}$, for two different sample sizes $n = 1{,}000$ and $n = 3{,}000$ and for all of the previously described models. Due to close similarity of some of the results, we report them only partially (for instance, we only list the results for $\alpha = 5\%$). The results are based on $N = 5{,}000$ repetitions, unless stated otherwise.

In Table 1, we present the results for TDC-Test 1 under 7×3 different null hypotheses. The s.r.p.s are stated in columns 3 ($n = 1{,}000$) and 6 ($n = 3{,}000$), respectively. The parameter k is determined by the plateau algorithm described in Section 3.5. The properties of this algorithm are summarized in columns 4 and 5 ($n = 1{,}000$) and 7 and 8 ($n = 3{,}000$), where we state the mean and the sample standard deviation of the estimate k^*. We observe an accurate approximation of the nominal level in all cases, with a tendency of a slight underestimation of the significance level in most of the cases. As already mentioned in Ssection 3, the additional initial estimation step of applying univariate filtering to the time series does not significantly influence the finite sample properties. The slight conservative behavior of the test can be explained by the constancy of the copula in most of our settings: defining $\hat{C}_n^\circ(s, u, v) = \frac{1}{n}\sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}(\hat{U}_i \le u, \hat{V}_i \le v)$ the test statistic $\mathcal{S}_n$ from Equation (8) can be rewritten as

$$\mathcal{S}_n = \left\{\hat{C}_n^\circ(1, k/n, k/n)\right\}^{-1} \int_0^1 \left[\sqrt{n}\left\{\hat{C}_n^\circ(s, k/n, k/n) - s\hat{C}_n^\circ(1, k/n, k/n)\right\}\right]^2 ds.$$

If k was chosen such that $u = k/n > 0$ is constant in n and if, additionally to the tail copula, the copula remained constant over time, it would follow from Corollary 3.3 (a) in Bücher and Volgushev (2013) that $\mathcal{S}_n$ weakly converges to $\{1 - C(u, u)\} \int_0^1 B^2(s)\, ds$, where B denotes a standard Brownian bridge. Since the critical values of TDC-Test 1 are the quantiles of $\int_0^1 B^2(s)\, ds$, we can easily see that the test rejects too rarely, provided that $C(u, u) > 0$. Note that this argument remains valid if the copula is constant over time only in a neighborhood of (u, u).

tail copula	$\Lambda(1,1)$	$n = 1{,}000$ s.r.p.	avg(k^*)	std(k^*)	$n = 3{,}000$ s.r.p.	avg(k^*)	std(k^*)
serial independence							
(Λ1)	0.25	0.046	52	23	0.044	97	49
	0.50	0.044	71	29	0.047	134	59
	0.75	0.039	127	46	0.038	237	97
(Λ2)	0.20	0.047	44	20	0.045	80	40
	0.40	0.042	61	26	0.047	113	52
	0.60	0.046	84	34	0.045	153	67
(Λ1), (Λ2)	0.20	0.053	47	21	0.056	86	44
	0.40	0.044	62	26	0.048	114	52
	0.60	0.047	87	33	0.048	159	69
AR(1) residuals							
(Λ1)	0.25	0.046	52	23	0.047	97	49
	0.50	0.049	72	29	0.049	134	60
	0.75	0.036	126	46	0.041	235	95
(Λ2)	0.20	0.042	44	20	0.049	81	40
	0.40	0.046	61	25	0.045	112	42
	0.60	0.043	83	34	0.045	154	67
GARCH(1,1) residuals							
(Λ1)	0.25	0.047	52	23	0.047	96	48
	0.50	0.045	72	29	0.048	134	59
	0.75	0.038	127	45	0.041	235	94
(Λ2)	0.20	0.043	44	20	0.046	81	42
	0.40	0.045	61	25	0.043	113	52
	0.60	0.044	85	33	0.047	154	68

Table 1: Simulated rejection probabilities of TDC-Test 1 under various null hypotheses H_0^Λ. In the AR(1) scenario the marginals X_i, $i = 1, \ldots, n$, are standard normally distributed and Y_i, $i = 1, \ldots, n$, are t_3-distributed, respectively. The parameters are set to $\beta_1 = 1/3$ and $\beta_2 = 2/3$. In the GARCH(1,1) setting, $\sqrt{3}Y_i$, $i = 1, \ldots, n$, are t_3-distributed.

A more enlightening view on this issue can be gained from the results in the third block of Table 1. Here, we first simulate the first half of the dataset from model (C1) whereas the second half is simulated from model (C2). The parameters are chosen in such a way that both models exhibit the same tail dependence coefficient. Hence, we are still simulating under the null hypothesis but this time the hybrid (copula) model is not constant (over time) at any point on the diagonal of the interior of the unit square. Within the serially independent setting we observe that this is the only case where the s.r.p.s (slightly) exceed some levels of significance.

In Table 2, we present simulation results for TDC-Test 1 under 8×3 different alternatives. We consider only the case of one break-point, which is either located at $\bar{s} = 0.25$ or at $\bar{s} = 0.5$, and of three different upward jumps. Note that, for symmetry reasons, the results are essentially the same for corresponding downward jumps at $1 - \bar{s}$. The second column of the table indicates the coefficient of tail dependence before and after the break-point. As one might have expected, higher jumps in the TDC are detected more frequently. Also, breaks at $\bar{s} = 0.5$ are more likely to be detected than breaks at $\bar{s} = 0.25$. Similar as for the null hypotheses presented in Table 1, the discrepancy between the corresponding results for the serially independent case and for the time series residuals appears to be negligible. Overall, one can conclude that TDC-Test 1 shows reasonable power properties.

tail copula	$\Lambda(1,1)$	$n = 1{,}000$ s.r.p.	avg(k^*)	std(k^*)	$n = 3{,}000$ s.r.p.	avg(k^*)	std(k^*)
serial independence, $\bar{s} = 0.5$							
(Λ1)	0.25 to 0.50	0.165	61	26	0.309	113	53
	0.25 to 0.75	0.563	76	30	0.845	140	64
	0.50 to 0.75	0.211	93	35	0.389	171	71
(Λ2)	0.20 to 0.40	0.175	52	22	0.306	94	45
	0.20 to 0.60	0.485	60	25	0.752	111	52
	0.40 to 0.60	0.139	71	29	0.258	130	59
serial independence, $\bar{s} = 0.25$							
(Λ1)	0.25 to 0.50	0.091	66	27	0.173	122	52
	0.25 to 0.75	0.340	95	36	0.650	174	74
	0.50 to 0.75	0.129	107	40	0.225	197	82
(Λ2)	0.20 to 0.40	0.092	56	24	0.161	104	49
	0.20 to 0.60	0.262	71	29	0.498	129	60
	0.40 to 0.60	0.079	77	31	0.148	144	63
AR(1) residuals, $\bar{s} = 0.5$							
(Λ1)	0.25 to 0.50	0.164	61	26	0.300	114	53
	0.25 to 0.75	0.567	76	30	0.849	138	62
	0.50 to 0.75	0.205	92	35	0.390	169	72
(Λ2)	0.20 to 0.40	0.170	52	22	0.295	96	48
	0.20 to 0.60	0.489	61	25	0.766	111	53
	0.40 to 0.60	0.148	71	29	0.268	131	59
GARCH(1,1) residuals, $\bar{s} = 0.5$							
(Λ1)	0.25 to 0.50	0.159	61	25	0.316	113	52
	0.25 to 0.75	0.575	76	30	0.849	141	63
	0.50 to 0.75	0.213	91	35	0.387	169	70
(Λ2)	0.20 to 0.40	0.174	52	23	0.317	95	45
	0.20 to 0.60	0.487	60	25	0.747	111	53
	0.40 to 0.60	0.147	71	29	0.262	132	61

Table 2: Simulated rejection probabilities of TDC-Test 1 under various alternatives H_1^λ. The marginal time series models are the same as in Table 1 except that $\beta_1 = 1/2 = \beta_2$.

Table 3 briefly presents simulation results for TDC-Test 2 and the TC-Test. For the sake of brevity, we only report the s.r.p.s for the Clayton tail copula model and the serially independent case, since the results for the other cases do not convey any additional insights. The s.r.p.s are based on $N = 1{,}000$ simulation runs, while the sample size is again either $n = 1{,}000$ or $n = 3{,}000$ with $B = 500$ bootstrap replications ($B = 300$ for the TC-Test) and multipliers $\xi_i^{(b)}$ that are uniformly distributed on the set $\{-1, 1\}$. In comparison to its competitor TDC-Test 1, we observe that with TDC-Test 2, there seems to be slight evidence that the rejection probabilities are higher both under the null hypothesis as well as under the alternative. Regarding the null hypothesis, a comparable observation can be made for the TC-Test, but the power under the alternative is even lower than that of TDC-Test 1.

The next results of this section, presented in Table 4, concern a setting where the tail dependence coefficient stays constant over time whereas the tail copula may change at points $(x, y) \neq (1, 1)$ (cf. third block of Table 1). From theory, one would expect that the TC-Test should be able to detect those breaks, whereas the TDC-Tests should hold the nominal size. We only consider breaks at $\bar{s} = 0.5$ and

scenario	$\Lambda(1,1)$	$n = 1{,}000$ s.r.p.	avg(k^*)	std(k^*)	$n = 3{,}000$ s.r.p.	avg(k^*)	std(k^*)
TDC-Test 2							
H_0^Λ	0.25	0.046	52	23	0.049	97	51
	0.50	0.052	73	28	0.052	135	62
	0.75	0.052	129	47	0.044	239	96
H_1^λ, $\bar{s} = 0.5$	0.25 to 0.50	0.179	61	26	0.320	113	52
	0.25 to 0.75	0.579	77	31	0.855	138	61
	0.50 to 0.75	0.256	93	34	0.425	171	72
TC-Test							
H_0^Λ	0.25	0.045	51	23	0.051	95	48
	0.50	0.044	70	29	0.039	135	58
	0.75	0.046	127	46	0.060	239	95
H_1^λ, $\bar{s} = 0.5$	0.25 to 0.50	0.152	61	25	0.312	116	52
	0.25 to 0.75	0.530	76	30	0.813	141	64
	0.50 to 0.75	0.169	91	35	0.302	169	72

Table 3: Simulated rejection probabilities of TDC-Test 2 and the TC-Test within the serial independence setting.

model (Λ2) (i.e., we simulate from (C2)) which will allow to construct tail copulas that are equal in $(1,1)$, but sufficiently different in other points. More precisely, for a given $\lambda \in \{0.2, 0.4, 0.6\}$, we choose $\psi_1 = \lambda$, $\psi_2 = 1$ and $\theta = 100$ for $s \leq \bar{s}$ and we set $\psi_1 = 1$, $\psi_2 = \lambda$ and $\theta = 100$ for $s > \bar{s}$. For $\lambda = 0.4$, the corresponding graphs of $t \mapsto \Lambda(2-2t, 2t)$ are shown in Figure 6 in the supplementary material. Note that, for fixed ψ_1, ψ_2, we have

$$\Lambda_\infty(1-t,t) := \lim_{\theta\to\infty} \Lambda(1-t,t) = \{\psi_1(1-t)\} \wedge (\psi_2 t).$$

The corresponding *limit* copula defined in (14) is the well-known Marshall–Olkin copula, whose TDC is given by $\min(\psi_1, \psi_2)$, see Segers (2012). With our choice of $\theta = 100$ in (Λ2), the difference between the TDC and $\min(\psi_1, \psi_2) = \lambda$ is less than the machine accuracy 10^{-16}.

The results in Table 4 confirm the expectations: the TC-Test (again based on $N = 1{,}000$ simulation runs and $B = 300$ bootstrap replications) has considerable power while TDC-Test 1 basically keeps the nominal size. As a conclusion, the developed testing procedures allow for empirically distinguishing between constant tail dependence coefficient and constant tail copula.

Next, we investigate a scenario (for sample size $n = 1{,}000$) where the simulated copula is constant at the center of the distribution throughout the sample period but exhibiting a significant structural break in the tail. For that purpose, we consider one break at $\bar{s} = 0.5$, and we simulate from the Clayton copula with $\lambda = 0.25$ before the break, and from the copula described in (C3), with $a = 0.1$, $p \in \{0, 0.25, 0.5, 0.75, 1\}$ and the Clayton copula with $\lambda = 0.25$, after the break. The results for TDC-Test 1 can be found in the right part of Table 5. As expected, the significant break in the tail is well detected by our methods.

Since our methods focus on the tail dependence, they should have, at least in this particular setting, more power than related tests for constancy of the whole copula. This is confirmed by the results in the left part of Table 5, which show the s.r.p.s for an L^2-type version of the tests for constancy of the copula proposed

scenario	$\Lambda(1,1)$	$n=1{,}000$ s.r.p.	avg(k^*)	std(k^*)	$n=3{,}000$ s.r.p.	avg(k^*)	std(k^*)
TDC-Test 1							
	0.20	0.048	45	21	0.053	95	46
$H_0^\lambda \cap H_1^\Lambda$	0.40	0.049	63	26	0.048	120	53
	0.60	0.053	89	35	0.050	168	72
TC-Test							
	0.20	0.183	45	20	0.566	92	44
$H_0^\lambda \cap H_1^\Lambda$	0.40	0.219	60	25	0.666	122	54
	0.60	0.160	91	34	0.509	169	72

Table 4: Simulated rejection probabilities of TDC-Test 1 and the TC-Test in the serial independence setting: The parameters are chosen such that the TDC remains constant over time while the tail copula does not.

scenario	p	**Copula-Test** s.r.p.	**TDC-Test 1** s.r.p.	avg(k^*)	std(k^*)
H_0^Λ	0.00	0.046	0.040	52	23
	0.25	0.060	0.128	67	32
H_1^λ, $\bar{s}=0.5$	0.50	0.070	0.340	84	43
	0.75	0.126	0.481	102	57
	1.00	0.230	0.542	121	73

Table 5: Simulated rejection probabilities of a test for constancy of the entire copula and TDC-Test 1 in the serial independence setting ($n = 1{,}000$) in which there is a structural break in the tail but not in the center of the distribution.

in Remillard (2010); Bücher and Ruppert (2013). More precisely, recalling that $\hat{C}_n^\circ(s,u,v) = \frac{1}{n}\sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}(\hat{U}_i \le u, \hat{V}_i \le v)$, the results are based on the test statistic

$$\mathcal{R}_n := n \int_{[0,1]^3} \left\{ \hat{C}_n^\circ(s,u,v) - s\hat{C}_n^\circ(1,u,v) \right\}^2 d(s,u,v),$$

along with the bootstrap approximation $\mathcal{R}_{n,\xi^{(b)}} := \int_{[0,1]^3} \mathbb{G}_{n,\xi^{(b)}}(s,u,v)^2 \, d(s,u,v)$ with $\mathbb{G}_{n,\xi^{(b)}}$ as defined in (10) based on the choice $k = n$. In practice, we approximate the integral through a sum over a finite grid. We use $N = 500$ repetitions and $B = 300$ bootstrap replications with multipliers that are uniformly distributed on the set $\{-1, 1\}$. It is clearly visible that, as expected, the power of the copula constancy test is lower than that of TDC-Test 1.

Finally, we investigate the detection of breaks in the tail dependence coefficient under the potential presence of breaks in the marginal laws. We restrict ourselves to a comparison of the TDC$^{\mathrm{MB}}$-Test to TDC-Test 1 in a serially independent case. Table 6 shows the s.r.p.s for the TDC$^{\mathrm{MB}}$-Test with level of significance $\alpha = 5\%$, based on samples from copula (C1) with or without a break in the TDC at $\bar{s} = 0.5$ and with marginal laws being either uniform on $[0,1]$ for the entire sample or being uniform on $[0,1]$ before $t_F = 0.25$ and $t_G = 0.5$ and uniform on $[5,6]$ after t_F and t_G, respectively. The marginal breaks are estimated by (13). Critical values are obtained by simulating 500 times from the respective limiting distribution, where

scenario	$\Lambda(1,1)$	$n = 1{,}000$ s.r.p.	avg(k^*)	std(k^*)	$n = 3{,}000$ s.r.p.	avg(k^*)	std(k^*)
TDC$^{\text{MB}}$-Test, constant mean							
H_0^Λ	0.25	0.061	52	24	0.063	98	49
	0.50	0.070	71	29	0.073	133	58
	0.75	0.072	120	44	0.075	231	95
H_1^λ, $\bar{s} = 0.5$	0.25 to 0.50	0.262	61	26	0.500	113	53
	0.25 to 0.75	0.807	73	30	0.963	138	62
	0.50 to 0.75	0.420	98	34	0.693	168	72
TDC$^{\text{MB}}$-Test, mean change							
H_0^Λ	0.25	0.061	53	23	0.065	99	50
	0.50	0.062	71	28	0.066	132	58
	0.75	0.053	120	43	0.054	231	94
H_1^λ, $\bar{s} = 0.5$	0.25 to 0.50	0.288	61	26	0.512	112	53
	0.25 to 0.75	0.847	73	31	0.968	136	61
	0.50 to 0.75	0.449	90	35	0.703	167	70

Table 6: Simulated rejection probabilities for the TDC$^{\text{MB}}$-Test in a serially independent setting with and without a mean change in the margins.

the Brownian bridge is simulated based on a grid of length 1/500. When comparing the simulations results in the case of a constant mean to the ones from TDC-Test 1 (see the first block of Table 1 and Table 2, respectively), the TDC$^{\text{MB}}$-Test has better power properties. On the other hand, it seems to be quite liberal compared to the slightly conservative TDC-Test 1. Moreover, the computational costs are substantially increased compared to TDC-Test 1: first, marginal breaks have to be estimated, and second, as the limiting distribution is not pivotal, additional estimation of the partial derivatives of Λ and simulations of a Brownian bridge are necessary. Finally, note that applying TDC-Test 1 to observations underlying a mean change in the marginal laws seems to be useless as under both H_0^Λ and H_1^λ all s.r.p.s are very close to 1.

5. Empirical applications

5.1. Energy sector In this section, we reinvestigate the bivariate dataset from Jäschke (2014) consisting of $n = 1{,}001$ daily closing quotes of WTI Cushing Crude Oil Spot and the Bloomberg European Dated Brent from October 2, 2006, to October 1, 2010, collected from Bloomberg's Financial Information Services. The analysis of the extremal dependence between the log-returns of the two time series in Jäschke (2014) is based on the implicit assumption that the tail dependence structure, more precisely their lower tail copula, remains constant over time. We are going to verify this assumption by applying the tests developed in the previous sections.

As pointed out in Jäschke (2014), the assumption of a serially independent sample is unrealistic. To account for autocorrelation and volatility clustering, it is shown that an ARMA(0,0)-EGARCH(2,3)-model including an explanatory variable (U.S. crude oil inventory) and the skewed generalized error distribution adequately describes the data generating process for the log-returns of the WTI time series. Regarding the daily Brent spot log-returns, an ARMA(1,1)-EGARCH(2,3)-model including U.S. crude oil inventory as an explanatory variable and the skewed generalized error distribution provides an adequate fit. In particular, there is no clear evidence against the assumption of marginal stationarity in both time series.

We calculate standardized residuals on the basis of the preceding time series models. A first view on the lower tail dependence between these residuals can be gained from the diagnostic plot in Figure 1. For various values of k such that k/n lies in the set $\{0.05, 0.06, \ldots, 0.15\}$, we depict the points in time where the pseudo-observations in both coordinates fall simultaneously below the value k/n. Note that these are exactly the joint extremal events inside the indicators in the definition of the empirical tail dependence coefficient. As the points are quite equally spaced in time, the picture suggests that the tail dependence remains rather constant.

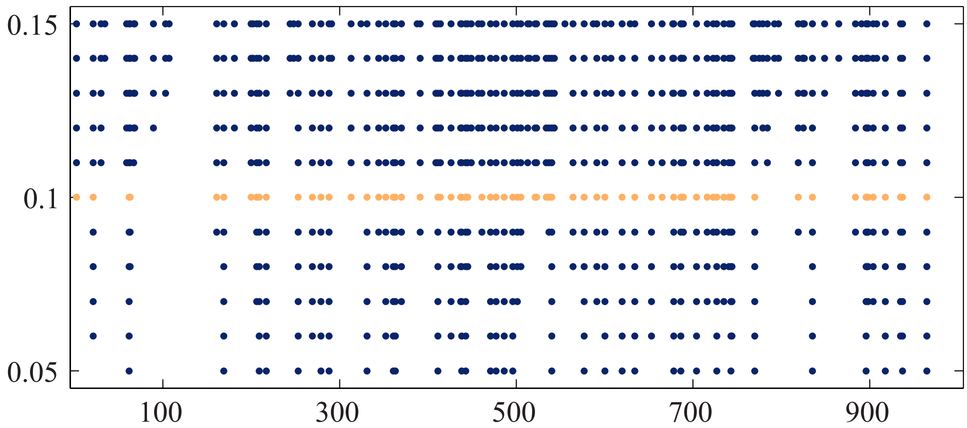

Figure 1: (WTI and Brent time series) Points in time where pseudo-observations in both coordinates fall simultaneously below the value k/n, for $k/n \in \{0.05, 0.06, \ldots, 0.15\}$. The yellow row corresponds to the plateau ratio $k^*/n = 104/1001 \approx 10\%$.

More formally, we proceed by checking the hypothesis H_0^λ of constancy of the tail dependence coefficients by an application of TDC-Test 1. First, in order to obtain a reasonable choice for the parameter k, we use the plateau algorithm from Section 3.5 with bandwidth $b = \lfloor 0.005n \rfloor = 5$. This yields a value of $k^* = 104$ (which is also depicted in yellow in Figure 1) and a plateau of length $m = 31$. Following Frahm et al. (2005), the average of the 31 empirical lower tail dependence coefficients on this plateau, given by $\hat{\lambda} = 0.732$, provides a good estimate for λ. Figure 7 in the supplementary material shows the corresponding standardized sequential empirical tail copula process $ns \mapsto \hat{\lambda}^{-1/2}\mathbb{G}_n(s, 1, 1)$ for $k^* = 104$. The graph seems to be indistinguishable from a simulated path of a one-dimensional standard Brownian bridge which indicates that the null hypothesis cannot be rejected. In Figure 2 we depict both the value of the Cramér-von Mises type test statistic $\mathcal{S}_n$ defined in (8) (yellow) as well as the corresponding p-values (blue) as a function of k. The dashed vertical line shows the outcomes for the plateau optimal $k^* = 104$, in which case we obtain $\mathcal{S}_n = 0.285$ with a resulting p-value of 0.15. Consequently, the null hypothesis H_0^λ cannot be rejected at a 5% level of significance. Moreover, Figure 2 shows that this conclusion is robust with respect to different choices of k. Results for TDC-Test 2 are very similar and are not depicted for the sake of brevity.

Finally, the assumption of a constant lower tail copula is verified by testing for the hypothesis H_0^Λ. We apply the TC-Test from Section 3.2 with $B = 2{,}000$ bootstrap replications using the plateau optimal $k^* = 104$. We obtain $\mathcal{T}_n = 0.069$

with a resulting p-value of 0.29. Again, the null hypothesis cannot be rejected at a 5% level of significance. Similar as for the tests for H_0^λ, this conclusion is robust with respect to different choices of k.

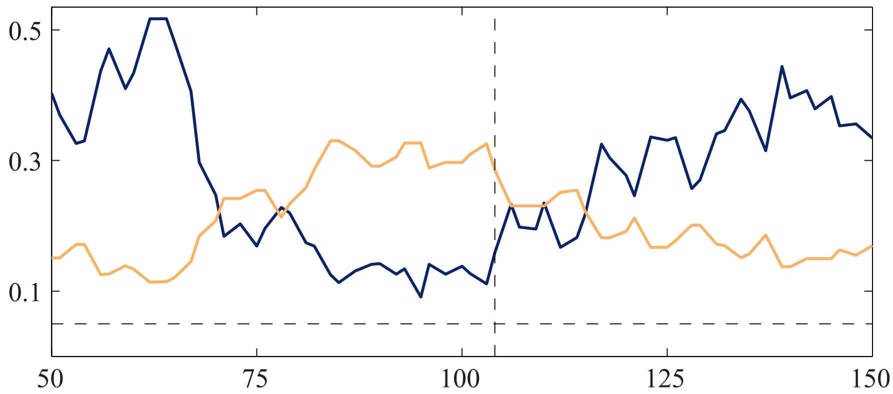

Figure 2: (WTI and Brent time series) Test statistics $\mathcal{S}_n$ (yellow) and corresponding p-values (blue) for different k. The horizontal line indicates the 5% level of significance, the vertical one the plateau $k^* = 104$.

5.2. Financial markets As an empirical application from the finance sector, we consider the Dow Jones Industrial Average and the Nasdaq Composite time series around Black Monday on 19th of October 1987. This dataset covers $n = 1{,}768$ log-returns from daily closing quotes between January 4, 1984, and December 31, 1990, collected from Datastream. Related studies in Wied et al. (2014) and Dehling et al. (2013) try to examine whether Black Monday constitutes a break in the dependence structure between the two time series. The outcomes of their studies do not provide a clear picture, as the answer depends on the applied test statistic. While the test for a constant Pearson correlation rejects the null hypothesis of constant correlation, the more robust (rank-based) tests for constant Spearman's rho and Kendall's tau yield no evidence for breaks. In these papers, the contrasting result is explained by the fact that the (unfiltered) time series contain several heavy outliers around Black Monday which seriously affect the Pearson-, but not the rank-based tests for Spearman's rho and Kendall's tau.

For our analysis, we begin by an investigation of the univariate time series. Applying the model selection and verification criteria from Jäschke (2014), we find that an ARMA(0,0)-GARCH(1,1)-model with t-distribution for the Dow Jones log-returns and an ARMA(1,0)-GARCH(1,1)-model with skewed t-distribution for the Nasdaq equivalent provide the best fits among a number of common stationary time series models. Details on the parameter estimation are given in Table 7 in the supplementary material. Note that more suitable models might be found by considering piecewise stationary models and by subsequently applying the tests from Section 3.7 where necessary. For our illustrative purposes, we restrict ourselves to the former models and to the tests from Sections 3.2 and 3.4 in the following.

Along the lines of Dehling et al. (2013) we first seek to answer the question whether Black Monday constitutes a break in the tail dependence between the two time series. A positive answer would indicate that the market conditions have substantially changed after this date. For the ease of a clear exposition, we restrict ourselves to an investigation of the lower tail dependence coefficient. A first visual description of the joint tail behavior similar to the one in Figure 1 can be found in Figure 3 which, however, does not provide a clear picture: even though there seems to be a tendency of stronger tail dependence for later dates in the time series, it is unclear whether this is due to a break on Black Monday (second dashed vertical line).

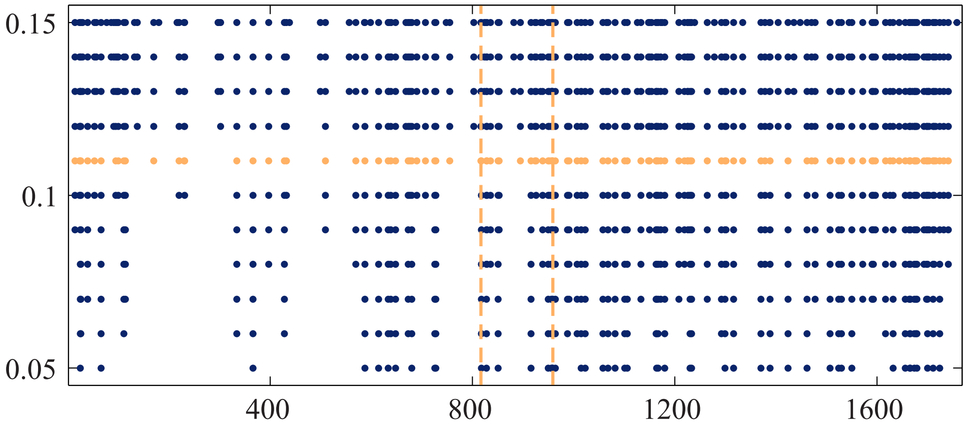

Figure 3: (Dow Jones and Nasdaq time series) Points in time where pseudo-observations in both coordinates fall simultaneously below the value k/n, for $k/n \in \{0.05, 0.06, \ldots, 0.15\}$. The yellow row corresponds to the plateau ratio $k^*/n \approx 11\%$. The first yellow vertical line reflects the argmax-estimator $\lfloor n\hat{s}^\lambda \rfloor = 817$, the second equivalent indicates Black Monday $\lfloor n\bar{s}_{\mathrm{B}M} \rfloor = 959$.

In the following, we examine this formally by applying the tests from Section 3, in particular the test from Section 3.4 for a specific break-point. First, a careful inspection of the plot $k \mapsto \mathrm{TDC}(k)$ and the statistics defining the plateau algorithm (which are not depicted for the sake of brevity) suggests that $k^* = 191$ is a reasonable choice for the parameter k, with a corresponding length of the plateau of $m = 41$. The average of the empirical lower tail dependence coefficients over the corresponding values $k \in \{171, \ldots, 211\}$ is given by $\hat{\lambda} = 0.620$.

Now, we apply the test from Section 3.4 for a specific break-point at $\lfloor n\bar{s}_{\mathrm{B}M} \rfloor = 959$, the date of Black Monday. The results are depicted in Figure 8 in the supplementary material, where we plot the p-values of the test against the parameter k. For $k^* = 191$, the resulting p-value of 0.082 does not allow for a clear rejection of the null hypothesis. In contrast to this, slightly lower values of k yield a rejection at the 5% level of significance, whence, as a summary, there seems to be some light, but disputable evidence against H_0. However, the rejection of the null hypothesis might be due to different reasons than a break precisely on Black Monday. To conclude upon the latter, one would have to accept the hypothesis of constancy of the lower tail dependence coefficient in the subsamples before and after Black Monday. Therefore, we perform the corresponding TDC-Test 1 in the subsamples, whose results are depicted in Figures 9 and 10 in the supplementary material in a similar manner as before; in particular, they are based on new (plateau-based) choices of k for the reduced samples. We can accept constancy after Black Monday, but have to reject it for the subsample before Black Monday. A summary of the results can also be found in the first two columns of Table 8 in the supplement.

In principal, one could now proceed by a refined analysis of the subsample before Black Monday in order to identify potential additional break-points. Motivated by the diagnostic plot in Figure 3, we prefer an application of TDC-Test 1 to the whole sample since this might reveal that a model with at most one break-point is also appropriate. In other words, we dismiss the initial guess of a break precisely on Black Monday and rather split the sample at an estimated break-point, hoping that the latter yields a simpler model with at most one break-point.

We do not depict the results of the corresponding test, since it clearly rejects the null hypothesis H_0^λ at the 1% level of significance for almost all choices of k. A short summary can be found in the last column of Table 8 in the supplementary material. More enlightening conclusions can be drawn from the plot of the the function $ns \mapsto |\hat{\lambda}^{-1/2}\mathbb{G}_n(s,1,1)|$ in Figure 4, for $k^* = 191$. The dashed vertical lines denote Black Monday $\lfloor n\bar{s}_{BM} \rfloor = 959$ (second line) and the value $\lfloor n\hat{s}^\lambda \rfloor = 817$ where the graph attains its maximum. The latter corresponds to the 27th of March 1987 and appears to be the argmax for most choices of k in a neighborhood of $k^* = 191$. We split the sample at this estimated break-point and conduct a refined analysis in the respective subsamples. The procedure is similar to what we have done before, whence we restrict ourselves to a brief summary of the results: in both subsamples, we cannot reject the null hypothesis for all reasonable choices of k, including the values obtained from the plateau algorithm, with p-values lying between 0.2 and 0.5. Similar to the values in Table 8 we find $\hat{\lambda} = 0.430$ for the first subsample ($k^* = 43$) and $\hat{\lambda} = 0.656$ for the second one ($k^* = 57$), respectively.

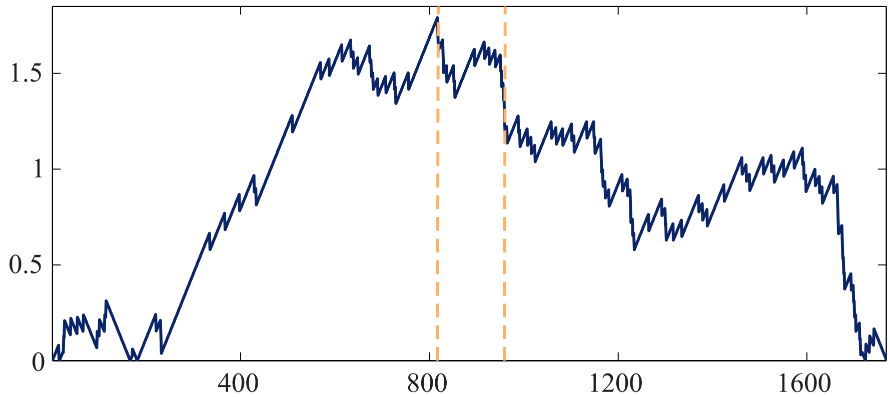

Figure 4: (Dow Jones and Nasdaq time series) Absolute standardized sequential empirical tail copula process $|\hat{\lambda}^{-1/2}\mathbb{G}_n(s,1,1)|$ for $k^* = 191$ with respect to ns, $s \in [0,1]$. The first yellow vertical line indicates the argmax estimator $\lfloor n\hat{s}^\lambda \rfloor = 817$, the second one shows Black Monday $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$.

We conclude this application with a short summary of the main findings:

(i) The test for a break on Black Monday does not yield entirely unambiguous results; in particular, we have to reject the null hypothesis of constant tail dependence in the subsample before Black Monday resulting in an overall model with more than one break-point.

(ii) Testing against the existence of *some* unspecified break-point in the full sample clearly rejects the null, with an estimated break-point at $\lfloor n\hat{s}^\lambda \rfloor = 817$. Since we cannot reject the null hypothesis in the corresponding subsamples, an overall model with only one break-point can be accepted.

6. Conclusion and Outlook

We developed new tests for detecting structural breaks in the tail dependence of multivariate time series, derived their theoretical properties, investigated their finite-sample performance and applied them to energy and financial market data.

Our work hints at interesting directions for further research. First of all, we did not give a formal proof for the conjecture derived from the simulation study, that the test statistics based on estimated residuals show the same asymptotic behavior as the ones based on i.i.d. samples. To the best of our knowledge, this problem is also unsolved for the estimation techniques described in Section 2: under what conditions does (or does not) the additional estimation step of forming *marginally almost i.i.d.* residuals influence the asymptotic behavior of the nonparametric estimators for the tail dependence? Second, extensions to the case of serially dependent datasets (e.g., to mixing sequences) would allow to check for constant tail dependence of the raw data which might also be of interest for practitioners. In particular with a view on the necessary (block) bootstrap procedure this could be a quite challenging task.

Finally, a deeper investigation of the results in Section 3.7 would be a worthwhile topic of future research: under what conditions can one replace the (unknown) marginal break points in Proposition 3.13 by their empirical counterparts? How can one treat the case of an unknown number of breaks in the marginals, and how can one adapt the bootstrap methodology to these settings?

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A. Online-Supplement: "Nonparametric tests for constant tail dependence with an application to energy and finance"

This supplement is organized as follows: in Section A.1, we give the proofs of all results in the main text. An auxiliary result is proven in Section A.2. Two tables concerning the empirical application in Section 5.2 are presented in Section A.3. Finally, Section A.4 completes this appendix with a couple of figures concerning Sections 4 and 5.

A.1. Proof of the results in the main text For all proofs, by asymptotic equicontinuity, we may redefine $\hat{U}_i = F_n(X_i)$ and $\hat{V}_i = G_n(Y_i)$. Now, for any $s \in [0,1]$ and $(x,y) \in \mathbb{E}$, let

$$\tilde{\Lambda}_n^{\circ}(s,x,y) = \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}(U_i \leq kx/n, V_i \leq ky/n). \tag{17}$$

Under H_0^{Λ}, this is a sequential (oracle) estimator for $\Lambda^{\circ}(s,x,y) = s\Lambda(x,y)$. The asymptotic behavior of $\tilde{\Lambda}_n^{\circ}$ can be derived under the following general condition, which allows for rather general changes of the tail copula Λ_i (see also Section 3.3).

Assumption A.1. *There exists some function $g : [0,1] \times \mathbb{E} \to \mathbb{R}$ such that $\Lambda_i(\cdot,\cdot) = g(i/n,\cdot,\cdot)$ and such that, for any $m \in \mathbb{N}$,*

$$\sup_{(s,x,y)\in S_m} \left| \frac{1}{n} \sum_{i=1}^{\lfloor ns \rfloor} g(i/n,x,y) - G(s,x,y) \right| = o(1), \quad n \to \infty, \tag{18}$$

where $G(s,x,y) = \int_0^s g(z,x,y)\,dz$.

Note that, under H_0^{Λ}, Assumption A.1 is trivially met with $g(z,x,y) = \Lambda(x,y)$, $G(s,x,y) = \Lambda^{\circ}(s,x,y) = s\Lambda(x,y)$ and with the expression on the left-hand side of (18) being of order $O(1/n)$. Now, consider the following sequential empirical process $\mathbb{B}_n$ defined as

$$\mathbb{B}_n(s,x,y) = \sqrt{k}\left\{\tilde{\Lambda}_n^{\circ}(s,x,y) - G(s,x,y)\right\},$$

and its corresponding centered version

$$\mathbb{B}_n'(s,x,y) = \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}(U_i \leq kx/n, V_i \leq ky/n) - C_i(kx/n, ky/n).$$

The proof of the following central lemma is given in Appendix A.2.

Lemma A.2. *Suppose that Assumptions 3.1, 3.2 (a), 3.2 (c) and A.1 hold. Then*

$$\mathbb{B}_n' \rightsquigarrow \mathbb{B}_g' \quad \textit{in } (\mathcal{B}_\infty([0,1]\times\mathbb{E}), d),$$

where $\mathbb{B}_g'$ denotes a tight, centered Gaussian process with covariance given by

$$\operatorname{Cov}\{\mathbb{B}_g'(s_1,x_1,y_1), \mathbb{B}_g'(s_2,x_2,y_2)\} = G(s_1 \wedge s_2, x_1 \wedge x_2, y_1 \wedge y_2).$$

If, additionally, Assumption 3.2 (b) is met and if the convergence in (18) *in Assumption 3.1 is of order* $o(k_n^{-1/2})$*, then* $d(\mathbb{B}'_n, \mathbb{B}_n) = o(1)$*.*

Note that, under H_0^Λ, the distribution of $\mathbb{B}'_g$ is equal to the distribution of $\mathbb{B}_\Lambda$ as defined in Proposition 3.3.

Proof of Proposition 3.3. Since the rank of X_i among $X_1, \dots, X_n$ is the same as the rank of U_i among $U_1, \dots, U_n$ (similar for the second coordinate) we may assume without loss of generality that (X_i, Y_i) is distributed according to C_i, i.e., $F(x) = G(x) = x$ for all $x \in [0, 1]$. Some thoughts reveal that

$$|\hat{\Lambda}_n^\circ(s, x, y) - \bar{\Lambda}_n^\circ(s, x, y)| \le 2/k,$$

uniformly in $(s, x, y) \in S_m$, where

$$\bar{\Lambda}_n^\circ(s, x, y) := \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\Big\{X_i \le F_n^-(kx/n), Y_i \le G_n^-(ky/n)\Big\}$$

and where F_n^- and G_n^- denote the generalized inverse functions of F_n and G_n, respectively. Note that $\bar{\Lambda}_n^\circ$ can be expressed in terms of $\tilde{\Lambda}_n^\circ$ as

$$\bar{\Lambda}_n^\circ(s, x, y) = \tilde{\Lambda}_n^\circ\left\{s, \frac{n}{k} F_n^-\left(\frac{kx}{n}\right), \frac{n}{k} G_n^-\left(\frac{ky}{n}\right)\right\}.$$

Now, we have $n/kF_n(kx/n) = \tilde{\Lambda}_n^\circ(1, x, \infty)$ and $n/kG_n(kx/n) = \tilde{\Lambda}_n^\circ(1, \infty, y)$, whence, by Hadamard-differentiability of the inverse mapping as stated in Bücher and Dette (2013),

$$\sup_{x \in [0,M]} \left|\frac{n}{k} F_n^-\left(\frac{kx}{n}\right) - x\right| = o_P(1), \quad \sup_{y \in [0,M]} \left|\frac{n}{k} G_n^-\left(\frac{ky}{n}\right) - y\right| = o_P(1) \tag{19}$$

for any $M > 0$ (this result can also be obtained by deducing weak convergence of $x \mapsto \mathbb{B}_n(1, x, \infty)$ as an element of the càdlàg space $D([0, M])$ with the Skorohod topology (from Lemma A.2), invoking a Skorohod construction and applying Vervaat's Lemma, see Vervaat (1972) or Lemma A.0.2 in de Haan and Ferreira (2006)). Therefore, by asymptotic equicontinuity of $\mathbb{B}_n$ from Lemma A.2, uniformly on S_m,

$$\begin{aligned}
\mathbb{G}_n(s, x, y) &= \sqrt{k}\left\{\bar{\Lambda}_n^\circ(s, x, y) - s\bar{\Lambda}_n^\circ(1, x, y)\right\} + O(k^{-1/2}) \\
&= \mathbb{B}_n\left\{s, \frac{n}{k} F_n^-\left(\frac{kx}{n}\right), \frac{n}{k} G_n^-\left(\frac{ky}{n}\right)\right\} \\
&\qquad - s\mathbb{B}_n\left\{1, \frac{n}{k} F_n^-\left(\frac{kx}{n}\right), \frac{n}{k} G_n^-\left(\frac{ky}{n}\right)\right\} + O(k^{-1/2}) \\
&= \mathbb{B}_n(s, x, y) - s\mathbb{B}_n(1, x, y) + o_P(1),
\end{aligned} \tag{20}$$

which converges weakly to $\mathbb{G}_\Lambda(s, x, y) = \mathbb{B}_\Lambda(s, x, y) - s\mathbb{B}_\Lambda(1, x, y)$ on $(S_m, \|\cdot\|_{S_m})$, for any $m \in \mathbb{N}$. The proposition is proven. □

Remark A.3. A crucial argument in the preceding proof is the decomposition (20) of $\mathbb{G}_n$ into a sum involving $\mathbb{B}_n$ from Lemma A.2. A similar decomposition is possible with $\mathbb{B}_n$ replaced by $\mathbb{B}'_n$ from Lemma A.2, and weak convergence of the latter holds without imposing Assumption 3.2 (b). Therefore, a relaxation of the assumptions for Proposition 3.3 seems to be possible. Indeed, a sufficient condition that makes occurring bias terms in an alternative version of (20) negligible and allows to dispense with Assumption 3.2 (b) is given by

$$\sup_{(s,x,y)\in S_m} \frac{\sqrt{k}}{n} \left| \sum_{i=1}^{\lfloor ns \rfloor} \frac{n}{k} C_i(kx/n, ky/n) - s \sum_{i=1}^{n} \frac{n}{k} C_i(kx/n, ky/n) \right| = o(1),$$

as $n \to \infty$. In case $C_i \equiv C$ is constant over time, this condition reduces to $\sqrt{k}/n = o(1)$, which is satisfied anyway since $k = o(n)$.

Proof of Corollary 3.4. It follows from Proposition 3.3 that

$$s \mapsto \left\{ \hat{\Lambda}_n^\circ(1,1,1) \right\}^{-1/2} \mathbb{G}_n(s,1,1)$$

converges to a standard Brownian bridge. Therefore, both assertions are simple consequences of the continuous mapping theorem. □

Proof of Proposition 3.6. Let us first fix a $b \in \{1, \ldots, B\}$ and show that $\mathbb{G}_{n,\xi^{(b)}}$ weakly converges to $\mathbb{G}_{\Lambda^{(b)}}$. For the sake of a clear notation, we omit the index b for the proof of this result. In light of the continuous mapping theorem, it is sufficient to prove that $\mathbb{B}_{n,\xi}$ weakly converges to $\mathbb{B}_\Lambda$. As in the proof of Proposition 3.3, we may assume that the marginal distributions are standard uniform. Let us suppose that we have proven $\tilde{\mathbb{B}}_{n,\xi} \rightsquigarrow \mathbb{B}_\Lambda$, where

$$\tilde{\mathbb{B}}_{n,\xi}(s,x,y) := \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i \left\{ \mathbb{1}\left(U_i \le kx/n, V_i \le ky/n\right) - \tilde{C}_n(kx/n, ky/n) \right\}$$

and where $\tilde{C}_n(u,v) := n^{-1} \sum_{i=1}^n \mathbb{1}(U_i \le u, V_i \le v)$. Then, by a similar reasoning as in the proof of Proposition 3.3,

$$\mathbb{B}_{n,\xi}(s,x,y) = \tilde{\mathbb{B}}_{n,\xi}\left\{ s, \frac{n}{k} F_n^-\left(\frac{kx}{n}\right), \frac{n}{k} G_n^-\left(\frac{ky}{n}\right) \right\} + O\left(k^{-1/2} \max_{i=1}^n |\xi_i| \right). \quad (21)$$

By (19) and asymptotic equicontinuity of $\tilde{\mathbb{B}}_{n,\xi}$, the first expression on the right-hand side weakly converges to $\mathbb{B}_\Lambda$ in $\ell^\infty(S_m)$, for any fixed S_m. In light of the fact that ξ_1 has finite moments of any order we have $\mathbb{P}(|\xi_1| > x) = O(x^{-q})$ for any $q \in \mathbb{N}$. Therefore, the estimation

$$\mathbb{P}(k^{-1/2} \max_{i=1}^n |\xi_i| > \varepsilon) \le n\mathbb{P}(|\xi_1| > \varepsilon\sqrt{k}/2) = nO(k^{-q/2})$$

shows that the O-term in (21) converges to 0 in probability, by choosing q sufficiently large. This proves that $\mathbb{G}_{n,\xi^{(b)}}$ weakly converges to $\mathbb{G}_{\Lambda^{(b)}}$.

It remains to be shown that $\tilde{\mathbb{B}}_{n,\xi} \rightsquigarrow \mathbb{B}_\Lambda$ in $\ell^\infty(S_m)$, for any fixed S_m. We have

$$\tilde{\mathbb{B}}_{n,\xi}(s,x,y) = A_{n1}(s,x,y) + A_{n2}(s,x,y) + A_{n3}(s,x,y),$$

where

$$A_{n1} := \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i \left\{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - C_i(kx/n, ky/n) \right\},$$

$$A_{n2} := \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i \left\{ C_i(kx/n, ky/n) - \Lambda(kx/n, ky/n) \right\},$$

$$A_{n3} := \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i \left\{ \Lambda(kx/n, ky/n) - \tilde{C}_n(kx/n, ky/n) \right\}.$$

The fact that $\tilde{C}_n(kx/n, ky/n) - \Lambda(kx/n, ky/n) = \sqrt{k}/n \times \mathbb{B}_n(s, x, y) = O_P(\sqrt{k}/n)$ from Lemma A.2 together with Donsker's invarance principle implies that the last term A_{n3} is of order $O_P(1/\sqrt{n}) = o_P(1)$, uniformly on each S_m. Furthermore, $C_i(kx/n, ky/n) - \Lambda(kx/n, ky/n) = k/n \times O(S(n/k))$ by Assumption 3.1, uniformly in i and uniformly on T_m, whence

$$\sup_{(s,x,y) \in S_m} |A_{n2}(s, x, y)| \le \frac{1}{n} \sum_{i=1}^{n} |\xi_i| \times \sqrt{k} O(S(n/k)).$$

The right-hand side is $o_P(1)$ by Assumption 3.2 (b). Hence, it remains to consider the leading term A_{n1}. Its conditional weak convergence follows from Theorem 11.19 in Kosorok (2008) and the proof of Lemma A.2 below. Further note that conditional weak convergence as considered by the last named author implies unconditional weak convergence.

Now, let us give the proof of the proposition. On each S_m, the sequence $(\mathbb{G}_n, \mathbb{G}_{n,\xi^{(1)}}, \dots, \mathbb{G}_{n,\xi^{(B)}})$ is jointly asymptotically tight by Lemma 1.4.3 in Van der Vaart and Wellner (1996). Hence, it remains to consider weak convergence of the finite-dimensional distributions. It suffices to consider the finite-dimensional distributions of $(\mathbb{B}_n, \mathbb{B}_{n,\xi^{(1)}}, \dots, \mathbb{B}_{n,\xi^{(B)}})$. By a similar argumentation as above in the case of a fixed $b \in \{1, \dots, B\}$, we may replace each coordinate $\mathbb{B}_{n,\xi^{(b)}}$ by

$$\frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i^{(b)} \left\{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - C_i(kx/n, ky/n) \right\}.$$

Then, the coordinates are uncorrelated and row-wise independent, whence the finite-dimensional distributions weakly converge to those of $(\mathbb{B}_\Lambda, \mathbb{B}_\Lambda^{(1)}, \dots, \mathbb{B}_\Lambda^{(B)})$ by the central limit theorem for row-wise independent triangular arrays. □

Proof of Corollary 3.7. For TDC-Test 1, this is a direct consequence of Corollary 3.4 (i). The proofs of TDC-Test 2 and TC-Test being essentially the same, we restrict ourselves to the proof of TDC-Test 2. For monotonicity reasons it suffices to consider $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Let K denote the c.d.f. of $\mathcal{S}$ and define

$$K_{n,B}(x) = B^{-1} \sum_{b=1}^{B} \mathbb{1}(\mathcal{S}_{n,\xi^{(b)}} \le x), \qquad K_B(x) = B^{-1} \sum_{b=1}^{B} \mathbb{1}(\mathcal{S}^{(b)} \le x),$$

where $\mathcal{S}^{(1)}, \dots, \mathcal{S}^{(B)}$ denote independent copies of $\mathcal{S}$. Then we can write $\mathbb{P}(\mathcal{S}_n \ge$

$\hat{q}_{\mathcal{S}_n,1-\alpha}) = \mathbb{P}\{K_{n,B}(\mathcal{S}_n) \geq 1-\alpha\}$. Let us first show that, for any $B \in \mathbb{N}$ fixed, we have

$$\lim_{n\to\infty} \mathbb{P}\{K_{n,B}(\mathcal{S}_n) \geq 1-\alpha\} = \mathbb{P}\{K_B(S) \geq 1-\alpha\}. \tag{22}$$

For that purpose, let $\varepsilon > 0$ be given. Define a map $\Psi : \mathbb{R}^{B+1} \to \mathbb{R}$ by $\Psi(t_0, \dots, t_B) = B^{-1}\sum_{b=1}^{B} \mathbb{1}(t_b \leq t_0)$ and note that Ψ is continuous at any point $(t_0, \dots, t_B)$ with pairwise different coordinates (i.e., $t_i \neq t_j$ for $i \neq j$). Then, observing that $(\mathcal{S}_n, \mathcal{S}_{n,\xi^{(1)}}, \dots, \mathcal{S}_{n,\xi^{(B)}}) \rightsquigarrow (\mathcal{S}, \mathcal{S}^{(1)}, \dots, \mathcal{S}^{(B)})$ with the limit having pairwise different coordinates, almost surely, the continuous mapping theorem implies that $K_{n,B}(\mathcal{S}_n) \rightsquigarrow K_B(\mathcal{S})$, for $n \to \infty$. The Portmanteau-Theorem implies that there exists some $n_0 = n_0(\varepsilon, B)$ such that

$$|\mathbb{P}\{K_{n,B}(\mathcal{S}_n) \geq 1-\alpha\} - \mathbb{P}\{K_B(\mathcal{S}) \geq 1-\alpha\}| < \varepsilon$$

(note that $\mathbb{P}(K_B(\mathcal{S}) = 1-\alpha) = 0$ since $\alpha \in \mathbb{R} \setminus \mathbb{Q}$), which proves (22).

It remains to be shown that

$$\lim_{B\to\infty} \mathbb{P}\{K_B(\mathcal{S}) \geq 1-\alpha\} = \alpha. \tag{23}$$

By the Glivenko-Cantelli Theorem, we may choose $B_0 = B_0(\varepsilon) \in \mathbb{N}$ such that

$$\mathbb{P}\left\{\sup_{x\in\mathbb{R}} |K_B(x) - K(x)| > \varepsilon\right\} \leq \varepsilon.$$

for all $B \geq B_0$. For all such B,

$$\mathbb{P}\{K_B(\mathcal{S}) \geq 1-\alpha\} \leq \mathbb{P}\{K(\mathcal{S}) \geq 1-\alpha-\varepsilon) + \varepsilon = \alpha + 2\varepsilon,$$

and similarly,

$$\mathbb{P}\{K_B(\mathcal{S}) \geq 1-\alpha\} \geq \mathbb{P}\{K(\mathcal{S}) \geq 1-\alpha+\varepsilon) = \alpha - \varepsilon,$$

which implies that

$$|\mathbb{P}\{K_B(\mathcal{S}) \geq 1-\alpha\} - \alpha| \leq 2\varepsilon.$$

This proves (23) and hence the Corollary. □

Proof of Proposition 3.8. The result is a special case of Proposition 3.11 which is proven below. □

Proof of Corollary 3.9. For TDC-Test 1, this is a direct consequence of Proposition 3.8 (*i*). The proofs for TDC-Test 2 and TC-Test being essentially the same, we only consider the TC-Test.

Let us first show that $\mathcal{T}_{n,\xi}$ is stochastically bounded. This follows if we prove that $\sup_{(s,x,y)\in S_m} |\mathbb{B}_{n,\xi}(s,x,y)| = O_P(1)$, for $n \to \infty$. By a similar reasoning as in (21) and the subsequent paragraph, it suffices to show the same for $\tilde{\mathbb{B}}_{n,\xi}(s,x,y)$.

Since

$$\sup_{(s,x,y)\in S_m} |\tilde{\mathbb{B}}_{n,\xi}(s,x,y)| \\ = \max\left\{ \sup_{s\le\bar{s},(x,y)\in T_m} |\tilde{\mathbb{B}}_{n,\xi}(s,x,y)|, \sup_{s\ge\bar{s},(x,y)\in T_m} |\tilde{\mathbb{B}}_{n,\xi}(s,x,y)| \right\}, \quad (24)$$

we can verify the claim for each of the suprema in the maximum separately. Let us first treat the notationally simpler first supremum. We can decompose $\tilde{\mathbb{B}}_{n,\xi}(s,x,y) = \sum_{\ell=1}^4 A_{n\ell}(s,x,y)$, where

$$A_{n1} := \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i \left\{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - C_i(kx/n, ky/n) \right\},$$

$$A_{n2} := \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i \left\{ C_i(kx/n, ky/n) - \Lambda^{(1)}(kx/n, ky/n) \right\},$$

$$A_{n3} := \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i (1-\bar{s}) \left\{ \Lambda^{(1)}(kx/n, ky/n) - \Lambda^{(2)}(kx/n, ky/n) \right\},$$

$$A_{n4} := \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i \left\{ \bar{s}\Lambda^{(1)}(kx/n, ky/n) + (1-\bar{s})\Lambda^{(2)}(kx/n, ky/n) - \tilde{C}_n(kx/n, ky/n) \right\}.$$

Since $s \le \bar{s}$, the first term A_{n1} converges weakly by the same arguments as in the proof of Proposition 3.6. Also as in that proof, $A_{n2} = o_P(1)$. Negligibility of A_{n3} follows from Donsker's invariance principle and the fact that $\Lambda(kx/n, ky/n) \le (x \wedge y) \times k/n$ for any tail copula Λ. Hence, it remains to consider A_{n4}. Again exploiting Donsker's invariance principle, it is certainly sufficient to show that $\Delta_n := \tilde{C}_n(kx/n, ky/n) - \bar{s}\Lambda^{(1)}(kx/n, ky/n) - (1-\bar{s})\Lambda^{(2)} = O_P(\sqrt{k}/n)$. This, however, follows from the fact that we can write

$$\frac{n}{\sqrt{k}}\Delta_n = \sqrt{k}\Bigg\{ \frac{1}{k} \sum_{i=1}^{\lfloor n\bar{s} \rfloor} \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - \bar{s}\Lambda^{(1)}(x,y) \\ + \frac{1}{k} \sum_{\lfloor n\bar{s} \rfloor+1}^{n} \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - (1-\bar{s})\Lambda^{(2)}(x,y) \Bigg\},$$

which is $O_P(1)$ by two suitable applications of Lemma A.2.

Regarding the second supremum on the left-hand side of (24), write

$$\tilde{\mathbb{B}}_{n,\xi}(s,x,y) = \tilde{\mathbb{B}}_{n,\xi}(\bar{s},x,y) \\ + k^{-1/2} \sum_{i=\lfloor n\bar{s} \rfloor+1}^{\lfloor ns \rfloor} \xi_i \left\{ \mathbb{1}(U_i \le kx/n, V_i \le ky/n) - \tilde{C}_n(kx/n, ky/n) \right\}. \quad (25)$$

The first term on the left-hand side has already been handled above, and the second one can be treated by a similar decomposition.

Now, fix $B \in \mathbb{N}$ and let $\varepsilon > 0$ be given. Then, since $\mathcal{T}_{n,\xi^{(b)}} = O_P(1)$ for each

$b = 1, \dots, B$, we may choose $K = K(\varepsilon, B) > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left(\max_{b=1}^{B} |\mathcal{T}_{n,\xi^{(b)}}| > K\right) \le \varepsilon.$$

Therefore, $\hat{q}_{\mathcal{T}_n,1-\alpha} > K$ with probability of at least ε, and since $\mathcal{T}_n \to \infty$ in probability, we get that

$$\liminf_{n\to\infty} \mathbb{P}(\mathcal{T}_n \ge \hat{q}_{\mathcal{T}_n,1-\alpha}) \ge \liminf_{n\to\infty} \{\mathbb{P}(\mathcal{T}_n \ge K) - \mathbb{P}(\hat{q}_{\mathcal{T}_n,1-\alpha} > K)\} \ge 1 - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the assertion is proven. □

Proof of Proposition 3.11. As in the proof of Proposition 3.3 we may assume without loss of generality that the marginals are standard uniform. We only prove (i), the proof of (ii) is completely analogous. By the continuous mapping theorem, it suffices to show that

$$\sup_{s\in[0,1]} \left| \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\left(\hat{U}_i \le k/n, \hat{V}_i \le k/n\right) - \int_0^s g(z,1,1)dz \right| = o_P(1).$$

As in the proof of Proposition 3.3, we may replace the indicators in the previous expression by $\mathbb{1}\{U_i \le F_n^-(k/n), V_i \le G_n^-(k/n)\}$. Now, we decompose

$$\frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} \mathbb{1}\Big\{U_i \le F_n^-(k/n), V_i \le G_n^-(k/n)\Big\} - \int_0^s g(z,1,1)dz = \sum_{\ell=1}^{4} A_\ell(s)$$

where

$$\begin{aligned}
A_1(s) &:= k^{-1/2}\mathbb{B}_n'\Big\{(n/k)F_n^-(k/n), (n/k)G_n^-(k/n)\Big\}, \\
A_2(s) &:= \frac{1}{k} \sum_{i=1}^{\lfloor ns \rfloor} C_i\Big\{F_n^-(k/n), G_n^-(k/n)\Big\} - C_i(k/n, k/n), \\
A_3(s) &:= \frac{1}{n} \sum_{i=1}^{\lfloor ns \rfloor} (n/k)C_i(k/n, k/n) - g(i/n, 1, 1), \\
A_4(s) &:= \frac{1}{n} \sum_{i=1}^{\lfloor ns \rfloor} g(i/n, 1, 1) - \int_0^s g(z,1,1)dz.
\end{aligned}$$

$A_1(s)$ converges to 0, uniformly in $s \in [0,1]$, by Lemma A.2. The second term is uniformly $o_P(1)$ by Lipschitz continuity of C_i and (19). By Assumption 3.2, the third term is of order $O(S(n/k)) = o(1)$. $A_4(s)$ goes to 0, uniformly in s, by the assumption in H_1^λ. □

Proof of Proposition 3.13. For $i = 1, \dots, \lfloor nt_F \rfloor$, the rank of X_i among $X_1, \dots, X_{\lfloor nt_F \rfloor}$ is the same as the rank of U_i among $U_1, \dots, U_{\lfloor nt_F \rfloor}$, and similar for the second subsample and for the second coordinate. Hence, we may assume without loss of generality that (X_i, Y_i) is distributed according to C_i, for all $i = 1, \dots, n$.

Moreover, by asymptotic equicontinuity, we may redefine $F_{(k+1):\ell}(x) := (\ell - k)^{-1} \sum_{j=k+1}^{\ell} \mathbb{1}(X_j \le x)$, and similar for the second coordinate. In the following, we suppose that $t_F \leq t_G$, the other case is treated similarly. We restrict ourselves to show weak convergence on $\ell^\infty([0,1]\times[\varepsilon,m]^2)$ for $0<\varepsilon<m<\infty$; the boundary cases can be treated similarly, following arguments in Bücher and Dette (2013) for x or y smaller than ε. Let n be large enough such that $t_F, t_G \in (1/n, 1-1/n)$. Define

$$\begin{aligned}\bar{\Lambda}_n^\circ(s,x,y;t_F,t_G) &= \frac{1}{k}\sum_{i=1}^{\lfloor n(s\wedge t_F)\rfloor} \mathbb{1}\Big\{X_i \leq F^-_{1:\lfloor nt_F\rfloor}(kx/n), Y_i \leq G^-_{1:\lfloor nt_G\rfloor}(ky/n)\Big\}\\ &+ \frac{1}{k}\sum_{i=\lfloor n(s\wedge t_F)\rfloor+1}^{\lfloor n(s\wedge t_G)\rfloor} \mathbb{1}\Big\{X_i \leq F^-_{\lfloor nt_F\rfloor+1:n}(kx/n), Y_i \leq G^-_{1:\lfloor nt_G\rfloor}(ky/n)\Big\}\\ &+ \frac{1}{k}\sum_{i=\lfloor n(s\wedge t_G)\rfloor+1}^{\lfloor ns\rfloor} \mathbb{1}\Big\{X_i \leq F^-_{\lfloor nt_F\rfloor+1:n}(kx/n), Y_i \leq G^-_{\lfloor nt_G\rfloor+1:n}(ky/n)\Big\}\end{aligned}$$

and note that

$$|\hat{\Lambda}_n^\circ(s,x,y) - \bar{\Lambda}_n^\circ(s,x,y;t_F,t_G)| = O(1/k),$$

uniformly in $(s,x,y)\in[0,1]\times[\varepsilon,m]^2$. Therefore, it suffices to prove weak convergence of

$$\sqrt{k}\left\{\bar{\Lambda}_n^\circ(s,x,y;t_F,t_G) - s\bar{\Lambda}_n^\circ(1,x,y;t_F,t_G)\right\}.$$

For $(t_1,t_2)\in[0,1]^2$ with $t_2 > t_1 + 1/n$ define

$$\alpha_n(t_1,t_2,x) = \frac{n}{k}F^-_{\lfloor nt_1\rfloor+1:\lfloor nt_2\rfloor}\left(\frac{kx}{n}\right), \qquad \beta_n(t_1,t_2,y) = \frac{n}{k}G^-_{\lfloor nt_1\rfloor+1:\lfloor nt_2\rfloor}\left(\frac{ky}{n}\right).$$

Recall the definition of $\tilde{\Lambda}_n^\circ$ in (17) and note that

$$\begin{aligned}&\bar{\Lambda}_n^\circ(s,x,y;t_F,t_G) = \tilde{\Lambda}_n^\circ\left\{s\wedge t_F, \alpha_n(0,t_F,x), \beta_n(0,t_G,y)\right\}\\ &+ \tilde{\Lambda}_n^\circ\left\{s\wedge t_G, \alpha_n(t_F,1,x), \beta_n(0,t_G,y)\right\} - \tilde{\Lambda}_n^\circ\left\{s\wedge t_F, \alpha_n(t_F,1,x), \beta_n(0,t_G,y)\right\}\\ &+ \tilde{\Lambda}_n^\circ\left\{s, \alpha_n(t_F,1,x), \beta_n(t_G,1,y)\right\} - \tilde{\Lambda}_n^\circ\left\{s\wedge t_G, \alpha_n(t_F,1,x), \beta_n(t_G,1,y)\right\}.\end{aligned}$$

In particular, we can write $\sqrt{k}\left\{\bar{\Lambda}_n^\circ(s,x,y;t_F,t_G) - s\bar{\Lambda}_n^\circ(1,x,y;t_F,t_G)\right\}$ as

$$\begin{aligned}&\mathbb{B}_n\left\{s\wedge t_F, \alpha_n(0,t_F,x), \beta_n(0,t_G,y)\right\}\\ &\quad + \mathbb{B}_n\left\{s\wedge t_G, \alpha_n(t_F,1,x), \beta_n(0,t_G,y)\right\} - \mathbb{B}_n\left\{s\wedge t_F, \alpha_n(t_F,1,x), \beta_n(0,t_G,y)\right\}\\ &\quad + \mathbb{B}_n\left\{s, \alpha_n(t_F,1,x), \beta_n(t_G,1,y)\right\} - \mathbb{B}_n\left\{s\wedge t_G, \alpha_n(t_F,1,x), \beta_n(t_G,1,y)\right\}\\ &- s\Big[\mathbb{B}_n\left\{1\wedge t_F, \alpha_n(0,t_F,x), \beta_n(0,t_G,y)\right\}\\ &\quad + \mathbb{B}_n\left\{1\wedge t_G, \alpha_n(t_F,1,x), \beta_n(0,t_G,y)\right\} - \mathbb{B}_n\left\{1\wedge t_F, \alpha_n(t_F,1,x), \beta_n(0,t_G,y)\right\}\\ &\quad + \mathbb{B}_n\left\{1, \alpha_n(t_F,1,x), \beta_n(t_G,1,y)\right\} - \mathbb{B}_n\left\{1\wedge t_G, \alpha_n(t_F,1,x), \beta_n(t_G,1,y)\right\}\Big]\\ &+ R_n \end{aligned} \tag{26}$$

where the remainder R_n is given by

$$\sqrt{k}\Big\{(s\wedge t_F - st_F)\Big[\Lambda\{\alpha_n(0,t_F,x),\beta_n(0,t_G,y)\} - \Lambda\{\alpha_n(t_F,1,x),\beta_n(0,t_G,y)\}\Big]$$
$$+(s\wedge t_G - st_G)\Big[\Lambda\{\alpha_n(t_F,1,x),\beta_n(0,t_G,y)\} - \Lambda\{\alpha_n(t_F,1,x),\beta_n(t_G,1,y)\}\Big]\Big\}.$$

The functional delta method applied to the inverse mapping (see the proof of Lemma A.1 in Bücher and Dette, 2013) shows that

$$\sup_{x\in[0,M]}\Big|\sqrt{k}\{\alpha_n(0,t_F,x)-x\} + t_F^{-1}\mathbb{B}_n(t_F,x,\infty)\Big| = o_P(1),$$
$$\sup_{x\in[0,M]}\Big|\sqrt{k}\{\alpha_n(t_F,1,x)-x\} + (1-t_F)^{-1}\{\mathbb{B}_n(1,x,\infty)-\mathbb{B}_n(t_F,x,\infty)\}\Big| = o_P(1),$$
$$\sup_{y\in[0,M]}\Big|\sqrt{k}\{\beta_n(0,t_G,y)-y\} + t_G^{-1}\mathbb{B}_n(t_G,\infty,y)\Big| = o_P(1),$$
$$\sup_{y\in[0,M]}\Big|\sqrt{k}\{\beta_n(t_G,1,y)-y\} + (1-t_G)^{-1}\{\mathbb{B}_n(1,\infty,y)-\mathbb{B}_n(t_G,\infty,y)\}\Big| = o_P(1),$$

for any $M\in\mathbb{N}$. In particular,

$$\sup_{x\in[0,M]}|\alpha_n(0,t_F,x)-x| = o_P(1), \qquad \sup_{x\in[0,M]}|\alpha_n(t_F,1,x)-x| = o_P(1)$$
$$\sup_{y\in[0,M]}|\beta_n(0,t_G,y)-y| = o_P(1), \qquad \sup_{y\in[0,M]}|\beta_n(t_G,1,y)-y| = o_P(1),$$

which implies, by asymptotic equicontinuity of $\mathbb{B}_n$, that the first six lines of the decomposition (26) are equal to $\mathbb{B}_n(s,x,y) - s\mathbb{B}_n(1,x,y)$, up to a term of uniform order $o_P(1)$. Regarding R_n, a Taylor expansion of Λ based on Assumption 3.12 shows that

$$\sqrt{k}\Big[\Lambda\{\alpha_n(0,t_F,x),\beta_n(0,t_G,y)\} - \Lambda\{\alpha_n(t_F,1,x),\beta_n(0,t_G,y)\}\Big]$$
$$= -\frac{\dot\Lambda_x(x,y)}{t_F(1-t_F)}\Big[(1-t_F)\mathbb{B}_n(t_F,x,\infty) - t_F\{\mathbb{B}_n(1,x,\infty)-\mathbb{B}_n(t_F,x,\infty)\}\Big] + o_P(1)$$
$$= -\frac{\dot\Lambda_x(x,y)}{t_F(1-t_F)}\{\mathbb{B}_n(t_F,x,\infty) - t_F\mathbb{B}_n(1,x,\infty)\} + o_P(1).$$

A similar calculation yields

$$\sqrt{k}\Big[\Lambda\{\alpha_n(t_F,1,x),\beta_n(0,t_G,y)\} - \Lambda\{\alpha_n(t_F,1,x),\beta_n(t_G,1,y)\}\Big]$$
$$= -\frac{\dot\Lambda_y(x,y)}{t_G(1-t_G)}\{\mathbb{B}_n(t_G,\infty,y) - t_G\mathbb{B}_n(1,\infty,y)\} + o_P(1).$$

Assembling terms yields the assertion. □

A.2. Proofs of additional results

Proof of Lemma A.2. First, consider the assertion regarding $\mathbb{B}_n'$. It suffices to fix one set S_m and to show weak convergence in $\ell^\infty(S_m)$. The latter can be accomplished by a suitable application of Theorem 11.16 in Kosorok (2008), see also

Bücher and Dette (2013) for a similar proof for the i.i.d. and non-sequential case. Write $\mathbb{B}_n'(s,x,y) = \mathbb{B}_n'(s,x,y,\omega)$ as

$$\mathbb{B}_n'(s,x,y,\omega) = \sum_{i=1}^n f_{n,i}(s,x,y,\omega) - \mathbb{E}[f_{n,i}(s,x,y,\cdot)],$$

where $f_{n,i}(s,x,y,\omega) = k^{-1/2}\mathbb{1}(U_i(\omega) \le kx/n, V_i(\omega) \le ky/n)\mathbb{1}(i \le \lfloor ns \rfloor)$. Moreover define envelopes $F_{n,i}$ for $f_{n,i}$ as

$$F_{n,i}(\omega) = k^{-1/2}\mathbb{1}(U_i(\omega) \le km/n \ \text{ or } \ V_i(\omega) \le km/n)\mathbb{1}(i \le \lfloor ns \rfloor).$$

By Theorem 11.16 in Kosorok (2008), the assertion in Lemma A.2 regarding $\mathbb{B}_n'$ is proven if we show that

(i) *The $f_{n,i}$ are manageable with envelopes $F_{n,i}$.*

(ii) *The limit $H((s_1,x_1,y_1),(s_1,x_1,y_1)) = \lim_{n\to\infty} \mathbb{E}[\mathbb{B}_n'(s_1,x_1,y_1)\mathbb{B}_n'(s_2,x_2,y_2)]$ exists for every $(s_1,x_1,y_1),(s_2,x_2,y_2) \in S_m$.*

(iii) $\limsup_{n\to\infty} \sum_{i=1}^n \mathbb{E}F_{n,i}^2 < \infty$.

(iv) $\lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}F_{n,i}^2 \mathbb{1}\{F_{n,i} > \varepsilon\} = 0$ *for all* $\varepsilon > 0$.

(v) *The limit $\lim_{n\to\infty} \rho_n((s_1,x_1,y_1),(s_2,x_2,y_2)) = \rho((s_1,x_1,y_1),(s_2,x_2,y_2))$ exists for all $(s_1,x_1,y_1),(s_2,x_2,y_2) \in S_m$, where*

$$\rho_n((s_1,x_1,y_1),(s_2,x_2,y_2)) = \left(\sum_{i=1}^n \mathbb{E}\Big|f_{n,i}(s_1,x_1,y_1,\cdot) - f_{n,i}(s_2,x_2,y_2,\cdot)\Big|^2\right)^{1/2}.$$

Furthermore, for all sequences $(s_{1n},x_{1n},y_{1n})_{n\in\mathbb{N}},(s_{2n},x_{2n},y_{2n})_{n\in\mathbb{N}}$ in S_m the convergence $\rho_n((s_{1n},x_{1n},y_{1n}),(s_{2n},x_{2n},y_{2n})) \to 0$ holds, provided that we have $\rho((s_{1n},x_{1n},y_{1n}),(s_{2n},x_{2n},y_{2n})) \to 0$.

(vi) $\{f_{n,1}(s,x,y,\omega),\dots,f_{n,n}(s,x,y,\omega) : (s,x,y) \in S_m\}$ *is almost measurable Suslin.*

For the proof of (i) note that we can write $f_{n,i}$, when indexed by the extended domain $[0,1]\times([0,m]\cup\{\infty\})^2$ instead of S_m, as a product of three non-decreasing functions in s, x and y, respectively. Manageability with respect to the envelopes $F_{n,i}$ then follows from the discussion on Page 221 in Kosorok (2008) and two applications of Theorem 11.17 (iv) in that reference. Then, also the restriction to S_m is manageable with envelopes $F_{n,i}$.

In the following, we omit the argument ω. For the proof of (ii), we have $\mathbb{E}[\mathbb{B}_n'(s_1,x_1,y_1)\mathbb{B}_n'(s_2,x_2,y_2)] = A_{n1} + A_{n2}$ where

$$A_{n1} := \frac{1}{k_n}\sum_{i=1}^{\lfloor n(s_1\wedge s_2)\rfloor} C_i\left(k(x_1\wedge x_2)/n, k(y_1\wedge y_2)/n\right),$$
$$A_{n2} := \frac{1}{k_n}\sum_{i=1}^{\lfloor n(s_1\wedge s_2)\rfloor} C_i(kx_1/n, ky_1/n)C_i(kx_2/n, ky_2/n).$$

Exploiting that $C_i(u,v) \le u\wedge v$, the second summand A_{n2} is uniformly bounded by $km/n = o(1)$. For the first summand, we write

$$A_{n1} = \frac{1}{n}\sum_{i=1}^{\lfloor n(s_1\wedge s_2)\rfloor} \Lambda_i(x_1\wedge x_2, y_1\wedge y_2) + \frac{1}{n}\sum_{i=1}^{n(s_1\wedge s_2)} \frac{n}{k}C_i(k(x_1\wedge x_2)/n, k(y_1\wedge y_2)/n) - \Lambda_i(x_1\wedge x_2, y_1\wedge y_2).$$

The second sum is of order $O(S(n/k)) = o(1)$ by Assumption 3.1, and the first sum converges to $G(s_1\wedge s_2, x_1\wedge x_2, y_1\wedge y_2) < \infty$.

For the proof of (iii) note that $\mathbb{E}F_{n,i}^2 = 2m/n - C_i(km/n, km/n)/k$. Therefore,

$$\sum_{i=1}^{n}\mathbb{E}\left[F_{n,i}^2\right] = 2m - \frac{1}{n}\sum_{i=1}^{n}\frac{n}{k}C_i(km/n, km/n).$$

As in the proof of (ii), the second sum converges to $\int_0^1 g(z,m,m)\,dz$.

For the proof of (iv), note that $\mathbb{E}[F_{n,i}^2\mathbb{1}(F_{n,i} > \varepsilon)] \le \mathbb{P}(F_{n,i} > \varepsilon)$, and the right-hand side is equal to 0 for sufficiently large n.

For the proof of (v), note that

$$\rho_n((s_1,x_1,y_1),(s_2,x_2,y_2))^2 = \frac{1}{n}\sum_{i=1}^{\lfloor ns_1\rfloor}\frac{k}{n}C_i(kx_1/n, ky_1/n) - \frac{2}{n}\sum_{i=1}^{\lfloor n(s_1\wedge s_2)\rfloor}\frac{k}{n}C_i(k(x_1\wedge x_2)/n, k(y_1\wedge y_2)/n) + \frac{1}{n}\sum_{i=1}^{\lfloor ns_2\rfloor}\frac{k}{n}C_i(kx_2/n, ky_2/n).$$

Similar calculations as before show that this expression converges uniformly (on S_m) to

$$\rho((s_1,x_1,y_1),(s_2,x_2,y_2))^2 = G(s_1,x_1,y_1) - 2G(s_1,\wedge s_2, x_1\wedge x_2, y_1\wedge y_2) + G(s_2,x_2,y_2).$$

Finally, the assertion in (vi) follows from separability of $\mathbb{B}_n'$ and Lemma 11.15 in Kosorok (2008).

Now, consider the assertion regarding $\mathbb{B}_n$. We have

$$\begin{aligned}|\mathbb{B}_n'(s,x,y) - \mathbb{B}_n(s,x,y)| &= \sqrt{k_n}\left|\frac{1}{k_n}\sum_{i=1}^{\lfloor ns\rfloor}C_i(k_nx/n, k_ny/n) - G(s,x,y)\right| \\ &\le \sqrt{k_n}\frac{\lfloor ns\rfloor}{n}\max_{i=1}^{n}\left|\frac{n}{k_n}C_i(k_nx/n, k_ny/n) - g(i/n,x,y)\right| \\ &\quad + \sqrt{k_n}\left|\frac{1}{n}\sum_{i=1}^{\lfloor ns\rfloor}g(i/n,x,y) - G(s,x,y)\right|.\end{aligned}$$

Since we assume that the convergence in Assumption 3.1 is of order $o(k_n^{-1/2})$, we immediately obtain negligibility of the second term on the right-hand side. By (9), the first term on the right-hand side is of order $O(\sqrt{k_n}S(n/k_n))$, uniformly on each S_m. Hence, by Assumption 3.2 (b), this term converges to 0 as well. □

A.3. Additional tables This section contains two additional tables regarding the empirical application in Section 5.2.

parameter	Dow Jones log-returns		Nasdaq log-returns	
	estimate	std error	estimate	std error
mean equation				
μ	0.0006	0.0002	0.0005	0.0002
θ_1			0.2714	0.0234
variance equation				
ω	0.0000	0.0000	0.0000	0.0000
α_1	0.0349	0.0084	0.1407	0.0179
β_1	0.9373	0.0080	0.7914	0.0182
distribution				
ξ			0.8531	0.0294
ν	4.2016	0.4390	5.3001	0.4135

Table 7: Maximum likelihood estimates together with their corresponding standard errors for the Dow Jones ARMA(0,0)-GARCH(1,1)-model with t-distribution and the Nasdaq ARMA(1,0)-GARCH(1,1)-model including the skewed t-distribution. All estimates but the additive constant ω are significant at the 1% level.

parameter	before Black Monday	after Black Monday	full sample size
n	958	810	1,768
k^*	48	169	191
m	30	28	41
$\hat{\lambda}$	0.449	0.678	0.620
$\mathcal{S}_n$	0.546	0.211	1.064
p-value	0.028	0.244	0.003

Table 8: Summary of results for TDC-Test 1 applied to the subsample before Black Monday, to the subsample after Black Monday and to the full sample.

A.4. Additional figures This section contains six additional figures concerning Sections 4 and 5.

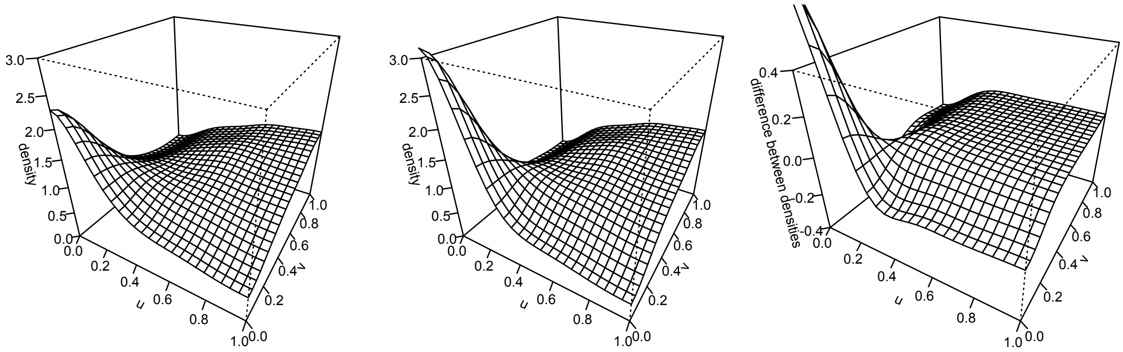

Figure 5: Left panel: (estimated) copula density from the Clayton copula with $\lambda = 0.25$. Middle: (estimated) copula density from the transformed Clayton copula described in (C3). Right: difference between the two densities.

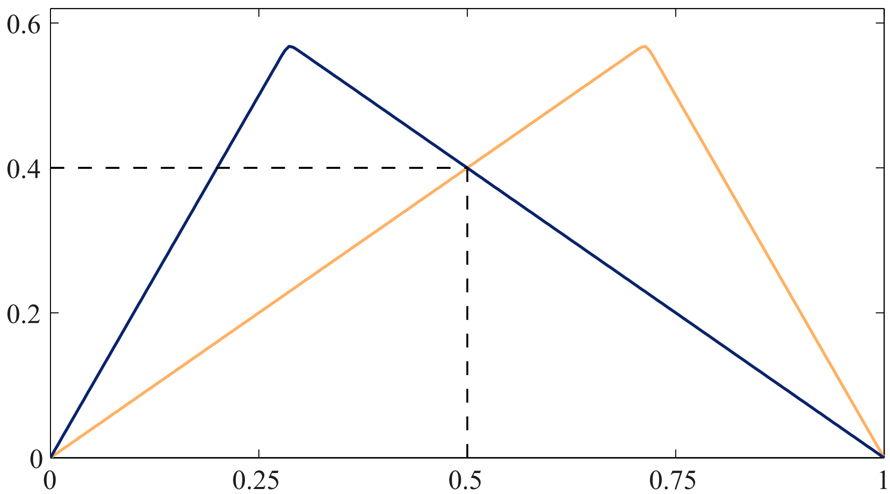

Figure 6: Negative asymmetric logistic model ($\Lambda 2$) for $\psi_1 = 0.4$, $\psi_2 = 1$, $\theta = 100$ (blue) and $\psi_1 = 1$, $\psi_2 = 0.4$, $\theta = 100$ (yellow) evaluated on the straight line $(2 - 2t, 2t)$, $t \in [0, 1]$. Both models exhibit the same tail dependence coefficient $\lambda = 0.4$.

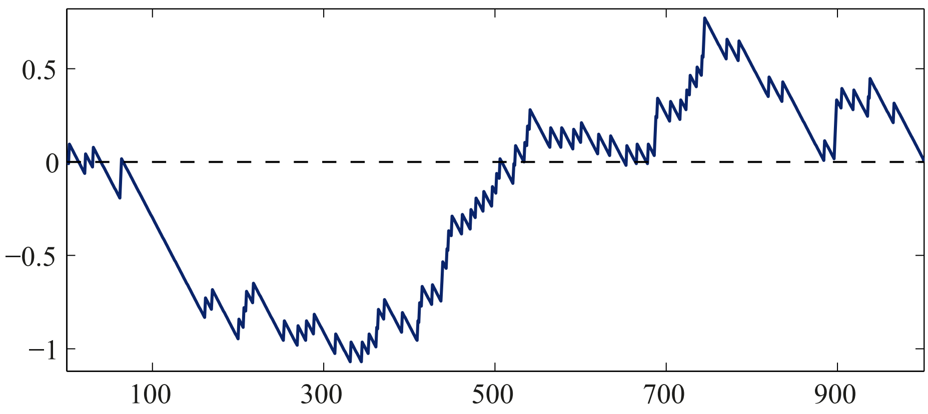

Figure 7: (WTI and Brent time series) Standardized sequential empirical tail copula process $\hat{\lambda}^{-1/2}\mathbb{G}_n(s, 1, 1)$ for $k^* = 104$ with respect to ns, $s \in [0, 1]$.

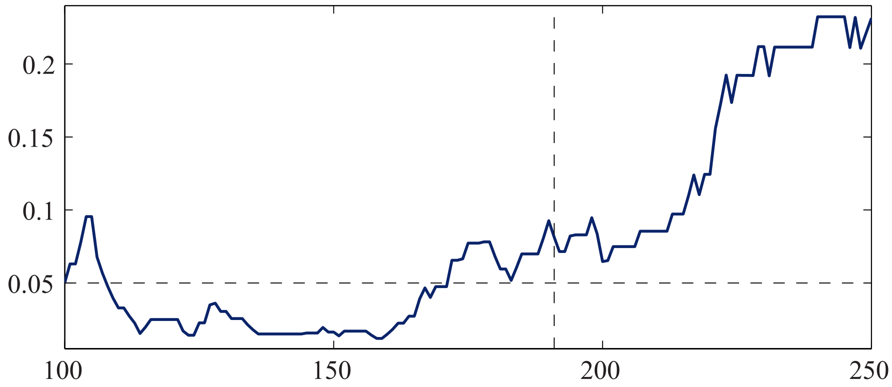

Figure 8: (Dow Jones and Nasdaq time series) Chi-squared test for a break at $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$: p-values for different k. The horizontal line indicates the 5% level of significance, the vertical one the plateau $k^* = 191$.

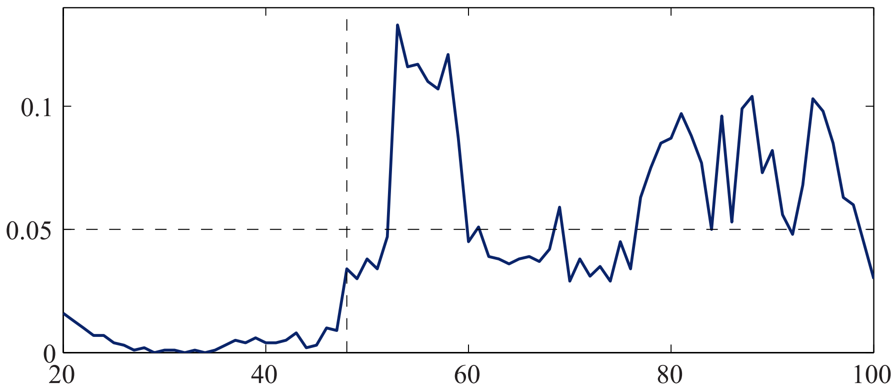

Figure 9: (Dow Jones and Nasdaq time series) TDC-Test 1 for the subsample before $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$: p-values for different k. The horizontal line indicates the 5% level of significance, the vertical one the plateau $k^* = 48$.

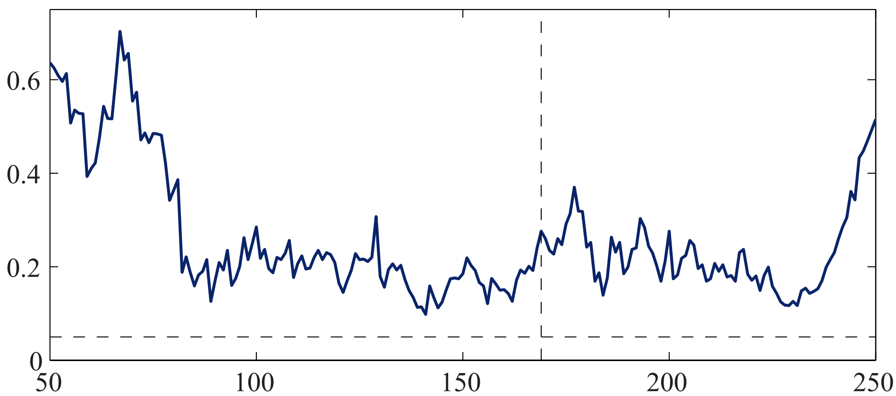

Figure 10: (Dow Jones and Nasdaq time series) TDC-Test 1 for the subsample after $\lfloor n\bar{s}_{\mathrm{BM}} \rfloor = 959$ (including Black Monday): p-values for different k. The horizontal line indicates the 5% level of significance, the vertical one the plateau $k^* = 169$.