# Monitoring correlation change in a sequence of random variables

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**Abstract**

We propose a monitoring procedure to test for the constancy of the correlation coefficient of a sequence of random variables. The idea of the method is that a historical sample is available and the goal is to monitor for changes in the correlation as new data become available. We introduce a detector which is based on the first hitting time of a CUSUM-type statistic over a suitably constructed threshold function. We derive the asymptotic distribution of the detector and show that the procedure detects a change with probability approaching unity as the length of the historical period increases. The method is illustrated by Monte Carlo experiments and the analysis of a real application with the log-returns of the Standard & Poor's 500 (S&P 500) and IBM stock assets.

**Keywords:** Correlation changes; Gaussian process; Online detection; Threshold function.

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## 1. Introduction

The correlation coefficient is the most widely used method to measure dependence between a sequence of two random variables. In the particular case of financial time series, the analysis of the correlations between returns are very important in risk management. Indeed, there is compelling empirical evidence that the correlation structure of financial returns cannot be assumed to be constant over time, see e.g. Longin and Solnik (1995) and Krishan et al. (2009). Consequently, in periods of financial crisis, investors are extremely concerned about changes on correlations because in such periods, the correlation often increases, a phenomenon which is referred to as “Diversification Meltdown” (Campbell et al., 2008).

In order to construct an adequate model and to forecast future data, structural stability is a key point. Testing for structural stability has recently become one of the principal objectives of statistical analysis. There are two distinctly different approaches to tackle this problem. On the one hand, the main goal of retrospective procedures is to look for the presence of change points given an historical dataset of fixed size. On the other hand, the main goal of sequential detection procedures is to detect as soon as possible the presence of a change point once new data become available. This article is concerned with the latter kind of procedures. We adopt the framework in Chu et al. (1996) in which a historical sample is available and the goal is to monitor for a change point as new data become available. In particular, we analyze the case of changes in the correlation structure of a sequence of random variables. Other papers analyzing related problems under this framework are Chu et al. (1996), Horváth et al. (2004), Aue et al. (2006), Aue et al. (2009a), Aue et al. (2009b) and Aue et al. (2011), among others.

The paper is organized as follows. Section 2 proposes a monitoring procedure for detecting a correlation change and presents its asymptotic properties under the null and alternative hypothesis as well. Section 3 analyzes the finite sample properties of the proposed procedure via Monte Carlo experiments. Section 4 illustrates the procedure by analyzing log-returns of the S&P 500 and IBM stock assets. Finally, all proofs are given in an appendix.

## 2. The monitoring procedure

Let $(X_t, Y_t)$, for $t \in \mathbb{Z}$, be a sequence of bivariate random variables with finite 4-th moments and correlation

$$\rho_t = \frac{\mathsf{Cov}(X_t, Y_t)}{\sqrt{\mathsf{Var}(X_t)\mathsf{Var}(Y_t)}}.$$

We are interested in the hypothesis of correlation stability of the sequence. For that, assume that we have observed a sequence of the bivariate random vector $(X_t, Y_t)$ of size $m$. Since we are interested in sequentially monitoring whether or not the correlation coefficient remains stable over time, we require that the correlation is constant over the historical period of length $m$, i.e.:

**Assumption 1.** $\rho_1 = \ldots = \rho_m$, *where $m$ is a positive integer.*

Although Assumption 1 may appear a strong assumption, in practice, if a sufficient amount of historical data is available, it can be analyzed with the retrospective change point method proposed by Galeano and Wied (2012). Given the results of this procedure, one can make necessary adjustments to ensure correlation stability. Now, we want to test the null hypothesis given by:

$$H_0 : \rho_1 = \ldots = \rho_m = \rho_{m+1} = \ldots.$$

versus the alternative $H_1$ that $\rho_t$ changes at some $t \geq m + 1$, i.e.:

$$H_1 : \exists k^* \geq 1 : \rho_1 = \ldots = \rho_m = \ldots = \rho_{m+k^*-1} \neq \rho_{m+k^*} = \rho_{m+k^*+1} = \ldots,$$

where $k^*$ is referred to as the change point and is assumed unknown.

Denote with $\hat{\rho}_k^l$ the empirical correlation coefficient calculated from the observations $k$ to

$l$ with $k < l$, given by:

$$\hat{\rho}_k^l = \frac{\sum_{t=k}^{l}(X_t - \overline{X}_{k,l})(Y_t - \overline{Y}_{k,l})}{\sqrt{\sum_{t=k}^{l}(X_t - \overline{X}_{k,l})^2}\sqrt{\sum_{t=k}^{l}(Y_t - \overline{Y}_{k,l})^2}}$$

where $\overline{X}_{k,l} = \frac{1}{l-k+1}\sum_{t=k}^{l} X_t$ and $\overline{Y}_{k,l} = \frac{1}{l-k+1}\sum_{t=k}^{l} Y_t$. The sequential procedure is based on the detector:

$$V_k = \hat{D}\frac{k}{\sqrt{m}}\left(\hat{\rho}_{m+1}^{m+k} - \hat{\rho}_1^m\right), \ k \in \mathbb{N}, \tag{1}$$

where $\hat{D}$ is an estimator which is calculated from the first $m$ observations and is given in the appendix, see also Wied et al. (2012). We stop and declare $H_0$ to be invalid at the first time $k$ such that the detector $V_k$ exceeds the value of a scaled threshold function $w$, therefore yielding the stopping rule:

$$\tau_m = \min\left\{k \leq [mT] : |V_k| > c \cdot w\left(\frac{k}{m}\right)\right\}, \tag{2}$$

where $T$ is a positive constant, $c$ is a suitably chosen constant such that under $H_0$, $\lim_{m\to\infty} \mathsf{P}(\tau_m < \infty) = \alpha$, with $\alpha \in (0,1)$, and $w$ is a positive and continuous function. Here, we write $\tau_m < \infty$ to indicate that the monitoring has been terminated during the testing period, i.e., the detector $V_k$ has crossed the boundary $c \cdot w(k/m)$ for some $k \leq [mT]$. We write $\tau_m = \infty$ if the detector has not crossed the boundary during the testing period (compare Aue et al., 2011). Note that the stopping time $\tau_m$ need not be the change point; in fact the change point might be before $\tau_m$. Some comments on the issue of estimating the change point once $H_0$ has been declared invalid will be given at the end of this section.

For deriving asymptotic results under $H_0$, some additional assumptions are necessary. The next three assumptions correspond to (A1), (A2) and (A3) in Wied et al. (2012).

**Assumption 2.** *For*

$$U_t := \left(X_t^2 - \mathsf{E}(X_t^2), \quad Y_t^2 - \mathsf{E}(Y_t^2), \quad X_t - \mathsf{E}(X_t), \quad Y_t - \mathsf{E}(Y_t), \quad X_tY_t - \mathsf{E}(X_tY_t)\right)'$$

*and* $S_j := \sum_{t=1}^{j} U_t$, *we have*

$$\lim_{m\to\infty} \mathsf{E}\left(\frac{1}{m} S_m S_m'\right) =: D_1 \text{ (finite and positive definite).}$$

**Assumption 3.** *The* $r$*-th absolute moments of the components of* $U_t$ *are uniformly bounded for some* $r > 2$.

**Assumption 4.** *The vector* $(X_t, Y_t)$ *is* $L_2$*-NED (near-epoch dependent) with size* $-\frac{r-1}{r-2}$, *where* $r$ *from Assumption 3, and constants* $(c_t), t \in \mathbb{Z}$, *on a sequence* $(V_t), t \in \mathbb{Z}$, *which is* $\alpha$*-mixing of size* $\phi^* := -\frac{r}{r-2}$, *i.e.*

$$||(X_t, Y_t) - \mathsf{E}\left((X_t, Y_t)|\sigma(V_{t-l}, \ldots, V_{t+l})\right)||_2 \leq c_t v_l$$

*with* $\lim_{l\to\infty} v_l = 0$, *such that*

$$c_t \leq 2||U_t||_2$$

*with* $U_t$ *from Assumption 3 and the* $L_2$*-norm* $||\cdot||_2$.

Furthermore, we impose a stationarity condition. This condition might be slightly relaxed to allow for some fluctuations in the first and second moments (see (A4) and (A5) in Wied et al., 2012), but for ease of exposition and because the procedure would remain exactly the same we stick to this notation.

**Assumption 5.** $(X_t, Y_t), t \in \mathbb{Z}$, *is weak-sense stationary.*

Our main result is then:

**Theorem 1.** *Under* $H_0$, *Assumptions 1, 2, 3, 4 and 5 and for any* $T > 0$,

$$\lim_{m\to\infty} \mathsf{P}(\tau_m < \infty) = \lim_{m\to\infty} \mathsf{P}\left(\sup_{0\leq b\leq T} \frac{|V_{[m\cdot b]+2}|}{w(b)} > c\right) = \mathsf{P}\left(\sup_{0\leq b\leq T} \frac{|G(b)|}{w(b)} > c\right), \qquad (3)$$

*where* $G(\cdot)$ *is a mean zero Gaussian process with covariance* $\mathsf{E}(G(k)G(l)) = \min(k,l) + kl$.

Theorem 1 establishes the asymptotic behavior of the monitoring procedure based on the stopping rule $\tau_m$ in Eq. (2). Following Aue et al. (2011), the limiting probability in Eq. (3) can be written in an alternative way that allows for finite sample statistical inference. First, it is easy to see that $\{G(b) : b \in [0,T]\} =_d \{W(b) + b\xi : b \in [0,T]\}$, where $\{W(b) : b \geq 0\}$ is a standard Brownian Motion independent of the standard Gaussian random variable $\xi$. Then, it is also easy to see that $\{W(b) + b\xi : b \in [0,T]\} =_d \{(1+b)\, W(b/(1+b)) : b \in [0,T]\}$ just by comparing their covariance structures. Therefore,

$$\sup_{0\leq b\leq T} \frac{|G(b)|}{w(b)} =_d \sup_{0\leq b\leq T} \frac{1+b}{w(b)} \left| W\left(\frac{b}{1+b}\right)\right|. \tag{4}$$

Eq. (4) leads to an obvious choice of the threshold function: take $w(b) = 1+b$, because in this case:

$$\sup_{0\leq b\leq T} \frac{|G(b)|}{w(b)} =_d \sup_{0\leq b\leq T} \left| W\left(\frac{b}{1+b}\right)\right|.$$

With this expression, quantiles of interest can be easily simulated with Monte Carlo methods. However, once there occurs a change point, it is very important to quickly detect it. Therefore, we consider a kind of generalization of this threshold function, previously considered in Horváth et al. (2004), which is given by:

$$w(b) = (1+b) \cdot \max\left\{ \left(\frac{b}{1+b}\right)^{\gamma}, \epsilon \right\} \tag{5}$$

where $0 \leq \gamma < \frac{1}{2}$ and $\epsilon > 0$ is a fixed constant which can be chosen arbitrarily small in applications.

Note that, if a correlation change occurs soon after the historical dataset, then, choosing $\gamma$ as large as possible, the stopping rule $\tau_m$ will stop nearly instantaneously. Note that $\gamma = 1/2$ is excluded, since $H_0$ would else be rejected with probability one regardless whether it is true or not because of the law of the iterated logarithm for Brownian Motions at zero, see Aue et al. (2009b). Using the threshold function in Eq. (5) and calling $u = b/(1+b)$, Eq. (4) leads to:

$$\sup_{0\le b\le T} \frac{|G(b)|}{w(b)} =_d \sup_{0\le u\le \frac{T}{T+1}} \frac{1}{\max\{u^\gamma, \epsilon\}} |W(u)|. \tag{6}$$

Finally, calling $s = u(T+1)/T$ and taking into account that $\{W(u), u \in [0, T/(T+1)]\}$ has the same covariance structure as $\left\{\sqrt{\frac{T}{1+T}} W(s), s \in [0,1]\right\}$, Eq. (6) transforms into:

$$\sup_{0\le b\le T} \frac{|G(b)|}{w(b)} =_d \left(\frac{T}{1+T}\right)^{\frac{1}{2}-\gamma} \sup_{0\le s\le 1} \frac{1}{\max\{s^\gamma, \epsilon((T+1)/T)^\gamma\}} |W(s)|. \tag{7}$$

Therefore, under the conditions in Theorem 1:

$$\lim_{m\to\infty} \mathsf{P}(\tau_m < \infty) = \mathsf{P}\left(\left(\frac{T}{1+T}\right)^{\frac{1}{2}-\gamma} \sup_{0\le s\le 1} \frac{1}{\max\{s^\gamma, \epsilon((T+1)/T)^\gamma\}} |W(s)| > c\right)$$

and Monte Carlo simulations can be used to obtain the constant $c(\alpha)$ such that:

$$\mathsf{P}\left(\left(\frac{T}{1+T}\right)^{\frac{1}{2}-\gamma} \sup_{0\le s\le 1} \frac{1}{\max\{s^\gamma, \epsilon((T+1)/T)^\gamma\}} |W(s)| > c(\alpha)\right) = \alpha,$$

for any $\alpha \in (0,1)$. In this way, the probability of a false alarm is approximately $\alpha$ if $m$ is large enough.

For a local power analysis, we impose the assumption

**Assumption 6.** $(X_t, Y_t), t \in \mathbb{Z}$, *is weak-sense stationary with the difference that* $\mathsf{Cov}(X_t, Y_t) = \frac{1}{\sqrt{m}} g\left(\frac{t}{m}\right)$ *with a bounded function* $g$ *that can be approximated by step functions such that* $g(z) = 0, z \in [0,1]$, *and* $\int_1^{T+1} |g(z)| dz > 0$.

Theorem 2 yields consistency of the monitoring procedure. Therefore, a correlation change will be detected with high probability if the historical period is large enough.

**Theorem 2.** *Under a sequence of local alternatives, Assumptions 1, 2, 3, 4, 5 and 6 and* *for any $T > 0$,*

$$\lim_{m\to\infty} \mathsf{P}(\tau_m < \infty) = \lim_{m\to\infty} \mathsf{P}\left(\sup_{0\le b\le T} \frac{|V_{[m\cdot b]+2}|}{w(b)} > c\right) = \mathsf{P}\left(\sup_{0\le b\le T} \frac{|G(b)+h(b)|}{w(b)} > c\right),$$

*where $G(\cdot)$ is as in Theorem 1 and $h(b) = H\left(\int_1^{b+1} g(z)dz - b \cdot \int_0^1 g(z)dz\right)$ for a constant $H$ depending on the data generating process.*

Once the presence of a correlation change is detected, an estimate of its location is provided by using the statistic proposed in Wied et al. (2012). The estimate of the change point is $\widehat{k} = \underset{1\le j\le \tau_m - 1}{\arg\max} D_{\tau_m}$ with

$$D_{\tau_m} := \hat{D}\frac{j}{\sqrt{\tau_m}}\left|\hat{\rho}_{m+1}^{m+j} - \hat{\rho}_{m+1}^{m+\tau_m-1}\right|. \tag{8}$$

Note that, except for the estimator $\hat{D}$, we do not use the historical period to compute the value of the statistic $D_{\tau_m}$ but only the observations from $m+1$ to $m+\tau_m-1$. Monte Carlo experiments have shown that the inclusion of the historical period severely distorts the estimates of the change point location. A theoretical analysis of this estimator is beyond the scope of this paper.

## 3. Simulations

In this section, we report the results of the Monte Carlo experiments that we have performed to assess the finite sample performance of the proposed monitoring procedure. In all the experiments, we consider three different values of the parameter $\gamma$ of the threshold function $w(b), b \in [0,T]$ in Eq. (5), $\gamma = 0$, 0.25 and 0.45, while we have taken $\epsilon = 10^{-10}$. Figure 1 shows the plot of the three threshold functions considered (which does not yet give concrete information about the critical values used in the procedure). Note that the larger the values of $\gamma$, the smaller the values of $w(b), b \in [0,T]$. The threshold function with $\gamma = 0.45$ is expected to allow for a quick detection of early change points. We consider three different values of the size of the historical sample, $m = 250$, 500 and

1000. Note that these values are specially designated for financial returns in which we can consider large historical samples. Finally, we consider four values of the parameter $T$, $T = 0.5$, 1, 2 and 4. Note that these values cover a large number of sample sizes of the generated bivariate series which is given by $n = m + [Tm]$. For instance, for $m = 500$, the sample sizes of the series generated are 750, 1000, 1500 and 2500, respectively.

**Figure 1 goes around here**

First, we obtain critical values to apply the monitoring procedure for the different values considered of $\gamma$ and $T$. Table 1 shows the critical values at level $\alpha = 0.05$ based on 10000 standard Brownian Motion processes approximated on a grid of 10000 equispaced points in the interval $[0, 1]$. Note that the critical values increases with $T$ and/or with $\gamma$ as expected.

**Table 1 goes around here**

Second, in order to obtain empirical sizes of the monitoring procedure, we generate 1000 bivariate series $(X_t, Y_t)$, for $t = 1, \ldots, n$, and any choice of $\gamma$, $T$ and $m$, as follows. Initially, we generate two series $\left(\widetilde{X}_t, \widetilde{Y}_t\right)$, for $t = 1, \ldots, n$, independently, following the GARCH$(1, 1)$ models given by:

$$\widetilde{X}_t = \sqrt{h_{1,t}}\epsilon_{1,t}$$
$$h_{1,t} = 0.01 + 0.05\widetilde{X}_{t-1}^2 + 0.8h_{1,t-1}$$

and,

$$\widetilde{Y}_t = \sqrt{h_{2,t}}\epsilon_{2,t}$$
$$h_{2,t} = 0.01 + 0.1\widetilde{Y}_{t-1}^2 + 0.75h_{2,t-1}$$

respectively, where $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are standard Gaussian distributed. Then, we transform the bivariate series $\left(\widetilde{X}_t, \widetilde{Y}_t\right)$ into $(X_t, Y_t)$ by multiplying each value of the pair $\left(\widetilde{X}_t, \widetilde{Y}_t\right)$ with $\Sigma^{1/2}$ where $\Sigma$ is a square symmetric matrix with ones in the main diagonal and with $\rho = 0.5$ outside the main diagonal. Then, the correlation between $X_t$ and $Y_t$ is $\rho = 0.5$. Afterwards, for each simulated dataset, we apply the monitoring procedure from time $m + 1$ until time $n$, with level $\alpha = 0.05$. Table 2 reports the simulated empirical sizes for the monitoring procedure based on the detector $V_k$. In most cases, the simulated empirical sizes slightly exceed the nominal sizes; for $\gamma = 0.45$ the empirical sizes are at least twice the nominal size. However, empirical and nominal sizes get closer as $m$ increases which is reasonable based on the results in Section 2. Also, larger empirical sizes are found as $\gamma$ gets larger and $m$ is small. Therefore, if a correlation change is expected to occur not shortly after the historical period and we want to minimize the type I error, the choice of the threshold function with $\gamma = 0$ appears to be appropriate. However, if a correlation change is expected to occur shortly after the historical period and we want to detect it as soon as possible even if a false alarm can happen, it is better to use the threshold function with $\gamma = 0.45$.

**Table 2 goes around here**

Third, in order to estimate the power of the monitoring procedure, the Monte Carlo setup is similar to the one described previously, but the series are generated with a single change point in the correlation at two different positions $k = [0.05mT]$ and $k = [0.5mT]$, in which $\rho = 0.5$ increases to $\rho = 0.75$. Therefore, the first $m$ observations have the same correlation coefficient, that changes after $k$ observations of the monitoring time. The first change point is at the initial 5% of the monitoring time, so that it is specially designated to estimate the power of the procedure in situations in which the change point occurs shortly after the historical period. The second change point is at the middle of the monitoring time, so that it is specially designated to estimate the power of the procedure in situations in which the change point does not occur shortly after the historical period. Tables 3, 4, 5, 6, 7 and 8 show the results for the three possible values of the $\gamma$ parameter, $\gamma = 0$, 0.25 and 0.45, and the two possible change points, $k = [0.05mT]$ and $k = [0.5mT]$. These tables show the empirical power of the procedure and a summary of both, the empirical stopping time distribution and the estimated change points, including the quartiles, the mean, the standard deviation and the coefficient of variation. The tables show that the power increases with $m$ and it can be large except in cases in which $m$ and $T$ are small. Besides, the power for early changepoints is larger than the power for changes at the middle of the monitoring period. Regarding the empirical stopping time distribution, if a change occurs shortly after the beginning of the monitoring period, then the threshold function with $\gamma = 0.45$ have the shortest detection delay time. However, for a change point at the middle of the monitoring period with $m = 250$ and $m = 500$, the first quartiles of the empirical stopping times with $\gamma = 0.45$ are very small indicating that is more likely to falsely detect a correlation change even before it occurred. On the other hand, regarding the change point estimates, we can observe that the estimates of the change point at the beginning of the monitoring period are upward biased, while the estimates of the change point at the middle of the monitoring period are downward biased. However, in both cases, the bias reduces substantially if $m$ and/or $T$ increases. In any case, the precision of the change point detection estimate is quite acceptable specially when the power is large.

In summary, if the bivariate series is going to be monitored for a long time and the type I error is to be avoided, or if a change in the correlation is expected to occur not shortly after the beginning of monitoring period, the threshold function with $\gamma = 0$ may be a good choice. However, if the focus is to detect a change point in the correlation as soon as possible, even if a false alarm is accepted, and if the change point is expected to occur shortly after the beginning of monitoring period, then it is better to use $\gamma = 0.45$. Alternately, the threshold function with $\gamma = 0.25$ appears to be a good compromise between the previous frameworks.

**Table 3 goes around here**

**Table 4 goes around here**

**Table 5 goes around here**

**Table 6 goes around here**

**Table 7 goes around here**

**Table 8 goes around here**

## 4. Real data example

In this section, we apply the proposed monitoring procedure discussed in Section 2 to a real data example. Galeano and Wied (2012) analyzed the log-return series of two U.S. assets: the Standard & Poors 500 Index and the IBM stock using a posteriori change point tests. In particular, Galeano and Wied (2012) considered the sample period starting from January 2, 1997 to December 31, 2010 consisting of $n = 3524$ observations, that are plotted in Figure 2. The binary segmentation procedure proposed in that paper detected a first change point at August 19, 1999 (observation number 664), that can be associated with the collapse of the dot-com bubble started at the end of the 1990s and the beginning of the 2000s, and a second change point at November 12, 2007 (observation number 2734), that can be associated with the beginning of the Global Financial Crisis around the end of 2007, which is considered by many economists the worst financial crisis since the Great Depression of the 1930s.

**Figure 2 goes around here**

Here, we apply the proposed monitoring procedure as follows. The analysis in Galeano and Wied (2012) indicated that the correlations between both log-returns remained constant for the period starting from January 2, 1997 to August 19, 1999. Then, we use the log-returns from January, 2, 1997 until May, 28, 1999, as the historical period, i.e., we take $m = 607$. If no correlation changes are found after $n-m = 2917$ observations (then, $T = 4.8056$) the procedure would be terminated. Otherwise, a change point is detected and a new historical period is defined with $m = 607$. Then, the monitoring procedure is applied again in a similar fashion. The results of our analysis are summarized in Table 9 for the three threshold functions with $\gamma = 0$, 0.25 and 0.45, for which the corresponding critical values at 5% level are 2.0510 for $\gamma = 0$, 2.2630 for $\gamma = 0.25$, and 2.7435 for $\gamma = 0.45$, respectively. The proposed procedure with the three values of the threshold functions detects four change points sequentially. Regarding the first hitting times, the procedure with $\gamma = 0.45$ has the shortest detection delay time whereas the procedure with $\gamma = 0$ the longest. This is in accordance with the Monte Carlo experiments in Section 3. Regarding the estimated change points, the procedure with the three values gives very similar estimates. Indeed, the first and the last detected change points coincide with the ones given in Galeano and Wied (2012). Finally, Table 10 shows the empirical correlations between the Standard & Poors 500 and IBM log-returns in the periods given by the monitoring procedure. As it can be seen, there are substantial differences between correlations at different periods.

**Table 9 goes around here**

**Table 10 goes around here**

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# A. Appendix

## A.1. The scalar $\hat{D}$ from the test statistic in Eq. (1)

The scalar $\hat{D}$ from our test statistic in Eq. (1) based on observations from $t = 1, \ldots, r$ can be written as

$$\hat{D} = (\hat{F}_1 \hat{D}_{3,1} + \hat{F}_2 \hat{D}_{3,2} + \hat{F}_3 \hat{D}_{3,3})^{-\frac{1}{2}}$$

where

$$\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \\ \hat{F}_3 \end{pmatrix} = \begin{pmatrix} \hat{D}_{3,1}\hat{E}_{11} + \hat{D}_{3,2}\hat{E}_{21} + \hat{D}_{3,3}\hat{E}_{31} \\ \hat{D}_{3,1}\hat{E}_{12} + \hat{D}_{3,2}\hat{E}_{22} + \hat{D}_{3,3}\hat{E}_{32} \\ \hat{D}_{3,1}\hat{E}_{13} + \hat{D}_{3,2}\hat{E}_{23} + \hat{D}_{3,3}\hat{E}_{33} \end{pmatrix},$$

$$\begin{aligned}
\hat{E}_{11} &= \hat{D}_{1,11} - 4\hat{\mu}_x \hat{D}_{1,13} + 4\hat{\mu}_x^2 \hat{D}_{1,33}, \\
\hat{E}_{12} &= \hat{E}_{21} = \hat{D}_{1,12} - 2\hat{\mu}_x \hat{D}_{1,23} - 2\hat{\mu}_y \hat{D}_{1,14} + 4\hat{\mu}_x \hat{\mu}_y \hat{D}_{1,34}, \\
\hat{E}_{22} &= \hat{D}_{1,22} - 4\hat{\mu}_y \hat{D}_{1,24} + 4\hat{\mu}_y^2 \hat{D}_{1,44}, \\
\hat{E}_{13} &= \hat{E}_{31} = -\hat{\mu}_y \hat{D}_{1,13} + 2\hat{\mu}_x \hat{\mu}_y \hat{D}_{1,33} - \hat{\mu}_x \hat{D}_{1,14} + 2\hat{\mu}_x^2 \hat{D}_{1,34} + \hat{D}_{1,15} - 2\hat{\mu}_x \hat{D}_{1,35}, \\
\hat{E}_{23} &= \hat{E}_{32} = -\hat{\mu}_y \hat{D}_{1,23} + 2\hat{\mu}_x \hat{\mu}_y \hat{D}_{1,44} - \hat{\mu}_x \hat{D}_{1,24} + 2\hat{\mu}_y^2 \hat{D}_{1,34} + \hat{D}_{1,25} - 2\hat{\mu}_y \hat{D}_{1,45}, \\
\hat{E}_{33} &= \hat{\mu}_y^2 \hat{D}_{1,33} + 2\hat{\mu}_x \hat{\mu}_y \hat{D}_{1,34} - 2\hat{\mu}_y \hat{D}_{1,35} + \hat{\mu}_x^2 \hat{D}_{1,44} + \hat{D}_{1,55} - 2\hat{\mu}_x \hat{D}_{1,45},
\end{aligned}$$

$$\hat{D}_1 = (\hat{D}_{1,ij})_{i,j=1}^5 = \sum_{t=1}^{r} \sum_{u=1}^{r} k\left(\frac{t-u}{\delta_r}\right) V_t V_u{}',$$

$$V_t = \frac{1}{\sqrt{r}} U_t^{***}, \delta_r = [\log r],$$

$$U_t^{***} = \left( X_t^2 - \overline{(X^2)}_r \quad Y_t^2 - \overline{(Y^2)}_r \quad X_t - \bar{X}_r \quad Y_t - \bar{Y}_r \quad X_tY_t - \overline{(XY)}_r \right)',$$

$$k(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & otherwise \end{cases},$$

$$\hat{\mu}_x = \bar{X}_r,\ \hat{\mu}_y = \bar{Y}_r,\ \hat{D}_{3,1} = -\frac{1}{2}\frac{\hat{\sigma}_{xy}}{\hat{\sigma}_y}\hat{\sigma}_x^{-3},\ \hat{D}_{3,2} = -\frac{1}{2}\frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x}\hat{\sigma}_y^{-3},\ \hat{D}_{3,3} = \frac{1}{\hat{\sigma}_x\hat{\sigma}_y}$$

and

$$\hat{\sigma}_x^2 = \overline{(X^2)}_r - (\bar{X}_r)^2,\ \hat{\sigma}_y^2 = \overline{(Y^2)}_r - (\bar{Y}_r)^2,\ \hat{\sigma}_{xy} = \overline{(XY)}_r - \bar{X}_r\bar{Y}_r.$$

This is the same expression as in Appendix A.1 in Wied et al. (2012).

### A.2. Proofs

*Proof of Theorem 1*

Let $D[d_1, d_2]$ be the space of càdlàg-functions on the interval $[d_1, d_2]$ equipped with the supremum norm.

The proof is mainly based on the fact that for fixed $c \geq 0$, and $m \to \infty$ the process $\{P_m(d), c \leq d \leq T\}$, defined by

$$P_m(d) = \hat{D}\frac{[m \cdot d] - [m \cdot c]}{\sqrt{m}}(\hat{\rho}_{[m\cdot c]}^{[m\cdot d]} - \rho_1),$$

converges in distribution to the process $\{W(d) - W(c), c \leq d \leq T\}$ on $D[c, T]$ with $W(\cdot)$ being a standard Brownian Motion. This result is a generalization of Lemma A.3 in Wied et al. (2012):

First, under Assumptions 2-5, we obtain with Lemma A.1 in Wied et al. (2012) that the process $\{Q_m(d), 0 \leq d \leq T\}$, defined by

$$Q_m(d) := \frac{1}{\sqrt{m}} \sum_{t=1}^{[m\cdot d]} U_t$$

with

$$U_t = \begin{pmatrix} X_t^2 - E(X_t^2) & Y_t^2 - E(Y_t^2) & X_t - E(X_t) & Y_t - E(Y_t) & X_tY_t - E(X_tY_t) \end{pmatrix}',$$

converges to $D_1 W_5(\cdot)$, where $W_5$ is a 5-dimensional Brownian Motion.

Let $c_2 \in [0,T]$ and $c_3 \in [0,T]$ be fixed. By applying the continuous mapping theorem with the continuous functional

$$(x(t))_{0\leq t\leq T} \to \begin{pmatrix} ((x(t) - x(c_1))\mathbf{1}_{\{t\geq c_1\}})_{0\leq t\leq T} \\ (x(c_3) - x(c_2)) \end{pmatrix}$$

we have for $d \geq c_1$

$$R_m(d) := \frac{1}{\sqrt{m}} \begin{pmatrix} \sum_{t=[m\cdot c_1]+1}^{[m\cdot d]} U_t \\ \sum_{t=[m\cdot c_2]+1}^{[m\cdot c_3]} U_t \end{pmatrix} = \begin{pmatrix} Q_m(d) - Q_m(c_1) \\ Q_m(c_3) - Q_m(c_2) \end{pmatrix} \Rightarrow_d \begin{pmatrix} D_1W_5(d) - D_1W_5(c_1) \\ D_1W_5(c_3) - D_1W_5(c_2) \end{pmatrix}$$

in $D([c_1, T], \mathbb{R}^5) \times \mathbb{R}^5$.

Now, on $\{R_m(d), c_1 \leq d \leq T\}$ we can (separately for both components) apply all calculations from the proofs of Lemma A.2 and A.3 (using the delta method argument) in Wied et al. (2012).

Then, we obtain, for $0 \leq b \leq T$, with $c_3 = 0$, $c_1 = c_2 = 1$, that

$$S_m(b) := \begin{pmatrix} \hat{D}\frac{[m\cdot b]+2}{\sqrt{m}} \left(\hat{\rho}_{m+1}^{m+[m\cdot b]+2} - \rho_1\right) \\ \hat{D}\sqrt{m}\left(\hat{\rho}_1^m - \rho_1\right) \end{pmatrix} \Rightarrow_d \begin{pmatrix} W(b+1) - W(1) \\ W(1) \end{pmatrix},$$

where $W(\cdot)$ is a one-dimensional Brownian Motion.

Consequently,

$$\begin{aligned}
V_{[m\cdot b]+2} &= \hat{D}\frac{[m\cdot b]+2}{\sqrt{m}}\left(\hat{\rho}_{m+1}^{m+[m\cdot b]+2}-\hat{\rho}_1^m\right)\\
&= \hat{D}\frac{[m\cdot b]+2}{\sqrt{m}}\left(\hat{\rho}_{m+1}^{m+[m\cdot b]+2}-\rho_1\right)-\hat{D}\frac{[m\cdot b]+2}{\sqrt{m}}\left(\hat{\rho}_1^m-\rho_1\right)
\end{aligned}$$

converges to the process

$$\{(W(b+1)-W(1))-bW(1), 0\leq b\leq T\} = \{W(b+1)-(b+1)W(1), 0\leq b\leq T\}.$$

Applying the continuous mapping theorem and calculating the covariance structure of the limit process proves the result. ■

*Proof of Theorem 2*

The proof uses the same arguments as Theorem 1 and is mainly based on the fact that for fixed $c \geq 0$, and $m \to \infty$ the process $\{P_m(d), c \leq d \leq T\}$ converges in distribution to the process $\left\{W(d) - W(c) + H\int_c^d g(z)dz, c \leq d \leq T\right\}$ on $D[c, T]$ with $W(\cdot)$ being a standard Brownian Motion. The constant $H$ is, up to a constant, the limit of $\hat{D}$ under the null hypothesis, compare the proof of Theorem 2 in Wied et al. (2012). This result is a generalization of arguments used in Theorem 2 in Wied et al. (2012), executed in basically the same way as presented in Theorem 1. ■

Figure 1: Threshold functions for different values of $\gamma$ and $\epsilon = 10^{-10}$

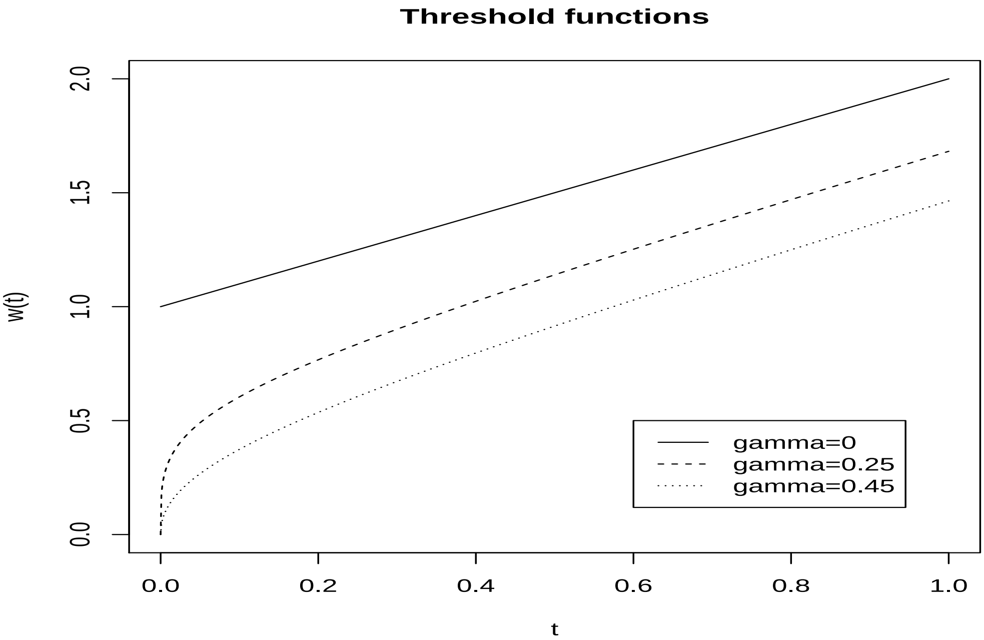


Table 1: Critical values.

| $T$ | $\gamma = 0$ | $\gamma = 0.25$ | $\gamma = 0.45$ |
|---|---|---|---|
| 0.5 | 1.2870 | 1.8001 | 2.6282 |
| 1 | 1.5578 | 1.9924 | 2.6844 |
| 2 | 1.8158 | 2.1684 | 2.7215 |
| 4 | 1.9980 | 2.2467 | 2.7660 |

Table 2: Empirical sizes.

| | $T$ | $m = 250$ | $m = 500$ | $m = 1000$ |
|---|---|---|---|---|
| $\gamma = 0$ | 0.5 | 0.059 | 0.058 | 0.050 |
| | 1 | 0.077 | 0.069 | 0.061 |
| | 2 | 0.066 | 0.054 | 0.057 |
| | 4 | 0.063 | 0.071 | 0.060 |
| $\gamma = 0.25$ | 0.5 | 0.075 | 0.079 | 0.047 |
| | 1 | 0.075 | 0.064 | 0.057 |
| | 2 | 0.087 | 0.063 | 0.052 |
| | 4 | 0.073 | 0.077 | 0.064 |
| $\gamma = 0.45$ | 0.5 | 0.169 | 0.125 | 0.109 |
| | 1 | 0.174 | 0.136 | 0.116 |
| | 2 | 0.164 | 0.138 | 0.109 |
| | 4 | 0.161 | 0.128 | 0.106 |

Figure 2: Log-retuns of S&P 500 and IBM indexes

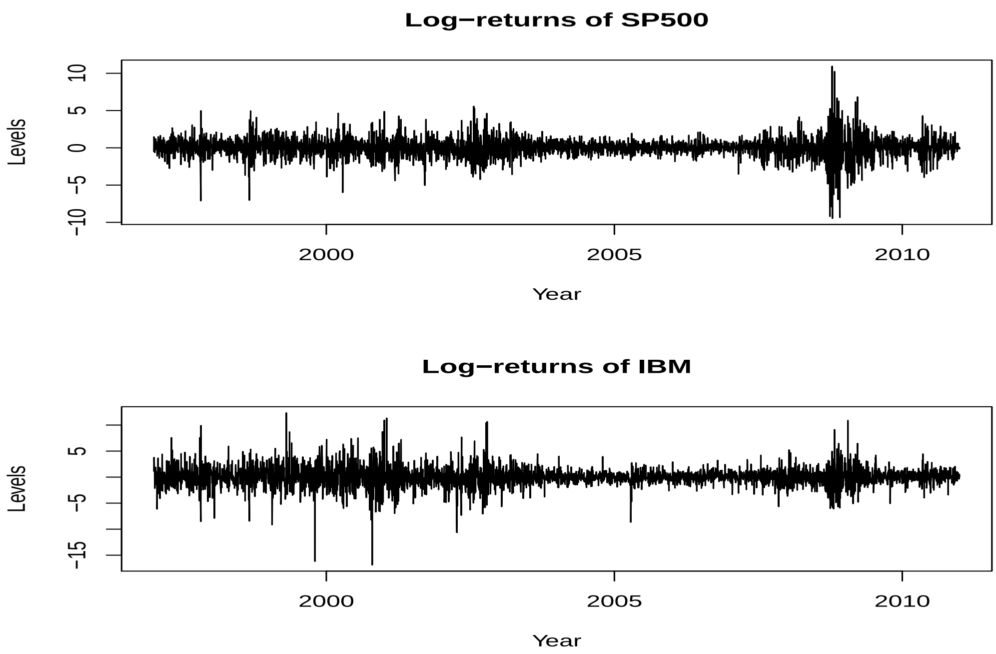

Table 3: Empirical power, stopping time and changepoint estimate for $\gamma = 0$ and $k = [0.05mT]$.

| $m = 250$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (6) | 0.884 | 65 | 80 | 96 | 81.49 | 20.27 | 0.24 | 6 | 22 | 40.25 | 26.58 | 22.56 | 0.84 |
| 1 (12) | 0.972 | 94 | 115 | 142.25 | 121.83 | 38.11 | 0.31 | 12 | 30 | 59 | 39.03 | 33.22 | 0.85 |
| 2 (25) | 0.993 | 131 | 160 | 202 | 173.73 | 61.61 | 0.35 | 25 | 39 | 67 | 52.36 | 40.77 | 0.77 |
| 4 (50) | 0.998 | 178 | 224 | 284.75 | 242.44 | 91.41 | 0.37 | 48 | 61 | 90 | 74.87 | 47.25 | 0.63 |
| $m = 500$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (12) | 0.997 | 95 | 112 | 137 | 118.50 | 31.78 | 0.26 | 12 | 30 | 58 | 37.99 | 31.67 | 0.83 |
| 1 (25) | 1 | 136 | 160 | 188 | 166.34 | 43.53 | 0.26 | 24 | 39 | 69 | 50.61 | 37.24 | 0.73 |
| 2 (50) | 1 | 191 | 224 | 264 | 231.52 | 55.53 | 0.23 | 47 | 58 | 83 | 69.03 | 36.81 | 0.53 |
| 4 (100) | 1 | 268 | 319 | 376.25 | 328.61 | 81.78 | 0.24 | 95 | 105 | 126 | 113.91 | 37.11 | 0.32 |
| $m = 1000$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (25) | 1 | 142.75 | 165 | 188.25 | 168.74 | 35.13 | 0.20 | 23 | 38 | 69 | 49.42 | 36.17 | 0.73 |
| 1 (50) | 1 | 198 | 228 | 259.25 | 232.47 | 46.74 | 0.20 | 48 | 61 | 90.25 | 72.77 | 39.66 | 0.54 |
| 2 (100) | 1 | 285 | 318 | 364 | 326.92 | 62.46 | 0.19 | 96 | 104 | 124 | 111.45 | 33.47 | 0.30 |
| 4 (200) | 1 | 422.75 | 475 | 537.25 | 483.86 | 85.81 | 0.17 | 191 | 200 | 214 | 203.85 | 37.22 | 0.18 |

Table 4: Empirical power, stopping time and changepoint estimate for $\gamma = 0$ and $k = [0.5mT]$.

| $m = 250$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (62) | 0.286 | 92 | 107 | 117 | 100.66 | 23.28 | 0.23 | 29 | 52 | 63 | 46.24 | 21.52 | 0.46 |
| 1 (125) | 0.397 | 182 | 212 | 232 | 202.87 | 39.51 | 0.19 | 96 | 118 | 127 | 108.09 | 33.67 | 0.31 |
| 2 (250) | 0.514 | 350.25 | 408 | 455.75 | 389.80 | 90.70 | 0.23 | 208.25 | 241.5 | 251 | 217.20 | 63.16 | 0.29 |
| 4 (500) | 0.586 | 695.5 | 809.5 | 901.5 | 772.19 | 184.68 | 0.23 | 452.25 | 491 | 502 | 448.89 | 111.89 | 0.24 |
| $m = 500$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (125) | 0.505 | 188 | 208 | 230 | 205.53 | 31.18 | 0.15 | 95 | 119 | 127 | 108.33 | 31.80 | 0.29 |
| 1 (250) | 0.740 | 357 | 404 | 448 | 396.81 | 70.90 | 0.17 | 211.75 | 243 | 250 | 223.17 | 51.46 | 0.23 |
| 2 (500) | 0.856 | 696 | 774.5 | 850 | 769.60 | 125.25 | 0.16 | 457 | 490 | 501 | 466.11 | 71.96 | 0.15 |
| 4 (1000) | 0.915 | 1323 | 1497 | 1679.5 | 1474.72 | 300.25 | 0.20 | 949 | 989 | 1000 | 930.13 | 166.93 | 0.17 |
| $m = 1000$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (250) | 0.833 | 363 | 403 | 438 | 398.32 | 56.30 | 0.14 | 215 | 242 | 251 | 227.90 | 41.92 | 0.18 |
| 1 (500) | 0.958 | 676 | 744 | 823 | 745.32 | 116.43 | 0.15 | 446.25 | 489 | 501 | 457.01 | 83.14 | 0.18 |
| 2 (1000) | 0.996 | 1271.5 | 1395 | 1542.5 | 1398.93 | 226.79 | 0.16 | 925 | 985.5 | 1000 | 926.62 | 155.20 | 0.16 |
| 4 (2000) | 1 | 2438.5 | 2688 | 2916.5 | 2665.75 | 435.85 | 0.16 | 1905 | 1976 | 1998 | 1870.88 | 300.12 | 0.16 |

Table 5: Empirical power, stopping time and changepoint estimate for $\gamma = 0.25$ and $k = [0.05mT]$.

| $m = 250$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (6) | 0.807 | 59.5 | 76 | 95 | 76.89 | 24.46 | 0.31 | 6 | 18 | 37 | 24.54 | 22.85 | 0.93 |
| 1 (12) | 0.964 | 80.75 | 105 | 139.25 | 113.18 | 43.78 | 0.38 | 11 | 27 | 53 | 36.99 | 33.07 | 0.89 |
| 2 (25) | 0.988 | 111 | 146 | 189 | 160.14 | 72.05 | 0.44 | 24 | 37 | 67 | 51.05 | 44.28 | 0.86 |
| 4 (50) | 0.998 | 150.25 | 196 | 254 | 215.77 | 96.08 | 0.44 | 46 | 55 | 78 | 67.80 | 44.56 | 0.65 |
| $m = 500$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (12) | 0.993 | 80 | 101 | 128 | 107.05 | 36.75 | 0.34 | 11 | 24 | 52 | 34.05 | 30.50 | 0.89 |
| 1 (25) | 1 | 111 | 135 | 172 | 144.28 | 49.03 | 0.33 | 24 | 36 | 61 | 45.88 | 32.29 | 0.70 |
| 2 (50) | 1 | 159 | 190 | 233 | 202.50 | 70.92 | 0.35 | 47 | 55 | 76 | 65.50 | 38.04 | 0.58 |
| 4 (100) | 1 | 227 | 275 | 332.25 | 285.95 | 83.72 | 0.29 | 92 | 102 | 117 | 106.95 | 36.24 | 0.33 |
| $m = 1000$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (25) | 1 | 113 | 135 | 163 | 140.62 | 38.53 | 0.27 | 23 | 34 | 58 | 42.99 | 30.75 | 0.71 |
| 1 (50) | 1 | 156.75 | 187 | 219 | 192.01 | 49.18 | 0.25 | 46.75 | 56 | 77.25 | 64.14 | 30.84 | 0.48 |
| 2 (100) | 1 | 235 | 275 | 319 | 281.93 | 64.80 | 0.22 | 92 | 101.5 | 117 | 106.19 | 31.44 | 0.29 |
| 4 (200) | 1 | 361 | 412 | 472 | 421.46 | 86.12 | 0.20 | 187.75 | 199 | 212 | 197.38 | 33.11 | 0.16 |

Table 6: Empirical power, stopping time and changepoint estimate for $\gamma = 0.25$ and $k = [0.5mT]$.

| $m = 250$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (62) | 0.241 | 84 | 102 | 115 | 91.17 | 33.62 | 0.36 | 18 | 50 | 61 | 42.75 | 24.16 | 0.56 |
| 1 (125) | 0.350 | 167 | 205 | 225 | 185.14 | 60.89 | 0.32 | 77.25 | 114 | 126 | 99.30 | 42.03 | 0.42 |
| 2 (250) | 0.472 | 350 | 409 | 455 | 384.27 | 103.04 | 0.26 | 210.75 | 241.5 | 252 | 217.57 | 63.89 | 0.29 |
| 4 (500) | 0.556 | 682.75 | 804 | 880.25 | 739.39 | 230.86 | 0.31 | 432.75 | 488 | 501 | 430.50 | 136.65 | 0.31 |
| $m = 500$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (125) | 0.497 | 180 | 209 | 230 | 198.31 | 46.30 | 0.23 | 92 | 119 | 128 | 106.29 | 36.84 | 0.34 |
| 1 (250) | 0.643 | 363 | 411 | 446 | 393.39 | 82.88 | 0.21 | 213 | 243 | 252 | 223.33 | 53.54 | 0.23 |
| 2 (500) | 0.803 | 697.5 | 789 | 876 | 771.40 | 157.22 | 0.20 | 453.5 | 490 | 501 | 457.12 | 92.34 | 0.20 |
| 4 (1000) | 0.897 | 1320 | 1503 | 1693 | 1459.73 | 362.23 | 0.24 | 936 | 988 | 1000 | 911.61 | 212.27 | 0.23 |
| $m = 1000$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (250) | 0.804 | 360 | 406 | 447 | 393.48 | 77.74 | 0.19 | 212 | 241.5 | 251 | 221.35 | 59.97 | 0.25 |
| 1 (500) | 0.951 | 670.5 | 750 | 833.5 | 744.29 | 133.66 | 0.17 | 453 | 489 | 500 | 456.21 | 89.41 | 0.19 |
| 2 (1000) | 0.993 | 1297 | 1427 | 1576 | 1422.85 | 256.28 | 0.18 | 943 | 986 | 999 | 935.72 | 152.72 | 0.16 |
| 4 (2000) | 1 | 2484.5 | 2710 | 2962.5 | 2698.17 | 458.50 | 0.16 | 1914 | 1979 | 1999 | 1881.67 | 298.38 | 0.15 |

Table 7: Empirical power, stopping time and changepoint estimate for $\gamma = 0.45$ and $k = [0.05mT]$.

| $m = 250$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (6) | 0.729 | 43 | 68 | 90 | 64.28 | 35.12 | 0.54 | 4 | 17 | 38.25 | 24.64 | 23.78 | 0.96 |
| 1 (12) | 0.922 | 67 | 101.5 | 143 | 105.05 | 61.50 | 0.58 | 10 | 27 | 55 | 38.11 | 36.54 | 0.95 |
| 2 (25) | 0.979 | 91 | 137 | 197 | 146.50 | 94.33 | 0.64 | 21 | 35 | 65 | 50.61 | 51.01 | 1.00 |
| 4 (50) | 0.993 | 130 | 194 | 284 | 218.15 | 153.63 | 0.70 | 43 | 56 | 86 | 73.00 | 62.28 | 0.85 |
| $m = 500$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (12) | 0.980 | 63 | 92 | 127 | 95.96 | 54.13 | 0.56 | 10 | 24 | 51 | 34.87 | 33.70 | 0.96 |
| 1 (25) | 0.998 | 92 | 124 | 166 | 129.43 | 66.50 | 0.51 | 21 | 33 | 57 | 43.29 | 38.22 | 0.88 |
| 2 (50) | 1 | 127 | 169 | 219 | 171.02 | 86.40 | 0.50 | 41 | 53 | 72 | 59.20 | 39.16 | 0.66 |
| 4 (100) | 1 | 203 | 260 | 325 | 259.13 | 120.38 | 0.46 | 86 | 100 | 114 | 98.22 | 46.35 | 0.47 |
| $m = 1000$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (25) | 1 | 87 | 116 | 150 | 117.69 | 55.93 | 0.47 | 19.75 | 30 | 53 | 40.03 | 33.73 | 0.84 |
| 1 (50) | 1 | 129 | 167 | 205 | 162.49 | 69.44 | 0.42 | 41.75 | 53 | 70 | 56.14 | 31.80 | 0.56 |
| 2 (100) | 1 | 200 | 242.5 | 291 | 238.43 | 95.57 | 0.40 | 86 | 99 | 113 | 97.02 | 40.77 | 0.42 |
| 4 (200) | 1 | 330 | 394 | 454 | 379.52 | 138.11 | 0.36 | 173 | 197 | 208 | 179.24 | 61.41 | 0.34 |

Table 8: Empirical power, stopping time and changepoint estimate for $\gamma = 0.45$ and $k = [0.5mT]$.

| $m = 250$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (62) | 0.231 | 3 | 8 | 96.5 | 40.12 | 47.99 | 1.19 | 2 | 4 | 51 | 22.11 | 26.54 | 1.20 |
| 1 (125) | 0.310 | 5 | 157 | 216 | 119.25 | 100.07 | 0.83 | 3 | 81.5 | 122 | 68.57 | 55.37 | 0.80 |
| 2 (250) | 0.416 | 6 | 371.5 | 440 | 264.38 | 202.17 | 0.76 | 6 | 217 | 248 | 156.63 | 112.02 | 0.71 |
| 4 (500) | 0.427 | 21.5 | 748 | 884 | 557.69 | 391.11 | 0.70 | 45.5 | 474 | 498.5 | 343.19 | 213.24 | 0.62 |
| $m = 500$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (125) | 0.403 | 7 | 189 | 224 | 140.70 | 97.82 | 0.69 | 4 | 95 | 124 | 76.57 | 56.52 | 0.73 |
| 1 (250) | 0.524 | 269.75 | 400 | 447.25 | 320.99 | 174.81 | 0.54 | 121 | 235 | 251 | 182.22 | 99.82 | 0.54 |
| 2 (500) | 0.703 | 646 | 775 | 891.5 | 679.03 | 305.66 | 0.45 | 422.5 | 486 | 501 | 406.55 | 172.94 | 0.42 |
| 4 (1000) | 0.820 | 1294 | 1544 | 1727.5 | 1333.82 | 613.14 | 0.45 | 891.5 | 983 | 1000 | 814.08 | 355.37 | 0.43 |
| $m = 1000$ | | | | | | | | | | | | | |
| | | Empirical first hitting time | | | | | | Changepoint estimate | | | | | |
| $T$ ($k$) | Power | 1st Q | Median | 3rd Q | Mean | Std | CV | 1st Q | Median | 3rd Q | Mean | Std | CV |
| 0.5 (250) | 0.718 | 336 | 406 | 448 | 347.23 | 155.39 | 0.44 | 171.5 | 234 | 249 | 192.20 | 90.57 | 0.47 |
| 1 (500) | 0.895 | 685.5 | 780 | 862.5 | 703.95 | 266.51 | 0.37 | 437.5 | 490 | 500 | 418.28 | 159.63 | 0.38 |
| 2 (1000) | 0.980 | 1287.25 | 1444 | 1631.25 | 1345.95 | 477.67 | 0.35 | 910.75 | 980 | 998 | 860.08 | 293.36 | 0.34 |
| 4 (2000) | 0.993 | 2570 | 2833 | 3114 | 2683.97 | 812.59 | 0.30 | 1898 | 1981 | 1999 | 1792.81 | 513.98 | 0.28 |

Table 9: Results of the monitoring procedure for three values of $\gamma$ (EFHT stands for empirical first hitting times).

| $\gamma = 0$ | | $\gamma = 0.25$ | | $\gamma = 0.45$ | |
|---|---|---|---|---|---|
| EFHT | Est. changepoints | EFHT | Est. changepoints | EFHT | Est. changepoints |
| 984 | 665 (1999/08/20) | 808 | 682 (1999/09/15) | 772 | 682 (1999/09/15) |
| 1580 | 1399 (2002/07/25) | 1554 | 1399 (2002/07/25) | 1529 | 1399 (2002/07/25) |
| 2222 | 2196 (2005/09/22) | 2209 | 2053 (2005/03/01) | 2208 | 2053 (2005/03/01) |
| 3014 | 2936 (2008/09/02) | 2945 | 2733 (2007/11/09) | 2890 | 2733 (2007/11/09) |

Table 10: Empirical correlations at different periods.

| Period | $\gamma = 0$ | $\gamma = 0.25$ | $\gamma = 0.45$ |
|---|---|---|---|
| 1 | 0.6274 | 0.6237 | 0.6237 |
| 2 | 0.5245 | 0.5264 | 0.5264 |
| 3 | 0.7249 | 0.7410 | 0.7410 |
| 4 | 0.6033 | 0.5364 | 0.5364 |
| 5 | 0.8021 | 0.7800 | 0.7800 |