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Supplementary material of "Dating multiple change points in the correlation matrix"

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1 Proofs

Proof of Theorem 1
The proof is similar to the proof of Theorem 1 in Galeano and Wied (2014). Denote $P^{**}_{l_1,l_2}(z) := P^{**,i,j}_{l_1,l_2}(z)_{1\le i<j\le p}$. Consider a fixed pair $(i,j), 1 \le i < j \le p$ and the process

$$\begin{aligned} B^{i,j}_{\eta(l_1),\tau(l_2)}(z) &:= \frac{\tau(z)-\eta(l_1)+1}{\tau(l_2)-\eta(l_1)+1} P^{ij}_{\tau(z),\eta(l_1),\tau(l_2)} \\ &:= \frac{\tau(z)-\eta(l_1)+1}{\tau(l_2)-\eta(l_1)+1}\left(\hat{\rho}^{ij}_{\eta(l_1),\tau(z)} - \rho_{i,j}\right) - \frac{\tau(z)-\eta(l_1)+1}{\tau(l_2)-\eta(l_1)+1}\left(\hat{\rho}^{ij}_{\eta(l_1),\tau(l_2)} - \rho_{i,j}\right) \end{aligned}$$

with $\rho_{i,j} := \frac{(m_{XX'})_{i,j}-\mu_i\mu_j}{\sqrt{\sigma_i^2\sigma_j^2}}$. Then, by arguments based on the strong law of large numbers, one can show that, uniformly for $z \in [l_1+\varepsilon, l_2]$ (for a small $0 < \varepsilon < l_2 - l_1$) $B^{i,j}_{\eta(l_1),\tau(l_2)}(z) \Rightarrow_d P^{**,i,j}_{l_1,l_2}(z)$. By means of arguments based on Theorem 4.2 in Billingsley (1968) one gets convergence uniformly in $[l_1,l_2]$. Then, uniformly in $[l_1,l_2]$,

$$B_{\eta(l_1),\tau(l_2)}(z) \to_{a.s.} P_{l_1,l_2}(z).$$

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The theorem follows by applying the argmax continuous mapping theorem (Kim and Pollard, 1990, Theorem 2.7). ■

Proof of Theorem 2
We assume without loss of generality that there is only one change point in $k_0 = [Tz_0]$ and that $[l_1, l_2] = [0, 1]$. Denote $P_{0,1}(z) =: P(z)$. Then $P(z)$ has a unique maximum in z_0. Similarly as in the proof of Proposition 2 in Bai (1997), we show that

$$\mathsf{P}_{M,T} := \mathsf{P}\left(\max_{|k-k_0|>M} B_{1,T}\left(\frac{k}{T}\right) - B_{1,T}\left(\frac{k_0}{T}\right) \geq 0\right)$$

becomes small for large M and T. That means that, for every $\varepsilon > 0$, there is a $M > 0$ and a $T_0 > 0$ such that, for all $T > T_0$, $\mathsf{P}_{M,T} < \varepsilon$.

Now, $B_{1,T}\left(\frac{k}{T}\right) - B_{1,T}\left(\frac{k_0}{T}\right) \geq 0$ is equivalent to

$$\left(B_{1,T}\left(\frac{k}{T}\right) - P\left(\frac{k}{T}\right)\right) - \left(B_{1,T}\left(\frac{k_0}{T}\right) - P\left(\frac{k_0}{T}\right)\right) + \left(P\left(\frac{k}{T}\right) - P\left(\frac{k_0}{T}\right)\right) \geq 0.$$

Assume for the moment $k > k_0$ and that the standard deviations of all random variables are equal to 1. (Divide each component of $P(\cdot)$ by the standard deviations if the latter assumption is not fulfilled.) We multiply the whole equation with $T/(k-k_0)$ and denote

$$A_1(k,k_0,T) = \frac{T}{k-k_0}\left(\left(B_{1,T}\left(\frac{k}{T}\right) - P\left(\frac{k}{T}\right)\right) - \left(B_{1,T}\left(\frac{k_0}{T}\right) - P\left(\frac{k_0}{T}\right)\right)\right)$$
$$A_2(k,k_0,T) = \frac{T}{k-k_0}\left(P\left(\frac{k}{T}\right) - P\left(\frac{k_0}{T}\right)\right).$$

Now, we use several observations in order to argue that the asymptotic behavior of $A_1(k,k_0,T)$ can be reduced to the behavior of

$$\sum_{1 \leq i < j \leq p} \frac{1}{k-k_0} \sum_{t=k_0}^{k} (X_{t,i}X_{t,j} - \mathsf{E}(X_{t,i}X_{t,j})).$$

This quantity is then arbitrarily small by the law of large numbers for sufficiently large M. The observations are the following:

1. $A_1(k,k_0,T)$ can be regarded as the sum of $p(p-1)/2$ components and each component can be treated separately.
2. For large T, with high probability and uniformly in $z \in \{k_0/T, k/T\}$, all components of $TP_{\tau(z),1,T}$ and $TP^*_{0,1}(z)$ have the same sign so that we consider the values without having applied the absolute value function.
3. For large T, with high probability and uniformly in $z \in \{k_0/T, k/T\}$, the successive variances in the denominators of the components of $A_1(k,k_0,T)$ are equal to their theoretical counterparts which are the same for $z = k/T$ and $z = k_0/T$, respectively.

4. In the numerators of the components of $A_1(k,k_0,T)$, we have expressions like

$$\frac{1}{k-k_0}\left(\left(\left(\sum_{t=k_0}^{k} X_{t,i}X_{t,j}\right) - T\left(P_{i,j}\left(\frac{k}{T}\right) - P_{i,j}\left(\frac{k_0}{T}\right)\right)\right.\right.$$
$$\left.\left. -\left(\frac{1}{k}\sum_{t=1}^{k} X_{t,i}\sum_{t=1}^{k} X_{t,j} - \frac{1}{k_0}\sum_{t=1}^{k_0} X_{t,i}\sum_{t=1}^{k_0} X_{t,j} - (k-k_0)\frac{1}{T}\sum_{t=1}^{T} X_{t,i}\frac{1}{T}\sum_{t=1}^{T} X_{t,j}\right)\right)\right).$$

Here, $P_{i,j}(\cdot)$ are the components of $P(\cdot)$.

5. It holds

$$T\left(P_{i,j}\left(\frac{k}{T}\right) - P_{i,j}\left(\frac{k_0}{T}\right)\right) = (k-k_0)\mathsf{E}(X_{k,i}X_{k,j}).$$

Then, after some tedious calculations, one sees that $A_1(k,k_0,T)$ is a random variable such that, for all $\varepsilon > 0$ and all $\eta > 0$, there is a $M > 0$ such that $\mathsf{P}(|A_1(k,k_0,T)| > \varepsilon) < \eta$ for $k > k_0 + M$ and $T > T_0$. This means that $A_1(k,k_0,T)$ is arbitrarily small whenever T and M are large. On the other hand, $A_2(k,k_0,T)$ does not converge to zero: $P\left(\frac{k}{T}\right) - P\left(\frac{k_0}{T}\right)$ is a finite sum of linear functions in k with negative slope (see Figure 1 in Galeano and Wied (2014)) so that it is equal to $C\left(\frac{k}{T} - \frac{k_0}{T}\right)$ for a $C < 0$ by Taylor's formula. Multiplied with $T/(k-k_0)$, the expression is equal to C. Then, with large probability, $A_1(k,k_0,T) + A_2(k,k_0,T)$ is strictly negative. For $k < k_0$, the argument is similar and the theorem is proven. ■

Proof of Theorem 3
Denote $Q_T^{l_1,l_2} := \sup_{z\in[l_1,l_2]} A_{\eta(l_1),\tau(l_2)}(z)$ the test statistic calculated from data from $\eta(l_1)$ to $\tau(l_2)$. Moreover, let $B \in \mathbb{N}$ a fixed number of bootstrap repetitions. Now, by Theorem 1, it holds

$$\frac{1}{a_T^k} B_{\eta(l_1),\tau(l_2)}(z) \to_p \infty$$

for any sequence $a_T^k = o\left(\sqrt{T}\right)$ if there is a change point in the interval $[l_1,l_2]$. Moreover, due to Assumption 9.b, the eigenvalues of $\hat{E}_{\eta(l_1),\tau(l_2)}$ are stochastically bounded. Consequently, the eigenvalues of $\hat{E}_{\eta(l_1),\tau(l_2)}^{-1/2}$ (remember that we assume its existence) are bounded away from zero and the matrix is positive definite. Therefore,

$$\frac{1}{a_T^k} Q_T^{l_1,l_2} \to_p \infty$$

(with respect to the measure $\mathsf{P}^\times$) for any sequence $a_T^k = o\left(\sqrt{T}\right)$ if there is a change point in the interval $[l_1,l_2]$. By Theorem 2, we moreover have

$$\frac{1}{a_T^k} Q_T^{\hat{z}_i,\hat{z}_{i+1}} \to_p \infty \tag{1}$$

(with respect to the measure $\mathsf{P}^{\times}$), where $\hat{z}_i$ and $\hat{z}_{i+1}$ for $i \in \mathbb{N}_0$ are two estimated change points in one of the iterations of the algorithms, as long as there is a change point in the interval $[z_i, z_{i+1}]$. This follows from the fact, that, by Theorem 2,

$$Q_T^{\hat{z}_i,\hat{z}_{i+1}} - Q_T^{z_i,z_{i+1}} = o_{\mathsf{P}^{\times}}\left(\frac{1}{\sqrt{T}}\right).$$

Moreover, with the same argument and with the results under the null hypothesis from Wied (2015),

$$Q_T^{\hat{z}_i,\hat{z}_{i+1}} = O_{\mathsf{P}^{\times}}(1) \tag{2}$$

if there is no change point in the interval $[z_i, z_{i+1}]$.

Then, due to (1), it holds $\lim_{T\to\infty} P^{\times}(\hat{\ell} < \ell) \to 0$ and due to (2), it holds $\lim_{T\to\infty} P^{\times}(\hat{\ell} > \ell) \to 0$, which proves the theorem (compare the proof of Proposition 11 in Bai, 1997). ■

Proof of Theorem 4

In this proof, we combine ideas of the proof of Theorem 3 in Galeano and Wied (2014) and of Theorem 3 in Wied (2015). Note that $\hat{z}^* = \mathrm{argmax}_{l_1 \le z \le l_2} B^*_{\eta(l_1),\tau(l_2)}(z)$ with

$$B_{\eta(l_1),\tau(l_2)}(z) := \frac{\tau(z) - \eta(l_1) + 1}{\sqrt{\tau(l_2) - \eta(l_1) + 1}} \left|\left|P_{\tau(z),\eta(l_1),\tau(l_2)}\right|\right|_1.$$

Then, one can show by means of the extended functional delta method from Wied et al (2012) that the process $\frac{\tau(z)-\eta(l_1)+1}{\sqrt{\tau(l_2)-\eta(l_1)+1}} P_{\tau(z),\eta(l_1),\tau(l_2)}$ converges to the process

$$E^{1/2} W^{\frac{p(p-1)}{2}}(z) - E^{1/2} W^{\frac{p(p-1)}{2}}(l_1) - \frac{z - l_1}{l_2 - l_1} E^{1/2}\left(W^{\frac{p(p-1)}{2}}(l_2) - W^{\frac{p(p-1)}{2}}(l_1)\right) + P^{**}_{l_1,l_2}(z).$$

This limit process has a unique maximum P-almost surely (Kim and Pollard, 1990, Lemma 2.6) and then the theorem follows by the argmax continuous mapping theorem (Kim and Pollard, 1990, Theorem 2.7). ■

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