Monitoring multivariate variance changes

Katharina Pape, Dominik Wied, Pedro Galeano*

TU Dortmund, Universität zu Köln and Universidad Carlos III de Madrid

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Abstract

We propose a model-independent multivariate sequential procedure to monitor changes in the vector of componentwise unconditional variances in a sequence of p-variate random vectors. The asymptotic behavior of the detector is derived and consistency of the procedure stated. A detailed simulation study illustrates the performance of the procedure confronted with different types of data generating processes. We conclude with an application to the log returns of a group of *DAX* listed assets.

Keywords: Multivariate sequences; Online detection; Threshold function; Variance changes.

JEL Classification: C12, C14

*K. Pape: TU Dortmund, Fakultät Statistik, 44221 Dortmund, Germany. E-Mail: pape@statistik.tu-dortmund.de, Phone: +49 231 755 3869. D. Wied: Universität zu Köln (Institut für Statistik und Ökonometrie) and TU Dortmund (Fakultät Statistik), E-Mail: dwied@uni-koeln.de, Phone: +49 221 470 4514 (D. Wied). P. Galeano: Universidad Carlos III de Madrid, Departamento de Estadística, 28903 Getafe, Madrid, Spain. E-mail: pedro.galeano@uc3m.es, Phone: +34 916 818 901. Financial support by Deutsche Forschungsgemeinschaft (SFB 823, project A1) and Ministerio de Ciencia e Innovación grant ECO2012-38442 is gratefully acknowledged.

1. Introduction

Variances and volatilities are of major interest in financial contexts since they can be used to evaluate the risk of financial instruments. From an empirical point of view, it is clear that, in general, variances of stock returns cannot be taken as constant over a long period of time and as they tend to increase heavily in times of instability, see, e.g., Schwert (2011) or Charles and Darné (2014). There are many papers which deal with models for time-varying *conditional* variances; a prominent one is Bollerslev (1986) which proposes the well-known $GARCH(p,q)$ model. However, it is far from clear whether also the *unconditional* variances should be modeled in a constant or time-varying way. For example, there might be a big, long-lasting increase in the fluctuations of financial returns which cannot be captured by a standard $GARCH$ model with constant parameters.

This paper is concerned with possible structural changes in variances. Also, the question of dating the breakpoint arises once such a possible change in the unconditional variance is detected. In particular, our aim is to monitor the vector of variances of a series of random vectors of moderate dimension p. In practice, it is important to get informed about changes in the model structure as soon as possible after their appearance, to be able to react to the change. Hence, a monitoring procedure could be of more practical relevance than a retrospective test. Specifically, we focus on changes in the individual variances. One reason for this is that, in financial contexts, the variances exhibit information about the idiosyncratic risk of the individual assets. Bissantz et al. (2011) show that the impact of fluctuations is distinctively larger for volatilities than for correlations. Furthermore, if constancy of the correlations is tested, constancy of the variances is a crucial issue. For example, the test by Wied et al. (2012b) does not allow for arbitrarily changing variances, while a test in the spirit of Dette et al. (2015) does. On the other hand, one can expect that Wied et al. (2012b) is more efficient than Dette et al. (2015) since the latter is based on nonparametric estimation. Additionally, the inclusion of the covariances would strongly increase the dimension of vectors and matrices that have to be estimated, which leads to a remarkable weakening of the procedure. This made us refrain from monitoring the whole covariance matrix as proposed by Aue et al. (2009b) in the retrospective case. If the

covariances were monitored as well, the vector of moments that are supervised would tend to be of unpropitious high dimension even if the time series itself were of moderate dimension. This would lead to unsatisfactory size and power properties.

The procedure is based on the monitoring technique proposed by Chu et al. (1996) who used a similar but univariate sequential method based on fluctuations to detect structural breaks in the parameter vector of a linear regression model. Their approach was refined and further investigated by Horváth et al. (2004), Aue et al. (2006) and Aue et al. (2009a), among others. Groen et al. (2013) expanded the approach to the multivariate case. Nevertheless, even if the main goal in Groen et al. (2013) is to monitor structural changes in multivariate sequences, as it is in our case, the focus is put on the parameters of the linear regression model and not in the individual variances of the components of the sequence, thus making their approach different to our own. While Berkes et al. (2004) and Aue et al. (2011) extended the field of applications to the monitoring of parameters in univariate $GARCH(p,q)$ models and high frequency portfolio betas, respectively, Wied and Galeano (2013) presented a model-independent monitoring procedure to detect changes in the correlation of bivariate time series. Retrospective methods to detect changes in the covariance or correlation structure of random vectors were proposed by Aue et al. (2009b) respectively by Wied (2015). Since we investigate the performance of the proposed procedure confronted with a special type of multivariate $GARCH$ model, it remains to mention that methods which address this model class have been proposed by Amado and Teräsvirta (2014) and Silvennoinen and Teräsvirta (2016), among others. They suggest to test constancy of the unconditional variance in univariate $GARCH(p,q)$ models using LM tests in a retrospective setting. We combine the sequential approach with the attempt to survey moments of multivariate processes.

Since whole random vectors often provide more information than single random variables, the additional information should be used to develop a procedure which enables monitoring the vector of variances of the individual components. Although we are only interested in the vector of variances, such a procedure could be able to detect changes in one or several variances more efficiently than using several univariate procedures similar to the correlation monitoring

procedure proposed by Wied and Galeano (2013) that could be adapted to the situation by using the Bonferroni-Holm method.

The rest of the paper is organized as follows. Section 2 introduces the proposed monitoring procedure for detecting a changepoint in the vector of variances of a multivariate random variable as soon as possible, and derives the asymptotic properties of the chosen detector. Sections 3 and 4 present a detailed simulation study and an application to real data that illustrate the behavior of the procedure in finite settings. Finally, section 5 provides some conclusions. All proofs are presented in Appendix A of the supplementary material. Appendix B of the supplementary material contains simulation results for the i.i.d.-setting, while Appendix C contains tables with the simulation results from Section 3.

2. The monitoring procedure

Let $(X_t,\ t \in \mathbb{Z})$ be a sequence of p dimensional random vectors whose elements possess finite fourth moments and cross moments. W.l.o.g. we assume that $\mathsf{E}(X_t) = 0$, $t \in \mathbb{Z}$. This assumption is natural in financial contexts when one considers daily log returns of financial assets. The value of interest is the vector of variances associated with the single components of the random vector $X_t = \left(X_{t1}, \ldots, X_{tp}\right)'$ denoted by

$$\sigma_t^2 = \left(\sigma_{t1}^2, \ldots, \sigma_{tp}^2\right)' \quad \text{with} \quad \sigma_{tj}^2 = \mathsf{Var}\left(X_{tj}\right) = \mathsf{E}\left(X_{tj}^2\right), \quad \text{for } j = 1, \ldots, p.$$

Since often a non time-varying variance structure cannot be assumed, we are interested in a monitoring procedure that supervises the vector of variances and reports a potential structural break as soon as possible after it has occurred. We estimate the variances from growing subsamples and compare them with estimators obtained from a reference data set that is assumed not being affected by a variance change. In the context of sequential testing, this implies using a historical sample to obtain a first estimator of the vector of variances. In the monitoring period the historical sample is successively extended by p dimensional data points that are used to update the chosen detector. This reflects the fact that data like daily asset or index prices is

observed step by step. The formal constancy assumption for the historical period of length m is

Assumption 1. $\sigma_1^2 = \ldots = \sigma_m^2$, *where m is a positive integer.*

The validity of this assumption can be checked by performing retrospective changepoint detection procedures on the historical data set, for instance a procedure similar to the one proposed by Wied et al. (2012b). In practice, it is usually possible to find a sufficient amount of historical data points with a stable variance structure.

In the following, we consider the null hypothesis of equal vectors of variances

$$\mathsf{H}_0 : \sigma_1^2 = \ldots = \sigma_m^2 = \sigma_{m+1}^2 = \ldots$$

versus the alternative H_1 that σ_t^2 changes at one or several unknown points in the monitoring period. Let $\mathcal{F}_B$ be the set of functions $f : [0, \infty) \to \mathbb{R}$ that are bounded and integrable on the interval $[0, B + 1]$ such that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{t=1}^{[m(B+1)]} f\left(\frac{t}{m}\right) = \int_0^{B+1} f(z)dz.$$

Throughout the paper the variable B indicates how much longer the monitoring period is compared to the historical data set. We consider the alternative H_1 that the individual variances can be decomposed as

$$\mathsf{Var}\left(X_{tj}\right) = \sigma_{tj}^2 = \bar{\sigma}_j^2 + g_j\left(\frac{t}{m}\right), \quad j = 1, \ldots, p, \quad t \in \mathbb{Z}, \tag{1}$$

with $\bar{\sigma}_j^2$, $j = 1, \ldots, p$, time-invariant constants and structural stability determining functions $g_j(\cdot) \in \mathcal{F}_B$, $j = 1, \ldots, p$. Then, $g_j(z) = 0$, for $z \in [0, 1]$ and for all $j = 1, \ldots, p$. However, for at least one $j \in \{1, \ldots, p\}$, $\int_1^{B+1} \left|g_j(z)\right| dz > 0$. i.e., the variance of the j-th vector component is affected by a structural change.

In order to derive asymptotic results concerning size and power of the procedure that will be presented below, some assumptions have to be imposed first. They are counterparts of the assumptions (A1)-(A3) in Wied et al. (2012a) and to (A2)-(A4) in Wied and Galeano (2013),

respectively.

Assumption 2. *For* $U_t := \left(X_{t1}^2 - \mathsf{E}\left(X_{t1}^2\right), \ldots, X_{tp}^2 - \mathsf{E}\left(X_{tp}^2\right)\right)'$ *and* $S_j := \sum_{t=1}^{j} U_t$, $j \in \mathbb{N}$, *we have*

$$\lim_{m\to\infty} \mathsf{E}\left(\frac{1}{m} S_m S_m'\right) =: D_p$$

where D_p *is a finite and positive definite matrix.*

Assumption 3. *The r-th absolute moments of the components of* U_t *are uniformly bounded for some* $r > 2$.

Assumption 4. *The process* $(X_t, t \in \mathbb{Z})$ *is* L_2*-near epoch dependent, see e.g.Davidson (1994), with size* $-\frac{r-1}{r-2}$, *where r is from Assumption 3, and constants* $(c_t), t \in \mathbb{Z}$, *on a sequence* $(Y_t), t \in \mathbb{Z}$, *which is* α*-mixing of size* $\phi^* := -\frac{r}{r-2}$, *i.e.*

$$\|X_t - \mathsf{E}\left(X_t | \sigma(Y_{t-l}, \ldots, Y_{t+l})\right)\|_2 \leq c_t v_l$$

with $\lim_{l\to\infty} v_l = 0$, *such that* $c_t \leq 2\|U_t\|_2$ *with* U_t *from Assumption 3 and* $\|\cdot\|_2$ *the* L_2*-norm.*

Assumption 4 allows for serial dependence insofar as it decays sufficiently fast. Compared to pure α-mixing, near epoch dependence with respect to an α-mixing process is the more general concept. Near epoch dependence can be stated for different types of processes, e.g. Gallant and White (1988) verified this property for finite order $ARMA(p, q)$ processes with roots lying outside the unit circle and Hansen (1991) for $GARCH(1, 1)$ processes.

The proposed procedure is inspired by the model-independent fluctuation test proposed by Wied and Galeano (2013) for the detection of changes in the correlation of two sequences of random variables. The fluctuations arise from the comparison of variance estimates calculated from several subsamples of the available data. Denote by $\left[\hat{\sigma}^2\right]_k^l$ the estimate of the vector of variances calculated from X_k to X_l, $k < l$:

$$\left[\hat{\sigma}^2\right]_k^l = \left(\left[\overline{X_1^2}\right]_k^l, \ddots, \left[\overline{X_p^2}\right]_k^l\right) \quad \text{with} \quad \left[\overline{X_j^2}\right]_k^l = \frac{1}{l-k+1} \sum_{t=k}^{l} X_{tj}^2, \text{ for } j = 1, \ldots, p.$$

Then, estimates of the vector of variances from growing samples are compared to estimates resulting from the historical data. Under the hypothesis of equal vectors of variances the estimate vectors should not differ too much. In Wied and Galeano (2013) the fluctuations could easily be defined as the absolute differences of the two correlation estimates. In the multivariate setting two vectors have to be compared. Let

$$V_k = \frac{k}{\sqrt{m}} \hat{D}_p^{-\frac{1}{2}} \left(\left[\hat{\sigma}^2 \right]_{m+1}^{m+k} - \left[\hat{\sigma}^2 \right]_1^m \right) = \frac{k}{\sqrt{m}} \hat{D}_p^{-\frac{1}{2}} \begin{pmatrix} \left[\overline{X_1^2} \right]_{m+1}^{m+k} - \left[\overline{X_1^2} \right]_1^m \\ \vdots \\ \left[\overline{X_p^2} \right]_{m+1}^{m+k} - \left[\overline{X_p^2} \right]_1^m \end{pmatrix}, \; k \in \mathbb{N},$$

with $\hat{D}_p$ a kernel-based estimator of the matrix D_p defined in Assumption 2 that is calculated from the first m observations. Define

$$\tilde{V}_t = \frac{1}{\sqrt{m}} \tilde{U}_t \quad \text{with} \quad \tilde{U}_t = \left(X_{t1}^2 - \left[\overline{X_1^2} \right]_1^m, \ldots, X_{tp}^2 - \left[\overline{X_p^2} \right]_1^m \right)'.$$

As $m \to \infty$, a consistent estimator of D_p is given by

$$\hat{D}_p = \sum_{t=1}^{r} \sum_{u=1}^{r} k\left(\frac{t-u}{\delta_r} \right) \tilde{V}_t \tilde{V}_u{}' \quad \text{with} \quad k(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & otherwise. \end{cases}$$

Here, $k(x)$ is the Bartlett kernel and δ_r the bandwidth that determines up to which lag outer products of the vectors $\tilde{V}_t$ are used to calculate the estimator. The choice of the kernel is motivated by the approach in Wied et al. (2012a). However, a different bandwidth was chosen since simulations show that $\delta_r = \left\lfloor r^{\frac{1}{4}} \right\rfloor$ is the most suitable one compared to alternative bandwidths. Consistency of the estimator $\hat{D}_p$ is necessary for deriving the asymptotic distribution of the detector that is presented later in Theorem 1.

As it is desirable to construct a one dimensional detector that can be compared to the values of a univariate threshold function, possible solutions are to use either the Euclidian norm or a quadratic form of the vector of differences. The latter was considered by Aue et al. (2009b)

in the retrospective setting. The detector used by our monitoring procedure is the Euclidean norm of V_k. The value of $\|V_k\|_2$ is calculated online for every k in the monitoring period. The procedure stops when the detector exceeds the value of a scaled threshold function $w(\cdot)$. As soon as this happens, the null hypothesis of no variance change cannot be taken as valid anymore and is rejected. Accordingly, the stopping rule can be defined as

$$\tau_m = \min\left\{k \leq [mB] : \|V_k\|_2 > c \cdot w\left(\frac{k}{m}\right)\right\}, \tag{2}$$

with $w(\cdot)$ a positive and continuous function and c a constant chosen such that under a valid null hypothesis $\lim_{m\to\infty} \mathsf{P}(\tau_m < \infty) = \alpha \in (0,1)$ is the test significance level. Along the lines of Aue et al. (2011) we write $\tau_m < \infty$ to indicate that the detector has exceeded the threshold function $cw(\cdot)$ in the monitoring period which implies a rejection of the hypothesis of equal vectors of variances. If $\|V_k\|_2$ does not exceed the corresponding value of the threshold function in the whole monitoring period, we write $\tau_m = \infty$, see Aue et al. (2011). This leads to our main result:

Theorem 1. *Under* H_0*, Assumptions 1-4 and for any* $B > 0$*,*

$$\lim_{m\to\infty} \mathsf{P}\left(\tau_m < \infty\right) = \lim_{m\to\infty} \mathsf{P}\left(\sup_{b\in[0,B]} \frac{\|V_{\lfloor m\cdot b\rfloor+2}\|_2}{w\left(b\right)} > c\right) = \mathsf{P}\left(\sup_{b\in[0,B]} \frac{\|\mathcal{G}(b)\|_2}{w\left(b\right)} > c\right), \tag{3}$$

where $\left\{\mathcal{G}(b) = \left(G_1(b), \ldots, G_p(b)\right)',\ b \in [0,B]\right\}$ *is a p-variate stochastic process whose component processes are p independent mean zero Gaussian processes* $\left\{G_j(b), b \in [0,B]\right\}$ *with covariance function* $\mathsf{E}\left(G_j(k)G_j(l)\right) = \min(k,l) + kl$*, for* $j = 1, \ldots, p$*.*

Theorem 1 establishes the asymptotic behavior of the monitoring procedure based on the stopping rule τ_m in (2). As argued in detail in Aue et al. (2011) and Wied and Galeano (2013) the limiting probability in (3) can be led back to the behavior of p independent standard Brownian motions $\left\{W_j\left(b\right) : b \in [0,1]\right\}$, $j = 1, \ldots, p$. Since $\left\{G_j\left(b\right) : b \in [0,B]\right\}$ has the same distribution

as $\left\{(1+b)\, W_j\left(b/\left(1+b\right)\right) : b \in [0, B]\right\}$, for $j = 1, \ldots, p$, we have

$$\sup_{b\in[0,B]} \frac{\|\mathcal{G}(b)\|_2}{w\left(b\right)} = \sup_{b\in[0,B]} \frac{\sqrt{\sum_{j=1}^{p}\left[G_j(b)\right]^2}}{w\left(b\right)} \stackrel{L}{=} \sup_{b\in[0,B]} \frac{(1+b)\,\sqrt{\sum_{j=1}^{p}\left[W_j\left(\frac{b}{1+b}\right)\right]^2}}{w\left(b\right)} \tag{4}$$

with $A_1 \stackrel{L}{=} A_2$ indicating that A_1 and A_2 possess the same distribution. As in Wied and Galeano (2013) the threshold function $w(\cdot)$ can be chosen as

$$w\left(b\right) = (1+b) \cdot \max\left\{\left(\frac{b}{1+b}\right)^{\gamma}, \epsilon\right\} \tag{5}$$

with $\gamma \in \left[0, \frac{1}{2}\right)$ and $\epsilon > 0$ a fixed constant that solely serves to guarantee the divisibility by $w(\cdot)$ and can be chosen arbitrarily small in applications. The parameter γ can be used to adjust the procedure to have the best performance in a certain expected situation. As discussed in Wied and Galeano (2013) in detail, there is a trade off between the aim to detect arisen structural breaks as soon as possible and the purpose to reduce the probability of type I errors to the significance level. A value of γ chosen closely to $\frac{1}{2}$ tends to cause a soon rejection of the null hypothesis. This is desirable if a structural change is expected to take place shortly after the beginning of the monitoring period, but also tends to produce type I errors, while the null is still valid. In contrast, using a smaller value for γ rather results in a reduction of type I errors but also leads to a testing routine that is less capable of indicating structural breaks arising early in the monitoring period. Further simulations show that large values of γ lead to unacceptable high percentages of falsely rejected null hypotheses, especially for higher dimensions of the random vectors under supervision. Hence, in the following no γ values larger than 0.25 are considered. Substituting $w(\cdot)$ from (5) and defining $u = \frac{b}{1+b}$ as well as $s = \frac{u(1+B)}{B}$ allows expression (4) to be written along the lines of Wied and Galeano (2013) as

$$\sup_{b\in[0,B]} \frac{\|\mathcal{G}(b)\|_2}{w\left(b\right)} \stackrel{L}{=} \sup_{u\in\left[0,\frac{B}{1+B}\right]} \frac{\sqrt{\sum_{i=1}^{p}\left[W_i\left(u\right)\right]^2}}{\max\left\{u^{\gamma}, \epsilon\right\}} \stackrel{L}{=} \sup_{s\in[0,1]} \left(\frac{B}{1+B}\right)^{\frac{1}{2}-\gamma} \frac{\sqrt{\sum_{i=1}^{p}\left[W_i\left(s\right)\right]^2}}{\max\left\{s^{\gamma}, \epsilon\left(\frac{1+B}{B}\right)^{\gamma}\right\}}.$$

Since, under the assumptions of Theorem 1,

$$\lim_{m\to\infty} \mathsf{P}(\tau_m < \infty) = \mathsf{P}\left(\sup_{s\in[0,1]} \left(\frac{B}{1+B}\right)^{\frac{1}{2}-\gamma} \frac{\sqrt{\sum_{i=1}^{p} [W_i(s)]^2}}{\max\left\{s^\gamma, \epsilon\left(\frac{1+B}{B}\right)^\gamma\right\}} > c \right),$$

Monte Carlo simulations can be used to obtain the constant $c = c(\alpha)$ such that

$$\mathsf{P}\left(\sup_{s\in[0,1]} \left(\frac{B}{1+B}\right)^{\frac{1}{2}-\gamma} \frac{\sqrt{\sum_{i=1}^{p} [W_i(s)]^2}}{\max\left\{s^\gamma, \epsilon\left(\frac{1+B}{B}\right)^\gamma\right\}} > c(\alpha) \right) = \alpha,$$

for any $\alpha \in (0,1)$. Thus, the probability of a false alarm is approximately α if m is large enough.

		$\alpha = 0.01$			$\alpha = 0.05$			$\alpha = 0.1$		
γ	B	$p = 2$	$p = 5$	$p = 10$	$p = 2$	$p = 5$	$p = 10$	$p = 2$	$p = 5$	$p = 10$
0	0.5	1.9062	2.3268	2.8462	1.5514	2.0265	2.5802	1.3991	1.8817	2.4146
	1	2.2924	2.8653	3.5217	1.9039	2.4659	3.1544	1.7003	2.3122	2.9439
	2	2.6246	3.3371	4.0214	2.1915	2.8704	3.6375	1.9737	2.6447	3.4005
0.25	0.5	2.5231	3.1579	3.8898	2.1439	2.7760	3.4385	1.9431	2.5872	3.2596
	1	2.8124	3.4880	4.2737	2.3881	3.0361	3.8051	2.1627	2.8457	3.6051
	2	2.9854	3.7461	4.5824	2.5351	3.2927	4.1315	2.3001	3.0523	3.8723

Table 1: Simulated critical values $c(\alpha)$.

Simulated critical values for all combinations of $p \in \{2, 5, 10\}$, $B \in \{0.5, 1, 2\}$, $\gamma \in \{0, 0.25\}$ and for significance levels of $\alpha \in \{0.01, 0.05, 0.1\}$ can be taken from Table 1. To obtain the values of c, 10.000 Brownian motions are simulated on a grid of 10.000 equidistant points.

Up to now, we have focused on the behavior of the detector under the null hypothesis. In the considered case the alternative is rather broad including scenarios with a single or multiple structural breaks in one or several vector components as well as variance changes of minor or major magnitude. This suggests investigating the testing power against local alternatives.

Assumption 5. *For the process* $(X_t,\ t \in \mathbb{Z})$ *with* $X_t = \left(X_{t1}, \ldots, X_{tp}\right)'$ *the variances of the individual vector components can be decomposed as*

$$\mathsf{Var}\left(X_{j,t}\right) = \bar{\sigma}_j^2 + \frac{1}{\sqrt{m}} g_j\left(\frac{t}{m}\right), \quad j = 1, \ldots, p,$$

with $\bar{\sigma}_j^2$ *and* $g_j(\cdot)$, $j = 1, \ldots, p$, *as in (1).*

Theorem 2. *Under a sequence of local alternatives, Assumptions 1-5 and for any* $B > 0$

$$\lim_{m\to\infty} \mathsf{P}\left(\tau_m < \infty\right) = \lim_{m\to\infty} \mathsf{P}\left(\sup_{b\in[0,B]} \frac{\|V_{\lfloor m\cdot b\rfloor+2}\|_2}{w\left(b\right)} > c\right) = \mathsf{P}\left(\sup_{b\in[0,B]} \frac{\|\mathcal{G}(b) + h(b)\|_2}{w\left(b\right)} > c\right),$$

with $\left\{\mathcal{G}(b) = \left(G_1(b) \ldots G_p(b)\right)',\ b \in [0, B]\right\}$ *as in Theorem 1 and* $h(\cdot) = H \cdot \left(h_1(\cdot), \ldots, h_p(\cdot)\right)'$. *Up to a constant,* H *is the limit of* $\hat{D}_p$ *under* H_0, *while the function* $h_j(b) \coloneqq \int_1^{b+1} g_i(u)du = 0$ *for all* $b \in [0, B]$ *if and only if the* j*-th component is not affected by a variance change.*

Theorem 2 yields that even a small variance change in just one single component can be detected with high probability if the historical period is large enough. To obtain general statements about the testing power, the magnitude of a variance change is assumed to tend to ∞. This can be modeled by defining one of the structural stability determining functions $g_j(\cdot)$, $j = 1, \ldots, p$, as a scaled function $g_*(\cdot)$ and assume the scaling factor to tend to ∞ implying an increasing magnitude of a shift in the respective component of the vector of variances.

Assumption 6. *At least one of the structural stability determining functions* $g_j(\cdot) \in \mathcal{F}_B$ *with* $\int_1^{b+1} \left|g_j(z)\right| dz > 0$ *can be decomposed as* $g_j(\cdot) = M \cdot g_*(\cdot)$ *with* $g_*(\cdot) \in \mathcal{F}_B$.

Theorem 3. *Under the alternative of at least one structural break in the vector of variances in the monitoring period and Assumption 6, let* $\mathsf{P}_{\mathsf{H}_1}(M)$ *be the probability that the detector exceeds the threshold function during the monitoring period for given* M. *Let* $g_*(\cdot) \in \mathcal{F}_B$ *be arbitrary but fixed. Under Assumptions 1-4 and 6, for every* $\varepsilon > 0$ *there exists an* M_ε *such that for all* $M > M_\varepsilon$

$$\lim_{m\to\infty} \mathsf{P}_{\mathsf{H}_1}(M) > 1 - \varepsilon.$$

Theorem 3 yields that a variance change of sufficiently high magnitude will be detected with given probability if the length of the historical period tends to ∞ even if just one single component is affected by the change or if multiple components experience contrary variance changes. If the detector actually exceeds the threshold function, the presence of a structural change is indicated. This leads to the challenge to determine the location of the changepoint. This does

not necessarily have to coincide with the first hitting time τ_m. In fact, an abrupt change of the variances will often take time to affect the detector strongly enough to get identified by the procedure. A possible estimator of the changepoint location is a multivariate equivalent to the one used by Wied et al. (2012a) and Wied and Galeano (2013):

$$\hat{k} = \underset{1\leq j\leq \tau_m - 1}{\arg\max} D_{j,\tau_m} \quad \text{with} \quad D_{j,\tau_m} := \frac{j}{\sqrt{\tau_m}} \hat{D}^{-\frac{1}{2}} \left\| \begin{pmatrix} \left[\overline{X_1^2}\right]_{m+1}^{m+j} - \left[\overline{X_1^2}\right]_{m+1}^{m+\tau_m-1} \\ \vdots \\ \left[\overline{X_p^2}\right]_{m+1}^{m+j} - \left[\overline{X_p^2}\right]_{m+1}^{m+\tau_m-1} \end{pmatrix} \right\|_2 .$$

This type of estimator led to satisfying results in the univariate case, hence we use it to estimate the location of an indicated changepoint. However, a detailed analysis of the estimator's properties lies beyond the scope of this paper. The performance of the proposed procedure as well as the properties of the first hitting times τ_m, the estimated changepoint locations $\hat{k}$ and the estimated location fractions $\hat{\lambda} = \frac{\hat{k}}{mB}$ will be investigated in the following section.

3. Simulations

This section is devoted to a performance analysis of the proposed monitoring procedure in finite samples. Each of the regarded scenarios is constructed using different tuning parameters. First, the dimension of the random vectors is chosen as $p \in \{2, 5, 10\}$. Since all of the asymptotics are based on the length of the historical period tending to ∞, large values of m are considered. We choose $m \in \{500, 1.000, 2.000\}$. In the context of financial data like asset returns, these values correspond to time periods of approximately 2, 4 and 8 years. It is important to note that smaller values of m may lead to noninvertible estimates of D_p in practice, especially for higher dimensions p. On the other hand, since the historical period must be assumed to be free from variance changes, larger values for m can hardly be found in practice. Furthermore, we choose $B \in \{0.5, 1, 2\}$ implying that the monitoring period is shorter, of the same length or longer than the historical period. Finally, the parameters in the threshold function $w(\cdot)$ have to be specified: ϵ is chosen as 10^{-6} in all of the following simulation settings and $\gamma \in \{0, 0.25\}$. These values

represent the aim to detect changes that are expected to occur earlier or later in the monitoring period. The theoretical size used for all of the simulations is $\alpha = 0.05$. In each case 10.000 time series are simulated. To simulate $c(\alpha)$, 10.000 Brownian motions are simulated on a grid of 10.000 equidistant points.

Monitoring scalar $BEKK$ time series

In Appendix B, we present simulation results for the i.i.d. case. There, the random vectors under consideration only show dependence between the individual vector components. In practice, time series that additionally exhibit serial dependence, which is permitted in moderate magnitude by Assumption 4, are of higher interest. In financial contexts, it may be desirable to detect changes in the vector of unconditional variances of random vectors whose conditional covariance matrices are expected to be time-varying. A common way to model this behavior and to explain volatility clusters that usually can be observed in financial time series is to use a multivariate $GARCH$ model. Assume

$$X_t = H_t^{\frac{1}{2}} \varepsilon_t, \tag{6}$$

where $(\varepsilon_t,\ t \in \mathbb{Z})$ is a sequence of i.i.d. $\mathbb{R}^p$-valued random vectors, $H_t^{\frac{1}{2}}$ is the square root of the conditional covariance matrix $H_t = Cov(X_t|\mathcal{I}_t)$ and $\mathcal{I}_t = \sigma(X_{t-1}, X_{t-2}, \ldots)$ is the information set at time t. Since Bollerslev (1986) states that even $GARCH$ models of low order are able to explain the behavior of many financial time series well, we will focus on models that are solely based on first order lagged conditional covariance matrices and observations. To specify the conditional covariance matrix we use the two parameter model, see Ding and Engle (2001), that arises from the scalar diagonal model when performing variance targeting as in Engle and Mezrich (1996) and that is a special case of the $BEKK(1,1,1)$ model proposed by Engle and Kroner (1995). We will refer to the scalar BEKK model in the following. The conditional

covariance matrix is recursively defined by

$$H_t = (1 - \alpha - \beta) H + \alpha X_{t-1} X_{t-1}' + \beta H_{t-1}, \tag{7}$$

where α and β are positive scalars with $|\alpha + \beta| < 1$ to guarantee stationarity and H is the unconditional covariance matrix of X_t, $t \in \mathbb{Z}$. The following lemma provides a useful help to check the validity of Assumption 3. Denote by $vec(\cdot)$ the operator that stacks the columns of a matrix in a vector of dimension p^2 and by $vech(\cdot)$ the operator that stacks only the lower triangular part including the main diagonal of a symmetric matrix in a vector of dimension $d := \frac{1}{2}p(p+1)$. Let D_p, L_p and K_{pp} with $vec(A) = D_p vech(A)$, $vech(A) = L_p vec(A)$ and $vec(A') = K_{pp} vec(A)$ for any $(p \times p)$ matrix A, be the duplication, elimination and commutation matrix, respectively. Furthermore, consider the matrix G_p that was defined in Hafner (2003) under the assumption that the innovations ε_t belong to the family of spherical distributions and possess fourth moments:

$$G_p := \frac{1}{3} E\left(\varepsilon_{1t}^4\right) \left[2 \left(L_p \otimes \left[D_p' D_p \right]^{-1} D_p' \right) \left(I_p \otimes K_{pp} \otimes I_p \right) \left(D_p \otimes D_p \right) + I_{d^2} \right]$$

The matrix G_p is used to transform the second order moments of the vector of conditional variances h_t into the second order moments of the conditional variances in vector ARMA representation, see Hafner (2003). Let $\Gamma := \mathsf{E}\left(X_t^2 X_t^{2\prime}\right)$ with $X_t^2 = vech(X_t X_t')$ be the matrix of fourth moments and cross moments of X_t.

Lemma 1. *If the innovations ε_t possess a spherical distribution with finite fourth moments, the matrix Γ exists if and only if all the eigenvalues of*

$$Z := \alpha^2 G_p + \left(2\alpha\beta + \beta^2 \right) I_{d^2} \tag{8}$$

are smaller than one in modulus.

The existence of finite fourth moments of the innovations is implied by Assumption 3. To allow for a moderate impact of past observations and variances on the conditional variances, we

chose the parameters as $(\alpha, \beta) = (0.03, 0.45)$. The innovation vectors ε_t are i.i.d. multivariate standard normal distributed and standardized t distributed with 8 degrees of freedom and thus possess a spherical distribution. All variable parameters are chosen as in the i.i.d. case. According to the dimension $p \in \{2, 5, 10\}$, the unconditional covariance matrix H is chosen as Σ_p from Appendix B. Results concerning the empirical size are presented in Tables 10 and 11 in Appendix C. To simplify the comparison to the i.i.d. case, these results are also illustrated in Figure 1 in Appendix B for $\gamma = 0$. The size is slightly higher in the case of serial dependence. The influence of parameter variations is similar to the i.i.d. case disregarding the fact that large values of B cause a size decrease when serial dependence is present.

Since each of the considered time series consists of a bundle of p univariate possibly correlated processes one could think about monitoring the single component series with the univariate equivalent of the procedure with detector

$$\|V_k\|_2 = \frac{k}{\sqrt{m}} \hat{D}_1^{-\frac{1}{2}} \left| \left[\hat{\sigma}_j^2\right]_{m+1}^{m+k} - \left[\hat{\sigma}_j^2\right]_1^m \right|, \quad j = 1, \ldots, p, \tag{9}$$

where $\hat{D}_1$ is a scalar. To fortify why the multivariate approach should be preferred to the univariate one we compare size and power when monitoring scalar $BEKK$ time series. To guarantee that asymptotically the probability of type I error, i.e. that one of the p detectors (9) exceeds the threshold function during the monitoring period, does not exceed $\alpha = 0.05$, the significance levels are adjusted by using the Bonferroni-Holm method. The simulated sizes are presented in Tables 10 and 11 in Appendix C. Also, the results for $\gamma = 0$ and $B = 1$ are illustrated in Figure 1 for the multivariate and the univariate procedure. The size is slightly lower for the univariate procedures, but the differences decline with m. Moreover, the problem of an increased error I probability when monitoring realizations of random vectors with heavy tailed distribution cannot be avoided by using univariate procedures.

Next, the multivariate and univariate procedure are confronted with alternative scenarios corresponding to those presented in Appendix B. The results for the multivariate procedure are given in Tables 12-15 and those for univariate procedures in Tables 16-17 in Appendix C. Figures 2

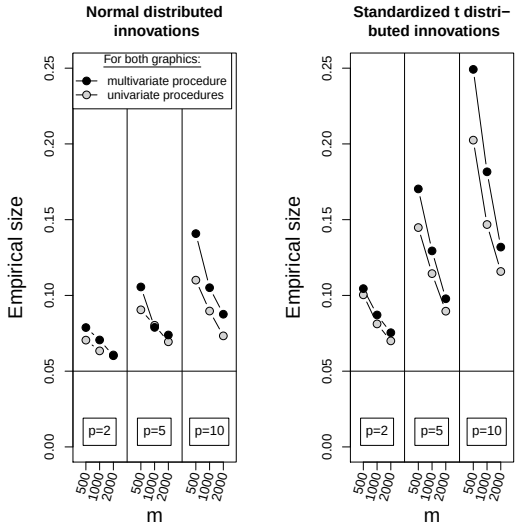

Figure 1: Size comparison: scalar BEKK time series under the use of a multivariate or several univariate procedures.

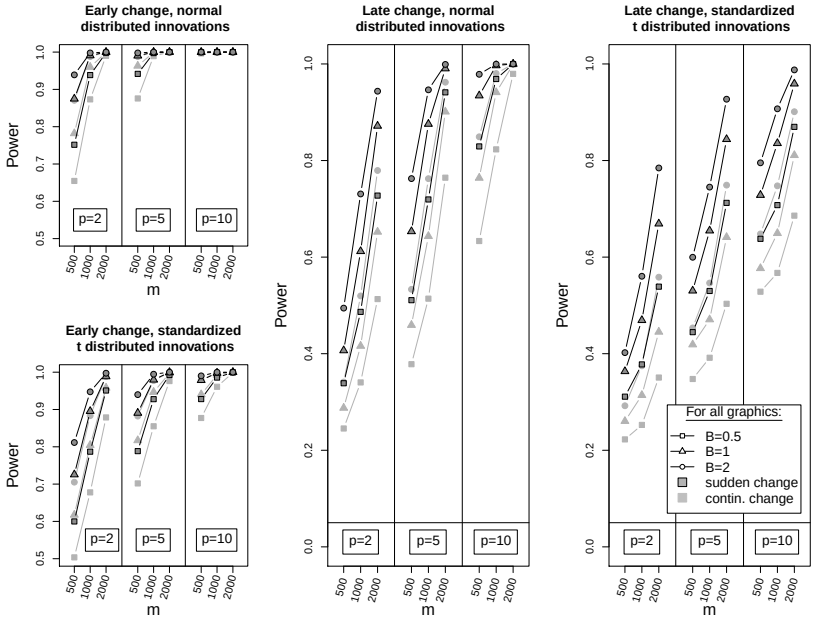

Figure 2: Power: scalar *BEKK* time series under the use of the multivariate procedure when all of the variances are affected by the change.

and 3 illustrate the performance of the multivariate procedure. Although the time series exhibit

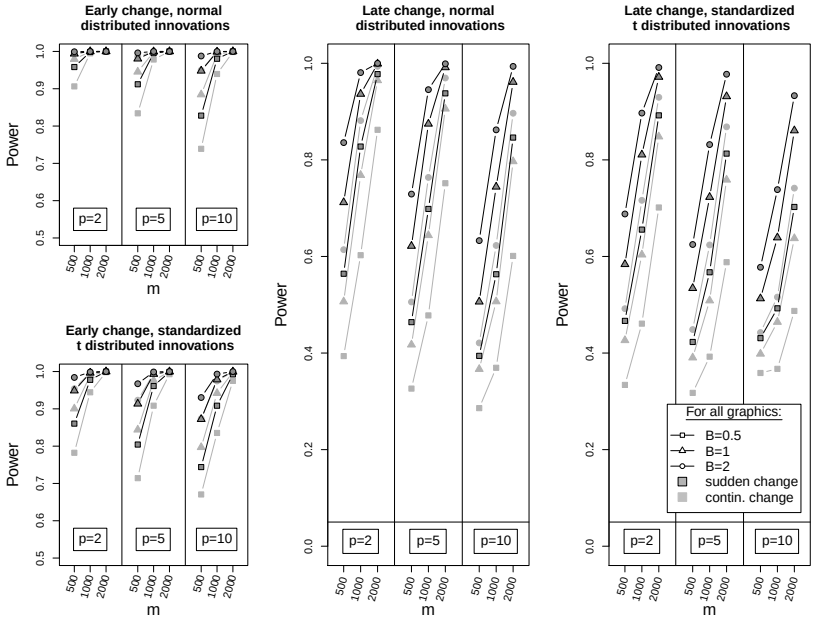

Figure 3: Power: scalar $BEKK$ time series under the use of the multivariate procedure when only one of the variances is affected by the change.

serial dependence, the results resemble those of the i.i.d. case very strongly. The power is slightly lower, but the impact of changes in the variable parameters remains the same.

The power results for the univariate procedure are illustrated in Figure 4. Since the results resemble strongly those of the multivariate procedure, the figure only shows the values for sudden changes and normal distributed innovations. To simplify the comparison, the graphic also contains the rejection frequencies for the multivariate procedure. As in the i.i.d. case, early changes can be detected reliably by both procedures while later changes are detected by the multivariate procedure more frequently. The latter one especially shows its strength when all of the variances experience a minor change or if just one of the variances is affected by a larger change but the historical period is rather short. Unfortunately, the higher power of the multivariate procedure goes along with a slightly increased size compared to the univariate procedure.

Since it is not only of interest to detect changes in the vector of variances but also to signalize their presence as soon as possible after they have occurred, we look closer at the properties of

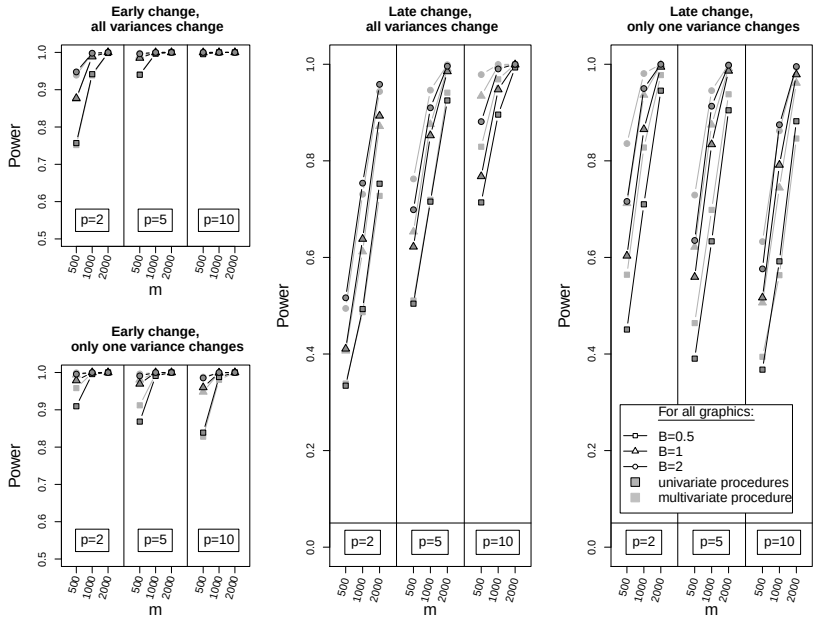

Figure 4: Power: scalar $BEKK$ time series with $N\left(0, \Sigma_p\right)$ distributed innovations under the use of several univariate procedures.

the first hitting times τ_m and the estimated changepoint locations $\hat{k}$ or location fractions $\hat{\lambda} = \frac{\hat{k}}{mB}$. While only the results for $\gamma = 0$ are visualized, the remaining parameters take the same values as before. To simplify the comparison for different sample lengths, Figures 5 and 6 illustrate the standardized delay times $d_m := \frac{\tau_m - k^*}{mB}$ and the bias of the location fraction estimator. Right under the boxplots, the graphics also show the means ± the standard deviations of the respective group. In general, the delay times decrease with growing length of the historical period and dimension. For small dimensions the procedure stops earlier if only a part of the variances is affected by a mayor change while for higher dimensions the delay time is shorter for smaller changes that affect more or all of the variances. This is in line with the power results discussed before.

Since the first hitting times determine which fraction of the data set is used to estimate the changepoint location, it is expected that the properties of the location fraction estimator resemble those of the first hitting times.

So far, we considered changes that affect the diagonal elements of the unconditional covariance matrix H directly. Now, assume that the variances of the innovation vectors ε_t jump from 1

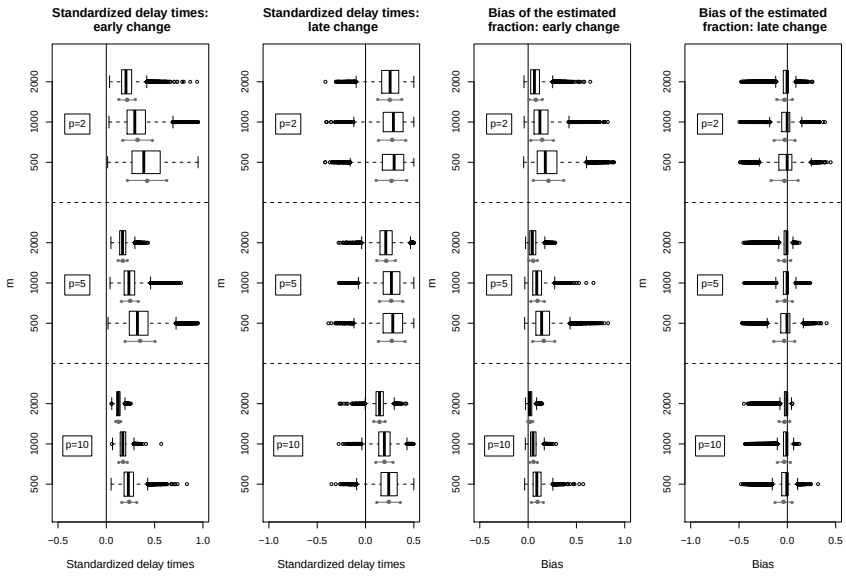

Figure 5: Properties of the standardized delay times and the estimated location fractions $\hat{\lambda}$ when monitoring scalar $BEKK$ time series with $N\left(0,\Sigma_p\right)$ distributed innovations and all of the variances are affected by the change.

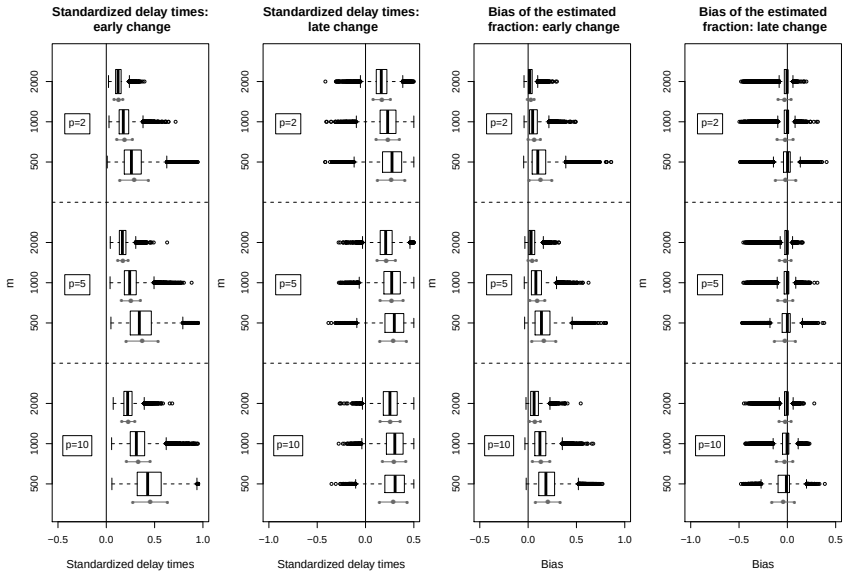

Figure 6: Properties of the standardized delay times and the estimated location fractions $\hat{\lambda}$ when monitoring scalar $BEKK$ time series with $N\left(0,\Sigma_p\right)$ distributed innovations and only one of the variances is affected by the change.

to 1.5. The power results are presented in Table 18 in Appendix C and illustrated in Figure 7 along with the results for comparable changes that affect the elements of the unconditional covariance matrix H directly. Changes in the innovations' variances can be detected almost as reliably as changes that affect the main diagonal entries of H. This is a plausible result

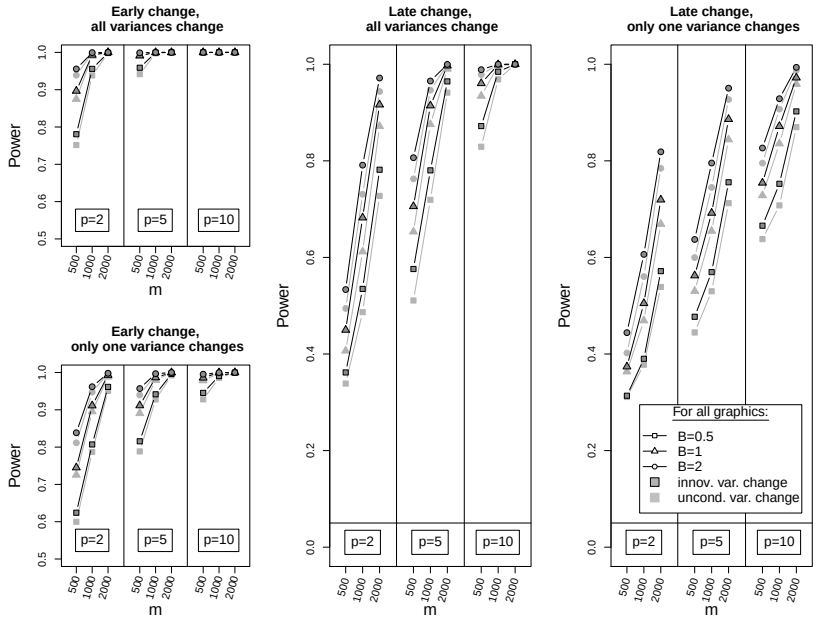

Figure 7: Power: scalar *BEKK* time series with $N\left(0,\Sigma_p\right)$ distributed innovations and the variances of the innovations change.

considering the model structure in (6).

To complete the simulation study, we illustrate the behavior under increasing magnitudes of the changes. Figure 8 shows the rejection frequencies of the procedure given different magnitudes of an earlier or later shift that affects all of the variances. Assume that all of the variances equal 1 before the change and experience a change of magnitude $\Delta \in \{-0.7, -0.6, \ldots, 0.6, 0.7\}$.

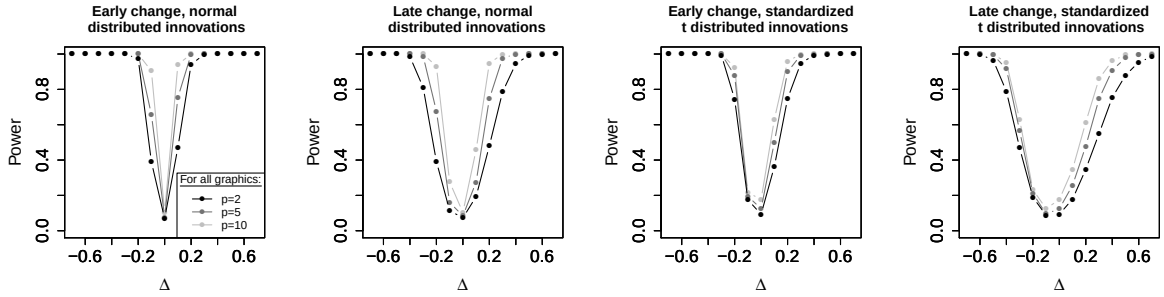

Figure 8: Power: scalar *BEKK* time series with $N\left(0,\sigma^2 I_p\right)$ or $t_\nu\left(0,\frac{\nu-2}{\nu}\sigma^2 I_p\right)$ distributed innovations and variance shifts of varying magnitude Δ.

The investigation is limited to the case of a historical period consisting of 1.000 observations, a monitoring period that is as long as the historical data set and scalar *BEKK* time series with multivariate normal or standardized *t* distributed innovation vectors. To ensure invertibility of

the covariance matrix, H is chosen as the identity and standardized identity matrix, respectively. In line with the previous results, the power approaches 1 with increasing absolute magnitude of the variance change and dimension. Besides, it is noticeable that for smaller absolute values of Δ the change is detected more frequently in the case of increasing variances compared to a decrease of the same amount.

4. Real data example

Finally, we use the proposed procedure to monitor a time series of log returns, namely those of the *DAX* listed assets of *Allianz*, *Bayer*, *DeutscheBank*, *RWE* and *Siemens* from 1979 to 2014. In a world without finite-sample size distortions, we would simply use the raw returns as input of our detector. However, simulations showed that there might occur size distortions in some cases, especially when $\alpha + \beta$ approaches 1, even though the assumptions are fulfilled. As a result, the procedure tends to falsely indicate the presence of a changepoint shortly after the beginning of the monitoring period with increasing probability as $\alpha + \beta \to 1$ which implies that the largest absolute eigenvalue of (8) tends to one and thus signifies an approach to a violation of Assumption 3. Fitting a scalar *BEKK* model to parts or the whole data set of log returns suggests that $\alpha + \beta$ is rather close to one. Therefore, we fit a scalar *BEKK* model to the data and consider the model residuals instead. The idea is that the residuals are close to the unobservable error terms and that the asymptotic distribution of our detector is the same for both choices. Indeed, additional simulations show that filtering multivariate *GARCH* time series and monitoring the residual vectors leads to empirical sizes close to those in the i.i.d. case indicating that the limit distributions based on the time series of *GARCH* residuals and based on the underlying innovation vectors are the same. Moreover, there is theoretical evidence from Kulperger and Yu (2005) who shows that a (univariate) CUSUM test based on *GARCH* residuals has the same limit behavior than a CUSUM test based on the underlying innovation vectors.

Note that there are a lot of different multivariate *GARCH* models that could have been fitted to the data. Since for most of the common multivariate *GARCH* models like vector *GARCH*

(Bollerslev et al. (1988)) or the models with constant (see Bollerslev (1990) or Jeantheau (1998)) or dynamic conditional correlation (see Engle (2002) or Tse and Tsui (2002)), methods for consistent parameter estimation have been proposed (see Jeantheau (1998), Engle and Sheppard (2001) and Bauwens et al. (2006)), we could have applied our procedure on the residuals that are obtained by fitting one of these models. A detailed proof that the limit distribution does not change in these cases is beyond the scope of the paper and left for future research.

In the following, the parameters α and β as well as the unconditional covariance matrix H are estimated from a historical data set of length $m \in \{500, 1.000\}$ via two stage quasi maximum likelihood estimation as described in Pedersen and Rahbek (2014). Since longer historical periods rather tend to be affected by variance changes, m is limited to a maximum of 1.000 observations. The significance level for all applications is $\alpha = 0.05$.

Since the parameters are estimated from the historical period, it must be ensured that this data is free from variance changes. To avoid missing a changepoint in the historical period, we perform a retrospective version of the procedure to $X_1, \ldots, X_m$. This procedure is similar to the method in Aue et al. (2009a) or a multivariate variant of Wied et al. (2012a) or Wied et al. (2012b) with detector

$$Q_k = \frac{k}{\sqrt{m}} \hat{D}_p^{-\frac{1}{2}} \left(\left[\hat{\sigma}^2 \right]_1^k - \left[\hat{\sigma}^2 \right]_1^m \right),$$

where

$$\lim_{m \to \infty} \mathsf{P} \left(\sup_{b \in [0,1]} \frac{\left\| Q_{\lfloor m \cdot b \rfloor + 2} \right\|_2}{w(b)} > c \right) = \mathsf{P} \left(\sup_{b \in [0,1]} \frac{\left\| B_p(b) \right\|_2}{w(b)} > c \right)$$

and $B_p(\cdot)$ is a p dimensional Brownian bridge whose component processes are p independent Brownian bridges. According to Aue et al. (2009a) the location of a detected changepoint can be estimated by $\hat{k}_r := \sup_{2 \leq k \leq m} Q_k$. Our approach is as follows:

(1) Apply the retrospective method to the subsample that consists of the first m data points.

(2a) If a changepoint is detected in the subsample, estimate the location of the changepoint and cut off all of the pre change observations. Then, restock the subsample to m observations and return to step (1). If there are not enough observations left to restock the subsample to m observations, terminate the procedure.

(2b) If no changepoint is detected in the subsample, take this sample as historical period and estimate the model parameters from it. Use the obtained estimates to calculate the model residuals for the whole dataset and apply the monitoring procedure to the residuals.

(3a) If the procedure detects a change, estimate the location of the changepoint. Then, cut off the pre change observations, use the first m observations of the resulting dataset as historical period and return to step (1). If there are not enough observations left to restock the subsample to m observations, terminate the procedure.

(3b) If no changepoint is detected in the monitoring period, terminate the procedure.

Unfortunately, performing several retrospective tests on only partially exchanged observations leads to an increased probability to commit a type I error. However, we neglect this problem as we need a changepoint-free historical period to perform the procedure properly. Choosing $\gamma = \{0, 0.25\}$ and B as the number of remaining data points after the historical period divided by m, we obtain the changepoints presented in Table 2.

$m = 500$							
$\gamma = 0$				$\gamma = 0.25$			
τ_m	$\hat{k}$	τ_m	$\hat{k}$	τ_m	$\hat{k}$	τ_m	$\hat{k}$
1983-03-24	1981-05-12		2002-06-13	1982-06-24	1981-05-13		2002-06-13
1984-10-22	1984-08-13		2003-04-15	1984-10-22	1984-08-13		2003-04-15
	1985-09-16		2003-11-24		1985-09-16		2003-11-24
1992-03-12	1988-07-22	2007-05-22	2006-06-14	1992-02-10	1988-07-22	2006-06-15	2006-03-03
1991-03-07	1990-10-24	2008-09-19	2008-09-10	1990-10-09	1990-09-12	2008-03-18	2008-03-14
1996-11-08	1996-02-12		2009-04-02	1997-09-23	1996-06-11	2013-01-08	2011-08-31
1999-12-24	1999-03-15	2012-07-23	2011-12-05	1998-09-30	1998-07-13	2014-05-30	2013-11-18
2001-10-15	2001-08-07	2014-11-20	2014-09-17	2003-02-26	2001-08-07		
$m = 1.000$							
1985-04-12	1984-08-13		2004-05-19	1984-10-30	1984-08-13	2003-12-22	2003-04-11
1992-03-26	1991-01-17	2008-11-25	2008-09-05	1991-12-24	1989-10-17		2004-05-19
1997-01-27	1996-06-11		2009-05-19		1991-03-20	2008-10-10	2008-09-05
	1998-07-13		2011-08-08	1996-12-04	1996-06-11		2009-05-19
2004-06-30	2003-04-16				1998-07-13		2011-08-08

Table 2: First hitting times and estimated changepoint locations when applying the monitoring procedure to asset returns of *Allianz*, *Bayer*, *Deutsche Bank*, *RWE* and *Siemens*. The gray dates indicate the location of changepoints that were detected in the historical data set.

Along with the log returns of the *Allianz* and *Siemens* assets, Figure 9 illustrates the changepoints that are detected using $\gamma = 0$ and $m = 1.000$. The time series are divided effectively in parts of higher or lower volatility by the procedure. The remaining time series show a similar behavior and will not be illustrated here for the sake of clarity. The reported changepoints are

used to split the time series in parts of constantly higher and lower variance. The sample standard deviations between two succeeding changepoints are presented in Table 3 and illustrated in Figure 10 for the *Allianz* and the *Siemens* asset. The results illustrated in Figures 9 and 10

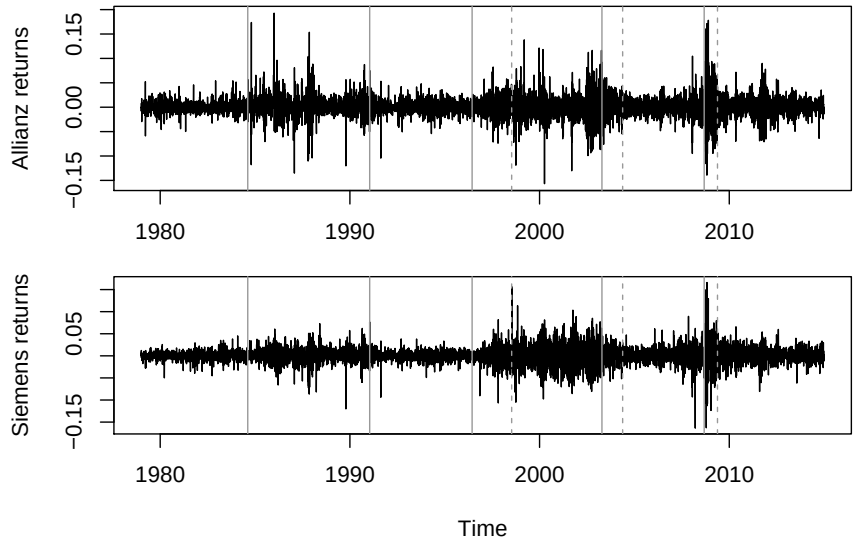

Figure 9: Returns of the *Allianz* and *Siemens* assets with detected changepoints ($\gamma = 0$ and $m = 1.000$). —— indicates that a changepoint was detected in the monitoring period; - - - indicates that a changepoint was detected in the historical period.

can be associated with distinctive events in the last 25 years. The late eighties were influenced strongly by the stock market crash and the Chernobyl catastrophe. The latter one is of interest since the asset of *RWE*, an energy generating company that relies on nuclear power since the seventies, is included in our sample. By the end of the nineties the volatilities increased in the

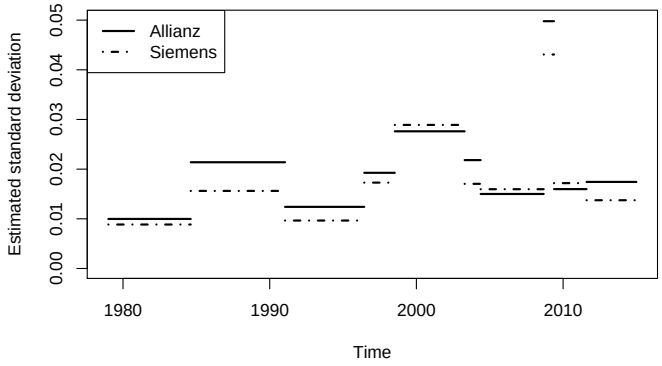

Figure 10: Sample standard deviations of the returns of the *Allianz* and *Siemens* assets between succeeding detected changepoints.

Estimation period	Allianz	Bayer	Deutsche Bank	RWE	Siemens
1979-01-02 to 1984-08-12	0.0100	0.0103	0.0096	0.0084	0.0089
1984-08-13 to 1991-01-16	0.0214	0.0154	0.0167	0.0167	0.0156
1991-01-17 to 1996-06-10	0.0124	0.0116	0.0105	0.0102	0.0097
1996-06-11 to 2003-03-20	0.0193	0.0187	0.0173	0.0179	0.0173
1998-07-13 to 2003-04-15	0.0276	0.0255	0.0261	0.0218	0.0289
2003-04-16 to 2004-05-18	0.0218	0.0199	0.0171	0.0155	0.0170
2004-05-19 to 2008-09-04	0.0150	0.0148	0.0146	0.0130	0.0160
2008-09-04 to 2009-05-18	0.0498	0.0311	0.0627	0.0325	0.0431
2009-05-19 to 2011-08-07	0.0160	0.0157	0.0213	0.0133	0.0172
2011-08-08 to 2014-12-31	0.0174	0.0166	0.0236	0.0197	0.0137

Table 3: Sample standard deviations calculated from the time periods between detected changepoints

course of the financial crises in Southeast Asia and Russia, a trend that was reinforced around the turn of the millennium by the bursting of the dotcom bubble and the beginning of the Iraq war. The following years of sinking volatility were interrupted by the Lehman bankruptcy and the following finance and debt crisis. Also, especially the asset of RWE was strongly influenced by the consequences of the nuclear incident in Fukushima in 2011.

5. Conclusion

We propose a multivariate monitoring procedure to detect changes in the vector of variances of a sequence of random vectors and analyzed its size and power properties. An application to a group of asset returns reported plausible changepoints that could be associated to past events that actually showed strong influence on the stock market.

In the paper, we refrain from monitoring the whole covariance matrix as proposed by Aue et al. (2009b) in the retrospective case and only focus on the variances instead. From a practitioner's point of view an application of the proposed procedure extended to the covariances to time series of higher dimension is problematic. Even for a moderate number of observation units, D_p is of unpropitious high dimension. The matrix has to be estimated and the quality of the estimate declines with p which shows strong influence on the performance of the procedure. To circumvent this problem, one should pursue different approaches, e.g., one could monitor the largest eigenvalue of covariance matrices. We leave this task for future research.

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