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Supplemental Appendix for the Journal of Empirical Economics: Testing for Relevant Dependence Change in Financial Data: A CUSUM Copula Approach

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Abstract We propose a new non-parametric test for detecting relevant breaks in copula functions. We assume that the data is driven by two non-equal copulas C_1 and C_2. Under the null hypothesis, the copula difference within an appropriate norm is smaller than a certain positive adjustable threshold Δ. Within the alternative hypothesis, the copula difference exceeds the fixed value Δ. The test is based on a cumulative sum approach of the empirical copula with sequentially estimated marginals. We propose a bootstrap procedure to compute critical values. The Monte Carlo simulation indicates that the test results in a reasonable sized and powered testing procedure. A real data application of the DAX30 up to cross sectional dimension $N = 30$ shows the test' ability to detect relevant break points.

Keywords Relevant change · Copula · Break testing · Bootstrap · CUSUM
JEL codes: C12, C13, C32

A Assumptions

For the theoretical justification we need some slightly adjusted assumptions following [Dette and Wied(2016)]:

A1) The marginals $F_i(\cdot)$ and its inverse $F_i^{-1}(\cdot)$ are assumed to be known for all $i \in \{1, ..., N\}$.

A2) Let $\{\mathbf{X}_{T,1}, ..., \mathbf{X}_{T,T}\}_{T\in\mathbb{N}}$ denote a triangular array of strong mixing random vectors and $\{\mathbf{U}_{T,1}, ..., \mathbf{U}_{T,T}\}_{T\in\mathbb{N}}$ its corresponding probability transform such that

$$\mathbf{U}_{T,1}, ..., \mathbf{U}_{T,\lfloor sT\rfloor} \sim C_1(\mathbf{u}) \ ; \quad \mathbf{U}_{T,\lfloor sT\rfloor+1}, ..., \mathbf{U}_{T,T} \sim C_2(\mathbf{u}).$$

A3) Consider the triangular array $\{\mathbf{U}_{T,j} \,|\, j = 1, ..., T\}_{T\in\mathbb{N}}$ and define for $1 \le s \le t$ the corresponding σ-field $\mathcal{F}_s^t(T) := \sigma(\{\mathbf{X}_{T,j} \,|\, s \le j \le t\})$ generate by the random variable $\{\mathbf{U}_{T,j} \,|\, s \le j \le t\}$. We denote by

$$\alpha(m) := \sup_{T\in\mathbb{N}} \sup_{1\le k\le T-m} \sup\{|P(A\cap B) - P(A)P(B)| \,|\, A \in \mathcal{F}_{m+k}^T(T), B \in \mathcal{F}_1^k(T)\},\ m \in \mathbb{N}$$

the strong mixing coefficients of the triangular array $\{\mathbf{U}_{T,1}, ..., \mathbf{U}_{T,T}\}$ and assume that for some $\eta > 0$

$$\alpha(T) = \mathcal{O}(T^{-(1+\eta)})$$

as $T \to \infty$.

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A4) For $l = 1, 2$ let $\{W_t(l)\}_{t\in\mathbb{Z}}$ denote sequences of strictly stationary processes, such that for each $T \in \mathbb{N}$

$$(\mathbf{U}_{T,1}, ..., \mathbf{U}_{T,\lfloor sT \rfloor}) \stackrel{d}{=} (W_1(1), ..., W_{\lfloor sT \rfloor}(1))$$
$$(\mathbf{U}_{T,\lfloor sT \rfloor+1}, ..., \mathbf{U}_{T,T}) \stackrel{d}{=} (W_1(2), ..., W_{T-\lfloor sT \rfloor}, (2))$$

where $\stackrel{d}{=}$ means equality in distribution. That means, there are two regimes $\{W_t(1)\}_{t\in\mathbb{Z}}$ and $\{W_t(2)\}_{t\in\mathbb{Z}}$ and the considered process switches from one regime to the other.

B Derivation and Asymptotic Distribution of the Test Statistic

We impose the Assumptions given in Appendix A to be valid. Then, the testing problem of no relevant change in the copula can be defined as follow:

$$H_0 : \|C_1(\mathbf{u}) - C_2(\mathbf{u})\|_{L^2} \leq \Delta$$

versus the alternative

$$H_1 : \|C_1(\mathbf{u}) - C_2(\mathbf{u})\|_{L^2} > \Delta,$$

where $\|.\|_{L^2}$ is the L^2-norm and $\Delta > 0$ fixed. For every $\mathbf{u} := (u_1, ..., u_N) \in [0, 1]^N$ and $t \in (0, 1)$ the CUSUM approach for detecting changes in the copula is then

$$\hat{\mathbb{U}}_T(t, \mathbf{u}) := t(1-t) \left(\frac{1}{\lfloor tT \rfloor} \sum_{i=1}^{\lfloor tT \rfloor} Z_i(\mathbf{u}) - \frac{1}{T - \lfloor tT \rfloor} \sum_{i=\lfloor tT \rfloor+1}^{T} Z_i(\mathbf{u}) \right), \tag{1}$$

where $Z_i(\mathbf{u}) := \mathbb{1}\{F_1(X_{i1}) \leq u_1, ..., F_N(X_{iN}) \leq u_N\}$, $i = 1, ..., T$ is the vector of marginal distributions at time i where $F_j(\cdot)$ is the known j-th marginal cumulative distribution function for all $j = 1, ..., N$. Before we start the calculation we compute the expected value of some showing up sums. Since Z_i is Bernoulli distributed for $i = 1, ..., N$ we have

$$\begin{aligned}
\mathbb{E}[Z_i(\mathbf{u})] &= P(F_1(X_{i1}) \leq u_1, ..., F_N(X_{iN}) \leq u_N) \\
&= P(X_{i1} \leq F_1^{-1}(u_1), ..., X_{iN} \leq F_N^{-1}(u_N)) \\
&= C(u_1, .., u_N), \\
\mathbb{E}[Z_i(\mathbf{u})Z_i(\mathbf{u})] &= \frac{C(u_1, .., u_N)[1 - C(u_1, .., u_N)]}{T} + C(u_1, .., u_N)^2.
\end{aligned}$$

Furthermore, we obtain from A3) and A4) $\mathbb{E}[Z_i(\mathbf{u})Z_j(\mathbf{u})] = C_i(\mathbf{u})C_j(\mathbf{u}) + o(1) \quad \forall i \neq j$. Due to readability we introduce the following abbreviations $C_i := C_i(\mathbf{u})$ and $Z_i := Z_i(\mathbf{u})$ for $i = 1, 2$. For fixed $s \in (0, 1)$, we compute $\lim_{T\to\infty} \mathbb{E}[\hat{\mathbb{U}}_T(t, \mathbf{u})]$. We first consider the case $t > s$

$$\begin{aligned}
\mathbb{E}[\hat{\mathbb{U}}_T(t, \mathbf{u})] &= t(1-t)\mathbb{E}\left[\frac{s}{t} \frac{1}{\lfloor sT \rfloor} \sum_{i=1}^{\lfloor sT \rfloor} Z_i + \frac{1}{\lfloor tT \rfloor} \sum_{i=\lfloor sT \rfloor+1}^{\lfloor tT \rfloor} Z_i - \frac{1}{T - \lfloor tT \rfloor} \sum_{i=\lfloor tT \rfloor+1}^{T} Z_i \right] \\
&= t(1-t) \left(\frac{s}{t} C_1 + \mathbb{E}\left[\frac{\lfloor tT \rfloor - (\lfloor sT \rfloor)}{\lfloor tT \rfloor} \frac{1}{\lfloor tT \rfloor - (\lfloor sT \rfloor)} \sum_{i=\lfloor sT \rfloor+1}^{\lfloor tT \rfloor} Z_i \right] - C_2 \right) \\
&= t(1-t) \left(\frac{s}{t} C_1 + \frac{t-s}{t} C_2 - C_2 \right) = s(1-t)\left(C_1 - C_2\right).
\end{aligned}$$

For $t \leq s$ we obtain

$$\begin{aligned}
\mathbb{E}[\hat{\mathbb{U}}_T(t, \mathbf{u})] &= t(1-t)\mathbb{E}\left[\frac{1}{\lfloor tT \rfloor} \sum_{i=1}^{\lfloor tT \rfloor} Z_i - \frac{1}{T - \lfloor tT \rfloor} \sum_{i=\lfloor tT \rfloor+1}^{\lfloor sT \rfloor} Z_i - \frac{1}{T - \lfloor tT \rfloor} \sum_{i=\lfloor sT \rfloor+1}^{T} Z_i \right] \\
&= t(1-t) \left(C_1 - \mathbb{E}\left[\frac{\lfloor sT \rfloor - (\lfloor tT \rfloor)}{T - \lfloor tT \rfloor} \frac{1}{\lfloor sT \rfloor - (\lfloor tT \rfloor)} \sum_{i=\lfloor sT \rfloor+1}^{\lfloor tT \rfloor} Z_i \right] - \frac{1-s}{1-t} \right) \\
&= t(1-t) \left(\frac{1-s}{1-t} C_1 - \frac{1-s}{1-t} C_2 \right) = t(1-s)\left(C_1 - C_2\right).
\end{aligned}$$

Considering both cases yields:

$$\mathbb{E}[\hat{\mathbb{U}}_T(t,\mathbf{u})] = \begin{cases} s(1-t)\,(C_1 - C_2) & \text{for } t > s \\ t(1-s)\,(C_1 - C_2) & \text{for } t \leq s. \end{cases} \tag{2}$$

The aim is to lose the quantile and time dimension $\mathbf{u}$ and t, respectively. As an intermediate step we consider $\mathbb{E}[(\hat{\mathbb{U}}_T(t,\mathbf{u}))^2]$ that can be decomposed into three partial sums A, B, C with

$$A := \left(\frac{1}{\lfloor tT \rfloor} \sum_{i=1}^{\lfloor tT \rfloor} Z_i \right)^2 \tag{3}$$

$$B := \frac{1}{\lfloor tT \rfloor} \frac{1}{T - \lfloor tT \rfloor} \sum_{i=1}^{\lfloor tT \rfloor} Z_i \sum_{j=\lfloor tT \rfloor + 1}^{T} Z_j \tag{4}$$

$$C := \left(\frac{1}{T - \lfloor tT \rfloor} \sum_{i=\lfloor tT \rfloor + 1}^{T} Z_i \right)^2. \tag{5}$$

We first consider the case where $t > s$ and subscript this accordingly.

$$\begin{aligned}
A^{t>s}: \quad & \mathbb{E}\left[\left(\frac{1}{\lfloor tT \rfloor} \sum_{i=1}^{\lfloor tT \rfloor} Z_i \right)^2 \right] = \mathbb{E}\left[\left(\frac{1}{\lfloor tT \rfloor} \sum_{i=1}^{\lfloor sT \rfloor} Z_i + \frac{1}{\lfloor tT \rfloor} \sum_{i=\lfloor sT \rfloor+1}^{\lfloor tT \rfloor} Z_i \right)^2 \right] \\
& = \mathbb{E}\left[\left(\frac{1}{\lfloor tT \rfloor} \sum_{i=1}^{\lfloor sT \rfloor} Z_i \right)^2 \right] + 2\mathbb{E}\left[\left(\frac{1}{\lfloor tT \rfloor} \sum_{i=\lfloor sT \rfloor+1}^{\lfloor tT \rfloor} Z_i \right) \left(\frac{1}{\lfloor tT \rfloor} \sum_{i=1}^{\lfloor sT \rfloor} Z_i \right) \right] + \mathbb{E}\left[\left(\frac{1}{\lfloor tT \rfloor} \sum_{i=\lfloor sT \rfloor+1}^{\lfloor tT \rfloor} Z_i \right)^2 \right] \\
& = \frac{1}{\lfloor tT \rfloor^2} \left[\lfloor sT \rfloor \left(\frac{C_1(1-C_1)}{\lfloor sT \rfloor} \right) + (\lfloor sT \rfloor C_1)^2 \right] + 2\frac{s(t-s)}{t^2} C_1 C_2 \\
& \quad + \frac{1}{\lfloor tT \rfloor^2} \left[\frac{\lfloor tT \rfloor - \lfloor sT \rfloor}{\lfloor tT \rfloor - \lfloor sT \rfloor} C_2(1 - C_2 + (\lfloor tT \rfloor - \lfloor sT \rfloor)^2 C_2^2 \right] + o(1) \\
& = \frac{s^2}{t^2} \left[\frac{C_1(1-C_1)}{T^2} + C_1^2 \right] + 2\frac{s(t-s)}{t^2} C_1 C_2 + \frac{C_2(1-C_2)}{\lfloor tT \rfloor^2} + \frac{(t-s)^2}{t^2} C_2^2 + o(1) \\
& = \frac{s^2}{t^2} C_1^2 + 2\frac{s}{t^2} C_1 (t-s) C_2 + \frac{(t-s)^2}{t^2} C_2^2 + o(1) \\
B^{t>s}: \quad & \mathbb{E}\left[\frac{1}{\lfloor tT \rfloor} \frac{1}{T - \lfloor tT \rfloor} \sum_{i=1}^{\lfloor tT \rfloor} Z_i \sum_{j=\lfloor tT \rfloor+1}^{T} Z_j \right] \\
& = \mathbb{E}\left[\frac{1}{\lfloor tT \rfloor} \frac{1}{T - \lfloor tT \rfloor} \sum_{i=1}^{\lfloor sT \rfloor} Z_i \sum_{j=\lfloor tT \rfloor+1}^{T} Z_j \right] + \mathbb{E}\left[\frac{1}{\lfloor tT \rfloor} \frac{1}{T - \lfloor tT \rfloor} \sum_{i=\lfloor sT \rfloor+1}^{\lfloor tT \rfloor} Z_i \sum_{j=\lfloor tT \rfloor+1}^{T} Z_j \right] \\
& = \frac{s}{t} C_1 C_2 + \frac{t-s}{t} C_2^2 + o(1) \\
C^{t>s}: \quad & \mathbb{E}\left[\left(\frac{1}{T - \lfloor tT \rfloor} \sum_{i=\lfloor tT \rfloor+1}^{T} Z_i \right)^2 \right] = \left(\frac{1}{T - \lfloor tT \rfloor} \right)^2 \left[\frac{T - \lfloor tT \rfloor}{T - \lfloor tT \rfloor} C_2(1-C_2) + [(T - \lfloor tT \rfloor) C_2]^2 \right] \\
& = C_2^2 + o(1).
\end{aligned}$$

Hence, we have

$$\frac{1}{t^2(1-t)^2}\mathbb{E}[\hat{\mathbb{U}}_T(t,\mathbf{u})^2] = \mathbb{E}\left[\underbrace{\left(\frac{1}{\lfloor tT\rfloor}\sum_{i=1}^{\lfloor tT\rfloor} Z_i\right)^2}_{A^{t>s}} \underbrace{-2\frac{1}{\lfloor tT\rfloor}\frac{1}{T-\lfloor tT\rfloor}\sum_{i=1}^{\lfloor tT\rfloor} Z_i \sum_{j=\lfloor tT\rfloor+1}^{T} Z_j}_{B^{t>s}} + \underbrace{\left(\frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor tT\rfloor+1}^{T} Z_i\right)^2}_{C^{t>s}}\right]$$

$$= \frac{s^2}{t^2}C_1^2 + 2\frac{s}{t^2}C_1(t-s)C_2 + \frac{(t-s)^2}{t^2}C_2^2 - 2\left[\frac{s}{t}C_1 + \frac{t-s}{t}C_2\right]C_2^2 + C_2^2 + o(1)$$

$$= \frac{s^2}{t^2}(C_1-C_2)^2 + o(1)$$

Now we consider the case where $t \leq s$

$$A^{t\leq s}:\quad \mathbb{E}\left[\left(\frac{1}{T}\sum_{i=1}^{\lfloor tT\rfloor} Z_i\right)^2\right] = \frac{1}{\lfloor tT\rfloor^2}\left(\frac{\lfloor tT\rfloor}{\lfloor tT\rfloor}[C_1(1-C_1)] + (\lfloor tT\rfloor C_1)^2\right) = C_1^2 + o(1)$$

$$B^{t\leq s}:\quad \mathbb{E}\left[\frac{1}{\lfloor tT\rfloor}\frac{1}{T-\lfloor tT\rfloor}\sum_{i=1}^{\lfloor tT\rfloor} Z_i \sum_{j=\lfloor tT\rfloor+1}^{T} Z_j\right]$$

$$= \mathbb{E}\left[\frac{1}{\lfloor tT\rfloor}\frac{1}{T-\lfloor tT\rfloor}\sum_{i=1}^{\lfloor tT\rfloor} Z_i \sum_{j=\lfloor tT\rfloor+1}^{\lfloor sT\rfloor} Z_j\right] + \mathbb{E}\left[\frac{1}{\lfloor tT\rfloor}\frac{1}{T-\lfloor tT\rfloor}\sum_{i=1}^{\lfloor tT\rfloor} Z_i \sum_{j=\lfloor sT\rfloor+1}^{T} Z_j\right] = \left[\frac{s-t}{1-t}C_1 + \frac{1-s}{1-t}C_2\right]C_1 + o(1)$$

$$C^{t\leq s}:\quad \mathbb{E}\left[\left(\frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor tT\rfloor}^{T} Z_i\right)^2\right] = \mathbb{E}\left[\left(\frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor tT\rfloor}^{\lfloor sT\rfloor} Z_i + \frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor sT\rfloor+1}^{T} Z_i\right)^2\right]$$

$$= \mathbb{E}\left[\left(\frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor tT\rfloor}^{\lfloor sT\rfloor} Z_i\right)^2\right] + 2\mathbb{E}\left[\left(\frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor tT\rfloor}^{\lfloor sT\rfloor} Z_i\right)\left(\frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor sT\rfloor+1}^{T} Z_i\right)\right] + \mathbb{E}\left[\left(\frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor sT\rfloor+1}^{T} Z_i\right)^2\right]$$

$$= \frac{1}{(T-\lfloor tT\rfloor)^2}\left(\frac{\lfloor sT\rfloor-\lfloor tT\rfloor}{\lfloor sT\rfloor-\lfloor tT\rfloor}[C_1(1-C_1)] + [(\lfloor sT\rfloor-\lfloor tT\rfloor)C_1]^2\right) + 2\frac{s-t}{1-t}C_1\frac{1-s}{1-t}C_2$$
$$+ \frac{1}{(T-\lfloor tT\rfloor)^2}\left(\frac{T-\lfloor sT\rfloor}{T-\lfloor sT\rfloor}[C_2(1-C_2)] + [(T-\lfloor sT\rfloor)C_2]^2\right) + o(1)$$

$$= \frac{(s-t)^2}{(1-t)^2}C_1^2 + +2\frac{s-t}{1-t}C_1\frac{1-s}{1-t}C_2 + \frac{(1-s)^2}{(1-t)^2}C_2^2 + o(1)$$

This yields for the expression $E[\hat{\mathbb{U}}_T(t,\mathbf{u})^2]$

$$\frac{1}{t^2(1-t)^2}\mathbb{E}[\hat{\mathbb{U}}_T(t,\mathbf{u})^2] = \mathbb{E}\left[\underbrace{\left(\frac{1}{\lfloor tT\rfloor}\sum_{i=1}^{\lfloor tT\rfloor} Z_i\right)^2}_{A^{t\leq s}} \underbrace{-2\frac{1}{\lfloor tT\rfloor}\frac{1}{T-\lfloor tT\rfloor}\sum_{i=1}^{\lfloor tT\rfloor} Z_i \sum_{j=\lfloor tT\rfloor+1}^{T} Z_j}_{B^{t\leq s}} + \underbrace{\left(\frac{1}{T-\lfloor tT\rfloor}\sum_{i=\lfloor tT\rfloor+1}^{T} Z_i\right)^2}_{C^{t\leq s}}\right]$$

$$= \frac{(s-t)^2}{(1-t)^2}C_1^2 + 2\frac{s-t}{1-t}C_1\frac{1-s}{1-t}C_2 + \frac{(1-s)^2}{(1-t)^2}C_2^2 - 2\left[\frac{s-t}{1-t}C_1 + \frac{1-s}{1-t}C_2\right]C_1 + C_1^2 + o(1)$$

$$= \frac{(1-s)^2}{(1-t)^2}(C_1-C_2)^2 + o(1)$$

Combining the previous calculations for $t > s$, $t \leq s$ and with the help of Fubini we obtain

$$L(t) := \lim_{T\to\infty}\mathbb{E}[\|\hat{\mathbb{U}}_T(t,\mathbf{u})\|_{L^2}^2] = \begin{cases} s^2(1-t)^2\|C_1(\mathbf{u}) - C_2(\mathbf{u})\|_{L^2}^2, & t > s \\ (1-s)^2t^2\|C_1(\mathbf{u}) - C_2(\mathbf{u})\|_{L^2}^2, & t \leq s. \end{cases}$$

By integrating out t a straightforward calculation yields

$$\int_0^1 L(t)dt = \frac{s^2(1-s)^2}{3}\|C_1(\mathbf{u}) - C_2(\mathbf{u})\|_{L^2}^2. \tag{6}$$

The next theorem provides the limiting distribution of the empirical centred counterpart $\hat{L}_T(t) := \|\hat{\mathbb{U}}_T(t,\mathbf{u})\|_{L^2}^2$

Theorem 1 *Under Assumptions A1)-A4)*

$$\sqrt{T}\left(\int\limits_0^1 \hat{L}_T(t)dt - \frac{1}{3}s^2(1-s)^2||C_1(\mathbf{u}) - C_2(\mathbf{u})||^2_{L^2}\right) \xrightarrow{d} N(0, \sigma^2_{C_1,C_2,s}), \tag{7}$$

with $\sigma^2_{C_1,C_2,s} = 4\int_0^1\int_0^1 \mathbb{E}\left[< \mathbb{U}(t_1,\mathbf{u}), A(t_1,\mathbf{u}) >_{L^2} < \mathbb{U}(t_2,\mathbf{u}), A(t_2,\mathbf{u}) >_{L^2}\right] dt_1 dt_2$ *and* $< \cdot,\cdot >_{L^2}$ *the* L^2 *inner product.*

Proof See Appendix C

Due to the high computational effort in high dimensions using the L^2-norm it could be reasonable to only test for specific quantiles (points) $\mathbf{q}$ in the copula. So similar to the L^2-norm testing we can test on fixed points $\mathbf{q} = (q_1, \ldots, q_N)'$ in the copula, using the previous notation and considering a constant functions $g := C(\mathbf{q})$, where $C(\mathbf{q})$ is the copula value at some fixed quantile $\mathbf{q}$.

Corollary 1 *Under Assumptions A1)-A4)*

$$\sqrt{T}\left(\int\limits_0^1 \hat{L}^{\mathbf{q}}_T(t)dt - \frac{1}{3}s^2(1-s)^2|C_1(\mathbf{q}) - C_2(\mathbf{q})|^2\right) \xrightarrow{d} N(0, \sigma^2_{C_1,C_2,s,\mathbf{q}}), \tag{8}$$

with $\hat{L}^{\mathbf{q}}_T(t) := (\hat{\mathbb{U}}_T(t,\mathbf{q}))^2$ *and* $\sigma^2_{C_1,C_2,s,\mathbf{q}} := 4\int_0^1\int_0^1 \mathbb{E}\left[\mathbb{U}(t_1,\mathbf{q}) \cdot A(t_1,\mathbf{q}) \cdot \mathbb{U}(t_2,\mathbf{q}) \cdot A(t_2,\mathbf{q})\right] dt_1 dt_2$ *for fixed* $\mathbf{q} \in [0,1]^N$.

$\hat{L}^{\mathbf{q}}_T(t)$ and $\sigma^2_{C_1,C_2,s,\mathbf{q}}$ are called the quantile version of $\hat{L}_T(t)$ and $\sigma^2_{C_1,C_2,s}$, respectively. The next Lemma shows that the test holds the size level and has considerable power.

Lemma 1 *The test*

$$\hat{\kappa}_T \geq \frac{1}{3}s^2(1-s)^2\Delta^2 + \frac{k_{1-\alpha}(s)}{\sqrt{T}} \tag{9}$$

is a consistent asymptotic α *test for all* $s > 0$*, where* $k_{1-\alpha}(s)$ *is the* $(1-\alpha)$*-quantile of the limiting normal distribution given in* (7) *and* $\hat{\kappa}_T = \int\limits_0^1 \hat{L}_T(t)dt$.

Proof Suppose $\delta := \|C_1(\mathbf{u}) - C_2(\mathbf{u})\|_{L^2} \leq \Delta$. Then

$$\begin{aligned} P_\delta(\hat{\kappa}_T \geq \frac{1}{3}s^2(1-s)^2\Delta^2 + \frac{k_{1-\alpha}(s)}{\sqrt{T}}) &= P(\sqrt{T}(\hat{\kappa}_T - \frac{1}{3}s^2(1-s)^2\delta^2) \geq \sqrt{T}\frac{1}{3}s^2(1-s)^2(\Delta^2 - \delta^2) + k_{1-\alpha}(s)) \\ &\leq P(\sqrt{T}(\hat{\kappa}_T - \frac{1}{3}s^2(1-s)^2\delta^2) \geq k_{1-\alpha}(s)) \\ &\underset{T\to\infty}{\longrightarrow} 1 - (1-\alpha) = \alpha. \end{aligned}$$

Otherwise, if $\delta > \Delta$

$$\begin{aligned} P_\delta(\hat{\kappa}_T \geq \frac{1}{3}s^2(1-s)^2\Delta^2 + \frac{k_{1-\alpha}(s)}{\sqrt{T}}) &= P(\sqrt{T}((\hat{\kappa}_T - \frac{1}{3}s^2(1-s)^2\delta^2) \geq \underbrace{\sqrt{T}\frac{1}{3}s^2(1-s)^2(\Delta^2 - \delta^2)}_{<0} + k_{1-\alpha}(s)) \\ &= 1 - P(\sqrt{T}(\hat{\kappa}_T - \frac{1}{3}s^2(1-s)^2\delta^2) < \sqrt{T}\frac{1}{3}s^2(1-s)^2(\Delta^2 - \delta^2) + k_{1-\alpha}(s)) \\ &\underset{T\to\infty}{\longrightarrow} 1 - 0 = 1. \end{aligned}$$

The test given in equation (9) is an exact level α test if Δ is chosen as the copula difference $\delta = ||C_1(\mathbf{u}) - C_2(\mathbf{u})||_{L^2}$. Otherwise the size is smaller than α.

C Proof of Theorem 1

We execute the proof of Theorem 1 stepwise. First, we start to consider only one partial sum of the process $\hat{\mathbb{U}}_T(\cdot,\cdot)$, i.e.

$$\hat{\mathbb{C}}_T(t,\mathbf{u}) := \frac{1}{T}\sum_{i=1}^{\lfloor tT \rfloor} Z_i(\mathbf{u}). \tag{10}$$

Second, by means of the continuous mapping theorem we obtain the limiting distribution of the process $\hat{\mathbb{U}}_T(\cdot,\cdot)$ and can then finally derive the limiting distribution given in Theorem 1. Again, for the computation of the expectation of $\hat{\mathbb{C}}_T(\cdot,\cdot)$ we have to distinguish two cases, i.e. either $t \le s$ or $t > s$. If $t \le s$, we have $\lim_{T\to\infty} \mathbb{E}[\hat{\mathbb{C}}_T(t,\mathbf{u})] = tC_1(\mathbf{u})$. For $t > s$ a straightforward calculation yields

$$\begin{aligned}\mathbb{E}[\hat{\mathbb{C}}_T(t,\mathbf{u})] &= \mathbb{E}\left[\frac{1}{T}\sum_{i=1}^{\lfloor sT \rfloor} Z_i(\mathbf{u}) + \frac{1}{T}\sum_{i=\lfloor sT \rfloor+1}^{\lfloor tT \rfloor} Z_i(\mathbf{u})\right]\\ &= sC_1(\mathbf{u}) + \frac{\lfloor tT \rfloor - \lfloor sT \rfloor}{T}C_2(\mathbf{u}) = sC_1(\mathbf{u}) + (t-s)C_2(\mathbf{u}) + o(1).\end{aligned}$$

Thus, the expectation of the partial sum $\hat{\mathbb{C}}_T(\cdot,\cdot)$ is given by

$$E_{C_1,C_2,s}(t,\mathbf{u}) := \lim_{T\to\infty} \mathbb{E}[\hat{\mathbb{C}}_T(t,\mathbf{u})] = (s\wedge t)C_1(\mathbf{u}) + (t-s)_+C_2(\mathbf{u}). \tag{11}$$

With the expectation (11) we derive the asymptotic distribution of the centred partial sum process (10), which leads to the following theorem.

Theorem 2 *Let Assumptions A1)-A4) hold. Then, a standardized version of the process* $\{\hat{\mathbb{C}}_T(t,\mathbf{u})\}_{t\in(0,1),\mathbf{u}\in[0,1]^N}$ *converges weakly in* $\ell^\infty((0,1)\times[0,1]^N)$*, i.e.*

$$\sqrt{T}\left\{\hat{\mathbb{C}}_T(t,\mathbf{u}) - E_{C_1,C_2,s}(t,\mathbf{u})\right\}_{t\in(0,1),\mathbf{u}\in[0,1]^N} \overset{d}{\Rightarrow} \left\{\mathbb{G}_{C_1,C_2,s}(t,\mathbf{u})\right\}_{t\in(0,1),\mathbf{u}\in[0,1]^N}.$$

Here, $\mathbb{G}_{C_1,C_2,s}$ *denotes a centred Gaussian process with covariance kernel*

$$\mathbb{E}[\mathbb{G}_{C_1,C_2,s}(t_1,\mathbf{u}_1)\mathbb{G}_{C_1,C_2,s}(t_2,\mathbf{u}_2)] = (t_1\wedge t_2\wedge s)k_1(\mathbf{u}_1,\mathbf{u}_2) + (t_1\wedge t_2 - s)_+k_2(\mathbf{u}_1,\mathbf{u}_2), \tag{12}$$

and the kernels k_1 *and* k_2 *are defined by*

$$k_l(\mathbf{u}_1,\mathbf{u}_2) = \sum_{i\in\mathbb{Z}} Cov[\mathbb{1}\{W_0(l)\le \mathbf{u}_1\}, \mathbb{1}\{W_i(l)\le\mathbf{u}_2\}], \quad l=1,2. \tag{13}$$

Proof Consider

$$\begin{aligned}\hat{\mathbb{C}}_T(t,u) - \mathbb{E}_{C_1,C_2,s}[t,\mathbf{u}] &= \frac{1}{T}\sum_{i=1}^{\lfloor tT \rfloor} Z_i(\mathbf{u}) - [(t\wedge s)C_1(\mathbf{u}) + (t-s)_+C_2(\mathbf{u})] + o_P(\frac{1}{\sqrt{T}})\\ &= \underbrace{\frac{1}{T}\sum_{i=1}^{\lfloor T(s\wedge t)\rfloor}[Z_i(\mathbf{u}) - C_1(\mathbf{u})]}_{\mathbf{X}_T^{(1)}(t,\mathbf{u}) := \sum_{i=1}^{\lfloor T(s\wedge t)\rfloor} Y_{T,i}(\mathbf{u})} + \underbrace{\mathbb{1}\{t>s\}\frac{1}{T}\sum_{i=\lfloor T(s\wedge t)\rfloor+1}^{\lfloor tT \rfloor}[Z_i(\mathbf{u}) - C_2(\mathbf{u})]}_{\mathbf{X}_T^{(2)}(t,\mathbf{u}) := \mathbb{1}\{t>s\}\sum_{i=\lfloor T(s\wedge t)\rfloor+1}^{\lfloor tT \rfloor} Y_{T,i}(\mathbf{u})} + o_P(\frac{1}{\sqrt{T}})\end{aligned}$$

$$\text{with } Y_{T,i}(\mathbf{u}) := \mathbb{1}\{i\le\lfloor sT\rfloor\}\frac{Z_i(\mathbf{u}) - C_1(\mathbf{u})}{T} + \mathbb{1}\{i>\lfloor sT\rfloor\}\frac{Z_i(\mathbf{u}) - C_2(\mathbf{u})}{T}$$

Then it follows by [Bücher et al.(2014)Bücher, Kojadinovic, Rohmer, and Segers] for $T\to\infty$

1. $\{\sqrt{T}\mathbf{X}_T^{(1)}(t,\mathbf{u})\}_{t\in[0,1],\mathbf{u}\in[0,1]^n} \overset{d}{\Longrightarrow} \mathbb{G}(t\wedge s,\mathbf{u})$
2. $\{\sqrt{T}\mathbf{X}_T^{(2)}(t,\mathbf{u})\}_{t\in[0,1],\mathbf{u}\in[0,1]^n} \overset{d}{\Longrightarrow} \mathbb{G}(t,\mathbf{u}) - \mathbb{G}(t\wedge s,\mathbf{u})$

where $\mathbb{G}(\cdot,\cdot)$ are tight centred Gaussian processes with covariance function

$$\mathrm{Cov}[\mathbb{G}(t_1\wedge s,\mathbf{u}_1), \mathbb{G}(t_2\wedge s,\mathbf{u}_2)] = (t_1\wedge t_2\wedge s)k_1(\mathbf{u}_1,\mathbf{u}_2) \tag{14}$$

and

$$\begin{aligned}
&\mathrm{Cov}[\mathbb{G}(t_1,\mathbf{u}_1)-\mathbb{G}(t_1\wedge s,\mathbf{u}_1),\mathbb{G}(t_2,\mathbf{u}_2)-\mathbb{G}(t_2\wedge s,\mathbf{u}_2)]\\
&=\mathrm{Cov}[\mathbb{G}(t_1,\mathbf{u}_1,\mathbb{G}(t_2,\mathbf{u}_2)]-\mathrm{Cov}[\mathbb{Z}(t_1,\mathbf{u}_1),\mathbb{G}(t_2\wedge s,\mathbf{u}_2)]-\\
&\quad\mathrm{Cov}[\mathbb{G}(t_1\wedge s,\mathbf{u}_1),\mathbb{G}(t_2,\mathbf{u}_2)]+\mathrm{Cov}[\mathbb{G}(t_1\wedge s,\mathbf{u}_1),\mathbb{G}(t_2\wedge s,\mathbf{u}_2)]\\
&=(t_1\wedge t_2)k_2(\mathbf{u}_1,\mathbf{u}_2)-(t_1\wedge t_2\wedge s)k_2(\mathbf{u}_1,\mathbf{u}_2)-(t_1\wedge t_2\wedge s)k_2(\mathbf{u}_1,\mathbf{u}_2)+(t_1\wedge t_2\wedge s)k_2(\mathbf{u}_1,\mathbf{u}_2)\\
&=(t_1\wedge t_2-t_1\wedge t_2\wedge s)k_2(\mathbf{u}_1,\mathbf{u}_2)\\
&=(t_1\wedge t_2-s)_+k_2(\mathbf{u}_1,\mathbf{u}_2).
\end{aligned}$$

Thus, the composition $\sqrt{T}\mathbf{X}_T:=\sqrt{T}\left(\mathbf{X}_T^{(1)}+\mathbf{X}_T^{(2)}\right)$ is asymptotically tight [cf. Section 1.5 [van der Vaart and Wellner(1996)]]. In order to prove convergence in distribution of $\sqrt{T}\mathbf{X}_T$ it remains to establish the weak convergence of the finite dimensional distributions. Therefore, we use the Cramér-Wold-device and show for all sequences $(t_1,\mathbf{u}_1),...,(t_n,\mathbf{u}_n)\in[0,1]\times[0,1]^n$

$$\sqrt{T}\{\sum_{i=1}^{k}a_j\mathbf{X}_T(t_j,\mathbf{u}_j)\}\overset{d}{\Longrightarrow}\sum_{j=1}^{k}a_j\mathbb{G}_{C_1,C_2,s}(t_j,\mathbf{u}_j)\tag{15}$$

with $\alpha_1,...,\alpha_k\in\mathbb{R}$ and $\mathbb{G}_{C_1,C_2,s}$ is the Gaussian process defined in Theorem 2. Now, we restrict ourselves to the case $k=2$ and begin with the calculation of the covariance of $\mathbf{X}_T^{(1)}(t_1,u_1)$ and $\mathbf{X}_T^{(2)}(t_2,u_2)$. Therefore, we consider four different cases.
$\underline{t_1\le t_2\le s}$:

$$T\mathrm{Cov}[\mathbf{X}_T^{(l)}(t_1,\mathbf{u}_1),\mathbf{X}_T^{(l)}(t_2,\mathbf{u}_2)]\overset{T\to\infty}{\to}\begin{cases}(t_1\wedge t_2\wedge s)k_1(\mathbf{u}_1,\mathbf{u}_2) & \text{if } l=1\\ 0 & \text{if } l=2.\end{cases}$$

$\underline{s\le t_1\le t_2}$:

$$T\mathrm{Cov}[\mathbf{X}_T^{(l)}(t_1,\mathbf{u}_1),\mathbf{X}_T^{(l)}(t_2,\mathbf{u}_2)]\overset{T\to\infty}{\to}\begin{cases}tk_1(\mathbf{u}_1,\mathbf{u}_2) & \text{if } l=1\\ (t_1\wedge t_2-s)_+k_2(\mathbf{u}_1,\mathbf{u}_2) & \text{if } l=2\end{cases}$$

$\underline{t_1< s\le t_2}$:

$$\begin{aligned}
T|\mathrm{Cov}[\mathbf{X}_T^{(l)}(t_1,\mathbf{u}_1),\mathbf{X}_T^{(l)}(t_2,\mathbf{u}_2)]|&=T|\mathrm{Cov}[\sum_{j=1}^{\lfloor t_1T\rfloor}Y_{T,i}(t_2,\mathbf{u}_2),\sum_{j=\lfloor sT\rfloor+1}^{\lfloor t_2T\rfloor}Y_{T,i}(t_2,\mathbf{u}_2)]|\\
&=\mathcal{O}(\frac{1}{T^{\eta+1}})=\mathcal{O}(\frac{1}{T^{\eta}})=o(1)
\end{aligned}$$

for all $\eta>0$.
In the case where $t_1=s\le t_2$ we use a sequence ϵ_T such that $\epsilon_T T\to\infty$ and $\epsilon_T^2T\to0$ and obtain by the same argument of strong mixing

$$\begin{aligned}
&T|\mathrm{Cov}[\mathbf{X}_T^{(l)}(t_1,\mathbf{u}_1),\mathbf{X}_T^{(l)}(t_2,\mathbf{u}_2)]|\\
&=T|\mathrm{Cov}[\sum_{i=1}^{\lfloor T(s-\epsilon_T)\rfloor}\mathbb{Y}_{T,i}(t_1,\mathbf{u}_1)+\sum_{i=\lfloor T(s-\epsilon_T)\rfloor+1}^{\lfloor sT\rfloor}\mathbb{Y}_{T,i}(t_1,\mathbf{u}_1),\sum_{i=\lfloor sT\rfloor+1}^{\lfloor T(s+\epsilon_T)\rfloor}\mathbb{Y}_{T,i}(t_2,\mathbf{u}_2)+\sum_{i=\lfloor T(s+\epsilon_T)\rfloor+1}^{\lfloor t_2T\rfloor}\mathbb{Y}_{T,i}(t_2,\mathbf{u}_2)]|\\
&=\mathcal{O}(\frac{1}{(\epsilon_T)T^{\eta}})+\mathcal{O}(T\epsilon_T^2)=o(1)
\end{aligned}$$

$$
\begin{aligned}
\sigma^2 &= \lim_{T\to\infty} \mathbb{V}[\sqrt{T}\sum_{j=1}^{2}\alpha_j \mathbf{X}_T(t_j,\mathbf{u}_j)] \\
&= \lim_{T\to\infty} \mathbb{V}[\alpha_1(\mathbf{X}_T^{(1)}(t_1,\mathbf{u}_1)+\mathbf{X}_T^{(2)}(t_1,\mathbf{u}_1))+\alpha_2(\mathbf{X}_T^{(1)}(t_2,\mathbf{u}_2)+\mathbf{X}_T^{(2)}(t_2,\mathbf{u}_2))] \\
&= \lim_{T\to\infty} T\{\alpha_1^2\mathrm{Cov}[\mathbf{X}_T^{(1)}(t_1,\mathbf{u}_1),\mathbf{X}_T^{(1)}(t_1,\mathbf{u}_1)]+2\alpha_1\alpha_2\mathrm{Cov}[\mathbf{X}_T^{(1)}(t_1,\mathbf{u}_1),\mathbf{X}_T^{(1)}(t_2,\mathbf{u}_2)] \\
&\quad + \alpha_1^2\mathrm{Cov}[\mathbf{X}_T^{(2)}(t_1,\mathbf{u}_1),\mathbf{X}_T^{(2)}(t_1,\mathbf{u}_1)]+2\alpha_1\alpha_2\mathrm{Cov}[\mathbf{X}_T^{(2)}(t_1,\mathbf{u}_1),\mathbf{X}_T^{(2)}(t_2,\mathbf{u}_2)] \\
&\quad + \alpha_2^2\mathrm{Cov}[\mathbf{X}_T^{(1)}(t_2,\mathbf{u}_2),\mathbf{X}_T^{(1)}(t_2,\mathbf{u}_2)]+\alpha_2^2\mathrm{Cov}[\mathbf{X}_T^{(2)}(t_2,\mathbf{u}_2),\mathbf{X}_T^{(2)}(t_2,\mathbf{u}_2)]\} \\
&= \alpha_1^2\left((t_1\wedge s)k_1(\mathbf{u}_1,\mathbf{u}_1)+(t_1-s)_+k_2(\mathbf{u}_1,\mathbf{u}_1)\right) \\
&\quad + a_2^2\left((t_2\wedge s)k_2(\mathbf{u}_2,\mathbf{u}_2)+(t_2-s)_+k_2(\mathbf{u}_2,\mathbf{u}_2)\right) \\
&\quad + 2\alpha_1\alpha_2\left((t_1\wedge t_2\wedge s)k_1(\mathbf{u}_1,\mathbf{u}_2)+(t_1\wedge t_2-s)_+k_2(\mathbf{u}_1,\mathbf{u}_2)\right) \\
&= \mathbb{V}[\alpha_1\mathbb{G}_{C_1,C_2,s}(t_1,\mathbf{u}_1)+\alpha_2\mathbb{G}_{C_1,C_2,s}(t_2,\mathbf{u}_2)]
\end{aligned}
\tag{16}
$$

with $\mathbb{E}[\mathbb{G}_{C_1,C_2,s}(t_1,\mathbf{u}_1)\mathbb{G}_{C_1,C_2,s}(t_2,\mathbf{u}_2)] = (t_1\wedge t_2\wedge s)k_1(\mathbf{u}_1,\mathbf{u}_2)+(t_1\wedge t_2-s)_+k_2(\mathbf{u}_1,\mathbf{u}_2)$ where the kernels for $i=1,2$ are given by

$$
k_i(\mathbf{u}_1,\mathbf{u}_2)=\sum_{k\in\mathbb{Z}}\mathrm{Cov}[\mathbb{1}\{W_0(i)\le\mathbf{u}_1\},\mathbb{1}\{W_k(i)\le\mathbf{u}_2\}]
$$

In order to prove asymptotic normality of $\sqrt{T}\sum_{j=1}^{2}\alpha_j\mathbf{X}_T(t_j,\mathbf{u}_j)$ we introduce the notation

$$
\mathbb{T}_T := \frac{\sqrt{T}}{\sigma}\sum_{j=1}^{2}\alpha_j\mathbf{X}_T(t_j,\mathbf{u}_j)=\sum_{j=1}^{T}S_{T,j}+o_P(1)
$$

with

$$
\begin{aligned}
S_{T,j} &= \frac{\alpha_1\mathbb{1}\{j\le\lfloor t_1T\rfloor\}}{\sigma\sqrt{T}}\left(\mathbb{1}\{\mathbf{U}_j\le\mathbf{u}_1\}-\mathbb{E}_{C_1,C_2,t}(t_1,\mathbf{u}_1)\right) \\
&+\frac{\alpha_2\mathbb{1}\{j\le\lfloor t_2T\rfloor\}}{\sigma\sqrt{T}}\left(\mathbb{1}\{\mathbf{U}_j\le\mathbf{u}_2\}-\mathbb{E}_{C_1,C_2,t}(t_2,\mathbf{u}_2)\right)
\end{aligned}
$$

and we use a central limit theorem for triangular arrays of strong mixing random variables [see Theorem 2.1 in [Liebscher(1996)], with $p=\infty$.] From the previous discussion it follows that $\lim_{T\to\infty}\mathbb{E}[\mathbb{T}_T^2]=1$ and thus, we have

$$
\lim_{T\to\infty}\sum_{j=1}^{T}(\operatorname*{ess\,sup}_{\omega\in\Omega}[\,|\,S_{T,j}\,|\,\mathbb{1}\{|S_{T,j}|>\epsilon\}])^2=0 \quad \text{a.s..}
$$

Similarly, it follows that the condition

$$
\lim_{T\to\infty}\sum_{j=1}^{T}(\operatorname*{ess\,sup}_{\omega\in\Omega}\,|\,S_{T,j}\,|\,)^2\le\text{const} \quad \text{a.s..}
$$

of Theorem 2.1 in [Liebscher(1996)] is also satisfied. Therefore this result shows that

$$
\sqrt{T}\sum_{j=1}^{2}\alpha_j\mathbf{X}_n(t_j,\mathbf{u}_j)=\frac{\sigma\mathbb{T}_T}{\sqrt{\mathbb{E}[\mathbb{T}_T^2]}}\overset{D}{\Longrightarrow}N(0,\sigma^2)
$$

where the asymptotic variance σ^2 is defined in (16). This proves the convergence of the finite dimensional distributions and completes the proof of the theorem.

Now, we can follow the asymptotic distribution of the centered $\hat{\mathbb{U}}_T(t,\mathbf{u})$, by using the continuous mapping theorem with $\hat{\mathbb{U}}_T(t,\mathbf{u})=\hat{\mathbb{C}}_T(t,\mathbf{u})-t\hat{\mathbb{C}}_T(1,\mathbf{u})$.

Corollary 2 *Under assumptions A1)-A4) we receive for* $t\in(0,1)$ *and* $\mathbf{u}\in[0,1]^N$

$$
\sqrt{T}\left(\hat{\mathbb{U}}_T(t,\mathbf{u})-\mathbb{U}(t,\mathbf{u})\right)\overset{d}{\Longrightarrow}\{A(t,\mathbf{u})\}_{t\in(0,1),\mathbf{u}\in[0,1]^N}, \tag{17}
$$

where $\hat{\mathbb{U}}_T(t,\mathbf{u}) = \hat{\mathbb{C}}_T(t,\mathbf{u}) - t\hat{\mathbb{C}}_T(1,\mathbf{u})$, $\mathbb{U}(t,\mathbf{u}) = \mathbb{E}_{C_1,C_2,s}(t,\mathbf{u}) - t\mathbb{E}_{C_1,C_2,s}(1,\mathbf{u})$ *and* $A(t,\mathbf{u}) = \mathbb{G}_{C_1,C_2,s}(t,\mathbf{u}) - t\mathbb{G}_{C_1,C_2,s}(1,\mathbf{u})$ *with covariance kernel*

$$a_{C_1,C_2,s}(t_1,\mathbf{u}_1,t_2,\mathbf{u}_2) = \mathbb{E}\left[A(t_1,\mathbf{u}_1)A(t_2,\mathbf{u}_2)\right]. \tag{18}$$

Now, we can complete the proof for Theorem 1. By Corollary 2 we have for $t \in (0,1)$ and $\mathbf{u} \in [0,1]^N$

$$\sqrt{T}\left(\hat{\mathbb{U}}_T(t,\mathbf{u}) - \mathbb{U}(t,\mathbf{u})\right) \overset{d}{\Longrightarrow} A(t,\mathbf{u}).$$

Thus, for every inner product space we have we can rewrite $\hat{L}_T(t) - L(t)$ for $t \in (0,1)$ as

$$\hat{L}_T(t) - L(t) = ||\hat{\mathbb{U}}_T(t,\mathbf{u}) - \mathbb{U}(t,\mathbf{u})||^2 + 2 < \mathbb{U}(t,\mathbf{u}), \hat{\mathbb{U}}_T(t,\mathbf{u}) - \mathbb{U}(t,\mathbf{u}) >_{L^2}.$$

Then, by Corollary 2 and the consistency of $\hat{\mathbb{U}}_T(\cdot)$ in (2) we get

$$\sqrt{T}\left(\hat{L}_T(t) - L(t)\right) \overset{\mathrm{d}}{\Longrightarrow} 2 < \mathbb{U}(t,\mathbf{u}), A(t,\mathbf{u}) >_{L^2}.$$

Thus, with the help of the continuous mapping theorem we receive

$$\sqrt{T}\left(\int_0^1 \hat{L}_T(t)dt - \int_0^1 L(t)dt\right) \overset{d}{\to} \int_0^1 2 < \mathbb{U}(t,\mathbf{u}), A(t,\mathbf{u}) >_{L^2} dt =: Q$$

$$\Leftrightarrow \sqrt{T}\left(\int_0^1 \hat{L}_T(t)dt - \frac{1}{3}s^2(1-s)^2||C_1(\mathbf{u}) - C_2(\mathbf{u})||^2\right) \overset{d}{\longrightarrow} Q,$$

where the random variable Q is normally distributed $N(0,\sigma^2_{C_1,C_2,s})$ with variance term

$$\sigma^2_{C_1,C_2,s} = 4\int_0^1\int_0^1 \mathbb{E}\left[< \mathbb{U}(t_1,\mathbf{u}), A(t_1,\mathbf{u}) >_{L^2} < \mathbb{U}(t_2,\mathbf{u}), A(t_2,\mathbf{u}) >_{L^2}\right] dt_1 dt_2.$$

D Covariance Bootstrap

Another approach next to the full bootstrap is to estimate the variance term of the limiting normal distribution[1]. Therefore, we have to estimate the covariance of the centred Gaussian process $d_{C_1,C_2,s}(t_1,t_2) = \mathbb{E}[D_{C_1,C_2,s}(t_1), D_{C_1,C_2,s}(t_2)]$ by using resampling, cf. Theorem 1. We also assume that our sample $\{\mathbf{X}_i\}_{i=1}^T$ is compounded of $\{\mathbf{X}_i\}_{i=1}^{\lfloor sT\rfloor}$ and $\{\mathbf{X}_i\}_{i=\lfloor sT\rfloor+1}^T$, such that there is only one breakpoint location in $\lfloor sT\rfloor$ with $s \in (0,1)$, i.e. $\{\mathbf{X}_i\}_{i=1}^{\lfloor sT\rfloor} \sim C_1(F(\mathbf{X}))$ and $\{\mathbf{X}_i\}_{i=\lfloor sT\rfloor+1}^T \sim C_2(F(\mathbf{X}))$. Then, the covariance bootstrap procedure suggests the following course of action:

i) Estimate the breakpoint location $\lfloor sT\rfloor$ with $\lfloor \hat{s}T\rfloor$, where $\hat{s}$ is determined by $\hat{s} := \underset{s\in(0,1)}{\operatorname{argmax}} \|\hat{\mathbb{U}}_T(s,\mathbf{u})\|_{L^2}$. Sample separately with replacement from $\{\mathbf{X}_i\}_{i=1}^{\lfloor \hat{s}T\rfloor}$ and $\{\mathbf{X}_i\}_{i=\lfloor \hat{s}T\rfloor+1}^T$ to obtain B bootstrap samples $\{\mathbf{X}_i^{(b)}\}_{i=1}^T$, for $b = 1,\dots,B$.

ii) Estimate B versions of the copula difference $\Delta_C^b = \|\hat{C}^{1:\hat{s}T}(\mathbf{u}) - \hat{C}^{\hat{s}T+1:T}(\mathbf{u})\|_{L^2}$, using the estimated break point location $\hat{s}T$ and re-sampled data $\{\mathbf{X}_i^{(b)}\}_{i=1}^T$, for $b = 1,\dots,B$.

iii) For $t_1, t_2 \in [0,1]$ compute separately

$$D_i^b(t_i) := < \sqrt{T}\left(\hat{\mathbb{U}}_T^b(t_i,\mathbf{u}) - \mathbb{U}^b(t_i,\mathbf{u})\right), \mathbb{U}^b(t_i,\mathbf{u}) >_{L^2}$$

for $i = 1,2$ using $\{\mathbf{X}_i^{(b)}\}_{i=1}^T$ for $b = 1,\dots,B$, where $\mathbb{U}^b(t_i,\mathbf{u}) = (\min\{\mathsf{\hat{s}},\mathsf{t_i}\} - \hat{s}t_i)\,\Delta_C^b$.

iv) Estimate the expected value given covariance of Theorem 1 for $t_1, t_2 \in (0,1)$ by the mean

$$\hat{d}_{C_1,C_2,\hat{s}}(t_1,t_2) := \frac{1}{B}\sum_{b=1}^B D_1^b(t_1)D_2^b(t_2).$$

v) Estimate the variance $\sigma^2_{C_1,C_2,s}$ from Theorem 1 by integrating out over t_1 and t_2, i.e

$$\hat{\sigma}^2_{C_1,C_2,\hat{s}} = 4\int_0^1\int_0^1 \hat{d}_{C_1,C_2,\hat{s}}(t_1,t_2)dt_1dt_2$$

and compute the q-quantile z_q of $N(0,\hat{\sigma}^2_{C_1,C_2,\hat{s}})$ where $q \in (0,1)$.

[1] Since we are only able to derive the limiting distribution in the case of known marginals, there is no theoretical evidence that the covariance bootstrap is applicable for sequentially estimated case.

The testing procedure is as follows: We reject the null of no relevant change $\|C_1(\mathbf{u}) - C_2(\mathbf{u})\|_{L^2} \leq \Delta$ if

$$\int_0^1 \hat{L}_T(t)dt > \frac{\hat{s}^2(1-\hat{s})^2}{3}\Delta^2 + \frac{z_q}{\sqrt{T}}. \tag{19}$$

The bootstrap and testing procedure can be easily adapted to the quantile case, i.e. $\mathbf{u}$ is fixed, by adapting step i) - iii). Note, the test given in equation (19) is an exact level α test if Δ is chosen as the copula difference $||C_1(\mathbf{u}) - C_2(\mathbf{u})||_{L^2}$ or $|C_1(\mathbf{q}) - C_2(\mathbf{q})|^2$. Otherwise the size is smaller than α. By the continuous mapping theorem we obtain that the left hand side of (19) converges weakly to a degenerated random variable if the copula difference is equal to zero (no break point). Consequently, the level of the proposed tests have practically size zero, whereas classical stationarity tests hold the asymptotic α-level. Thus, the power of the classical tests is usually larger than the power of the relevant change tests cosndiered here. For practitioners we suggest to run a classical test first, e.g. [Bücher and Ruppert(2013)] for the case of known marginals and [Bücher et al.(2014)Bücher, Kojadinovic, Rohmer, and Segers] in the case of sequentially estimated marginals. If the test rejects the null of stationarity, i.e. the copula difference is significantly larger than zero, estimate the break fraction and apply the proposed relevant change test.

E Simulations for the Covariance Bootstrap

The data generating process (DGP) is similar to the DGP used in the main paper. We recap the description of the DGP since we want the Supplement Appendix to be autonomous readable. Let

$$\mathbf{X}_t = [X_{1t}, X_{2t}]' = N_2\left(\mathbf{0}, \Sigma_t(\rho)\right), \tag{20}$$

where $N_2(\mathbf{0}, \Sigma_t(\rho))$ with $t = 1, ..., T$ describes the bivariate normal distribution with expectation vector zero and covariance matrix $\Sigma_t(\rho) = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and $\rho \in [-1, 1]$. We set ρ equal to -0.3 for $t = 1, ..., \frac{T}{2}$ and $\rho = 0.8$ for $t = \frac{T}{2} + 1, ..., T$. Thus, the breakpoint sT is chosen at $\frac{T}{2}$. We restrict the size analysis in this subsection to the two dimensional case $N = 2$. The following size study presents both L^2-norm based results and an analysis where we consider the specific point $\mathbf{u} = (0.6, 0.6)$. Note, the closer the quantile is to its boundaries, i.e. 0 or 1, the more observations are needed. Critical values of our tests are computed using the bootstrap algorithms from Sections D with $B = 300$ bootstrap replications. The tests are performed at the $\alpha = 0.05, 0.1$ significance level using 301 Monte Carlo replications. The computations were implemented in Matlab, parallelized and performed using CHEOPS, a scientific High Performance Computer at the Regional Computing Center of the University of Cologne (RRZK).
Table 1 presents the results of the relevant change tests under the null with Δ chosen as the estimated copula difference $|C_1(\mathbf{u}) - C_2(\mathbf{u})|$, where C_1 and C_2 are estimated by the consistent copula estimator

$$\hat{C}(\mathbf{u}) = \frac{1}{t_2 - t_1} \sum_{i=t_1}^{t_2} \mathbb{1}\{F_1(X_{i1}) \leq u_1, \ldots, F_N(X_{iN}) \leq u_N\}, \tag{21}$$

using realizations $\{\mathbf{X}_1, \ldots \mathbf{X}_{\lfloor \hat{s}T \rfloor}\}$ and $\{\mathbf{X}_{\lfloor \hat{s}T \rfloor + 1}, \ldots \mathbf{X}_T\}$. The breakpoint $\lfloor \hat{s}T \rfloor$ is estimated by

$$\hat{s} := \underset{s \in (0,1)}{\operatorname{argmax}} |\hat{\mathbb{U}}_T(s, \mathbf{u})|. \tag{22}$$

Table 1 reports the results of the relevant change tests under the null, where the functional difference between the copulas is determined by the L^2-norm. Similar to the quantile case we consider for the size analysis $\Delta := \|C_1(\mathbf{u}) - C_2(\mathbf{u})\|_{L^2}$ and accordingly $\hat{s} := \underset{s \in (0,1)}{\operatorname{argmax}} \|\hat{\mathbb{U}}_T(s, \mathbf{u})\|_{L^2}$. Collectively, the tests show good size properties and converges to the predetermined rejection level α if T gets larger. Overall, the covariance bootstrap shows good size properties for both, the quantile version

Table 1 Size using quantile version for Covariance Bootstrap

Copula with known marginals				
	$T = 300$	$T = 500$	$T = 750$	$T = 1000$
q_{95}	0.099	0.083	0.059	0.046
q_{90}	0.142	0.106	0.109	0.109

Table 1 reports the rejection rate of the relevant change test for data generated with the DGP described in (20) for known marginal distributions and sequential estimated marginals using the two distribution estimation methods, using $B = 300$ bootstrap replications. The copula difference is evaluated at $\mathbf{u} = (0.6, 0.6)$. In total, we conducted 301 Monte Carlo replications.

[2] For a detailed description of the quantile version of the test statistic we refer to the main paper.

Table 2 Size using the L^2-norm

Copula with known marginals (Covariance Bootstrap)				
	$T = 1000$	$T = 2000$	$T = 3000$	$T = 4000$
q_{95}	0.085	0.063	0.046	0.066
q_{90}	0.156	0.122	0.113	0.102

Table 2 reports the rejection rate of the relevant change test for data generated with the DGP described in (20) for known marginal distributions and sequential estimated marginals using the two distribution estimation methods, using $B = 300$ bootstrap replications. The copula difference is determined using the L^2-norm. In total, we conducted 301 Monte Carlo replications.

of the test and the test given in (19) by a moderate rate of bootstrap replications.

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