A Monitoring Procedure for Detecting Structural Breaks in Factor Copula Models*

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Abstract

We propose a new monitoring procedure based on moving sums (MOSUM) for detecting single or multiple structural breaks in factor copula models. The test compares parameter estimates from a rolling window to those from a historical data set and analyzes the behavior under the null hypothesis of no parameter change. The case of multiple breaks is also treated. In the model, the joint copula is given by the copula of random variables which arise from a factor model. This is particularly useful for analyzing data with high dimensions. Parameters are estimated with the simulated method of moments (SMM). We analyze the behavior of the monitoring procedure in Monte Carlo simulations and a real data application. We consider an online procedure for predicting the day-ahead Value-at-risk based on the suggested monitoring procedure.

Keywords: Factor Copula Model, Monitoring procedure, Simulated Method of Moments, Value at Risk

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1. INTRODUCTION

Analyzing time-variant parameters in models for financial data is a research topic of wide importance. In this paper, we consider factor copula models which have been recently proposed by Oh and Patton (2017) and Krupskii and Joe (2013), and we focus on the first approach. In such models, the joint copula between random variables is given by the copula of random variables which arise from a factor model. The time-varying parameters are factor loadings and the parameters describing the distributions of the common and idiosynchratic factors.

The advantage of these models is that they can be used in relatively high dimensional applications and nevertheless capture the dependence structure by a fairly low number of parameters. Alternative copula models suitable for high-dimensional data are hierarchical Archimedean copulas (see Savu and Trede, 2010) and vine copulas (see Bedford and Cooke, 2002). We focus on factor copula models to have both considerable model flexibility and parsimonious parametrizations that allow for reliable statistical inference.

For the estimation of the model parameters, we use the simulated method of moments (SMM), which is different to standard method of moments applications, since the theoretical moment-counterparts are simulated and not as usual analytically derived. This makes asymptotic theory such as deriving consistency and asymptotic distribution results of the estimators more difficult. The reason is that the objective function is not continuous and furthermore not differentiable in the parameters and standard asymptotic approaches cannot be used here.

There are many papers which deal with monitoring procedures for detecting structural changes, for instance Hoga and Wied (2017), who construct a sequential monitoring procedure for changes in the tail index and extreme quantiles of beta-mixing random variables, which can be based on a large class of tail index estimators. Furthermore Pape, Wied, and Galeano (2017) propose a model-independent multivariate sequential procedure to monitor changes in the vector of component wise unconditional variances in a sequence of p-variate random vectors, where Galeano and Wied (2013) developed a monitoring procedure to test for the constancy of the correlation coefficient of a sequence of random variables. Here the basic idea is that an initial training sample with constant parameters is available and the goal is to monitor for changes in the correlation as new data become available. All the previously proposed monitoring procedures have in common that they are of a non-parametric kind. A parametric approach for detecting structural breaks is proposed, e.g., in Kurozumi (2017), where a monitoring test for parameter change in linear regression models with endogenous regressors is proposed. The test is of the CUSUM-type and relies on instrumental variable (IV) estimation.

The aim of this paper is to construct a new parametric monitoring procedure, based on moving sums (MOSUM), for the parameters in factor copula models. Rolling window parameter estimates are compared to the parameter estimates of an initial training sample for which we can assume constant parameter values. By using rolling window parameter estimates based on moving sums, new data has a stronger impact on the estimated parameter compared to CUSUM-type statistics. This approach can be expected to yield better power properties. Concerning the assumption of constant parameters for the initial training period, we suggest applying the retroperspective changepoint test in Manner, Stark, and Wied (2017) to pre-test this crucial assumption. These two tests complement each other in the sense that the monitoring procedure proposed here is meant for real-time monitoring of change-points, whereas the test in Manner et al. (2017) detects structural change in factor copulas in a retrospective way.

We study the asymptotic properties of the test and suggest a bootstrap procedure to approximate its resulting asymptotic distribution. We then analyze size and power properties of our procedure in single and multi break situations in Monte Carlo simulations. Finally, we use the monitoring procedure in a real-data application for a data set covering the last financial crisis. We also propose an online procedure for predicting the 1-day ahead Value-at-risk using simulations from the considered factor model accounting for the detected change-points. The rest of the paper is structured as follows: Section 2 presents the model and the monitoring procedure, whereas in Section 3 we study its asymptotic distribution under the setting of simulated method of moments estimation. Results from the Monte Carlo simulations can be found in Section 4. Section 5 presents our empirical application and Section 6 concludes the paper. The main proof can be found in the appendix.

2. MODEL, NULL HYPOTHESIS, DETECTORS AND MONITORING

In this section we present the factor copula model (Section 2.1), followed by our testing problem and the monitoring procedure (Section 2.2).

2.1. Factor copula model

We consider the same class of data-generating process as in Manner et al. (2017), i.e. the factor copula model proposed by Oh and Patton (2017). In this class the dynamics of the marginal distributions are determined by a parameter vector ϕ_0 and each variable can have time varying conditional mean $\boldsymbol{\mu}_t(\phi_0) := [\mu_{1t}(\phi_0), \dots, \mu_{Nt}(\phi_0)]'$ and variance $\boldsymbol{\sigma}_t(\phi_0) := \mathsf{diag}\{\sigma_{1t}(\phi_0), \dots, \sigma_{Nt}(\phi_0)\}$. The dependence function of the joint distribution of the innovations $\boldsymbol{\eta}_t$, namely the copula $C(\cdot, \theta_t)$, depends on the unknown parameter vector θ_t for $t = 1, \dots, T$, which we allow to be time-varying in general. The data-generating process is given by

$$[Y_{1t}, \dots, Y_{Nt}]' =: \boldsymbol{Y}_t = \boldsymbol{\mu}_t(\phi_0) + \boldsymbol{\sigma}_t(\phi_0)\boldsymbol{\eta}_t,$$

with $[\eta_{1t}, \dots, \eta_{Nt}] =: \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} \boldsymbol{F}_\eta = C(F_1(\eta_1), \dots, F_N(\eta_N); \theta_t)$, with marginal distributions F_i, where $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are $\mathcal{F}_{t-1}$-measurable and independent of $\boldsymbol{\eta}_t$. $\mathcal{F}_{t-1}$ is the sigma field containing information from the past $\{\boldsymbol{Y}_{t-1}, \boldsymbol{Y}_{t-2}, \dots\}$. Note that the $r \times 1$ vector ϕ_0 is assumed to be

$\sqrt{T}$ consistently estimable, which is fulfilled by most commonly used time series models, e.g. ARMA and GARCH models, and the corresponding estimator is denoted as $\hat{\phi}$.

For the contemporaneous dependence of the vector $\boldsymbol{\eta}_t$ we assume the factor copula model $C(\cdot, \theta_t)$, which is implied by the following factor structure

$$[X_{1t}, \ldots, X_{Nt}]' =: X_t = \boldsymbol{\beta}_t \boldsymbol{Z}_t + \boldsymbol{q}_t, \tag{2.1}$$

with $X_{it} = \sum_{k=1}^{K} \beta_{ik}^t Z_{kt} + q_{it}$, where $q_{it} \overset{iid}{\sim} F_{\boldsymbol{q}_t}(\alpha_{\boldsymbol{q}_t})$ and $Z_{kt} \overset{iid}{\sim} F_{\boldsymbol{Z}_{kt}}(\gamma_{kt})$ for $i = 1, \ldots, N$, $t = 1, \ldots, T$ and $k = 1, \ldots, K$. Note that Z_{kt} and q_{it} are independent $\forall i, k, t$ and the Copula for $\boldsymbol{X}_t$ is given by

$$\boldsymbol{X}_t \sim \boldsymbol{F}_{\boldsymbol{X}_t} = C(G_{1t}(x_{1t}; \theta_t), \ldots, G_{Nt}(x_{Nt}; \theta_t); \theta_t),$$

with marginal distributions $G_{it}(\cdot, \theta_t)$ and $\theta_t = \left[\{\{\beta_{ik}^t\}_{i=1}^N\}_{k=1}^K, \alpha_{\boldsymbol{q}_t}', \gamma_{1t}', \ldots, \gamma_{Kt}'\right]'$.

We assume that the dependence structure of $\boldsymbol{\eta}_t$, estimated by the time-series of residuals $\{\hat{\boldsymbol{\eta}}_t := \boldsymbol{\sigma}_t^{-1}(\hat{\boldsymbol{\phi}})[\boldsymbol{Y}_t - \boldsymbol{\mu}_t(\hat{\phi})]\}_{t=1}^T$, is determined by the copula $C(\cdot, \theta_t)$, where we are only interested by the implied factor copula from the factor structure (2.1) and completely ignore the marginal distributions $G_{it}(\cdot, \theta_t)$, which are in general different from the marginals $F_{it}(\cdot)$. The advantage of these models is that they can be applied in high dimensions and nevertheless capture the dependence structure by a relatively low numbers of parameters. Through the choice of the distributions of the common factor F_{Z_k} and the idiosyncratic error distribution F_q one can adapt asymmetry and tail dependence properties to the copula, which is useful when dealing with financial data. A (block-) equidependence structure can be accommodated by placing appropriate restrictions on θ_t. Note that we have a model with time-varying parameters in mind, where the type of time-variation is specified in the next subsection. The estimation of the the $p \times 1$ vectors $\theta_t \in \Theta$ of the copula is based on the simulated method of moments described in Section 3.1 below.

2.2. Null Hypothesis and Detectors

In this paper we want to test the null hypothesis of no parameter change of the factor copula model that is assumed to describe the residual dependence. The main idea is to compare parameter estimates from a training sample of size $\lfloor mT \rfloor$(that we call "initial sample" for the remainder of the paper), for which constant dependence is assumed, to sequentially estimated parameters from a rolling data window of the same size. Here T is the length of the monitored time series and m a value in $(0, 1]$.

Since we are interested in sequentially monitoring whether or not the parameter θ_t changes in $t = \lfloor mT \rfloor + 1, \ldots, T$, we assume that the parameters remain constant over the initial sample $t = 1, \ldots, \lfloor mT \rfloor$, meaning that:

Assumption 1.

$$\theta_1 = \cdots = \theta_{mT}. \tag{2.2}$$

In practice, if a sufficient amount of initial data is available, this assumption can be tested by using the test for parmeter constancy in factor copulas proposed in Manner et al. (2017). We are interested in testing the null hypothesis

$$H_0: \quad \theta_1 = \cdots = \theta_{mT} = \theta_{mT+1} = \ldots$$

versus the alternative

$$H_1: \quad \theta_1 = \cdots = \theta_{mT} = \cdots = \theta_{mT+k^*-1} \neq \theta_{mT+k^*} = \theta_{mT+k^*+1} = \ldots,$$

by using the detector

$$D_T(s) := m^2 T(\hat{\theta}_{1+(s-m)T:sT} - \hat{\theta}_{1:mT})'(\hat{\theta}_{1+(s-m)T:sT} - \hat{\theta}_{1:mT}), \tag{2.3}$$

where $k^* \geq 1$ and $\lfloor mT \rfloor + k^*$ is the unknown change point and $\hat{\theta}_{t_1:t_2}$ a consistent estimator for θ that is based on the subsample ranging from t_1 to t_2.

We stop our monitoring procedure if the detector defined in (2.3) exceeds the appropriately chosen critical value c for the first time k. This yields the stopping rule

$$\tau_T := \inf_k \left\{ k \leq T : D_T\left(\frac{k}{T}\right) > c \right\},$$

where τ_T is the stopping time of the monitoring procedure. Here c is chosen in a way that under H_0 the monitoring procedure holds the size level $\lim_{T,S\to\infty} P(\tau_T < \infty | H_0) = \alpha$, with $\alpha \in (0,1)$. We write $\tau_T < \infty$ to indicate that the monitoring has been terminated during the testing period, meaning that the detector crossed the boundary value c at a time point $k \leq T$. On the other hand, we write $\tau_T = \infty$, if D_T does not cross the boundary value during the testing period.

Note that the detected stopping time τ_T is not meant to be an estimator of change point, as the actual change point is likely to be earlier. This is due to the fact the monitoring procedure needs a sufficient number of observations after a change point before it can be detected. In the next chapter we present a procedure for estimating the change point conditional on H_0 having been rejected.

3. ESTIMATION AND ASYMPTOTICS

In this section we describe our theoretical results. The estimation of the factor copula model by the simulated method of moments (SMM) is reviewed in Section 3.1, whereas the asymptotic behaviour of our monitoring procedures is studied in Section 3.2. A bootstrap algorithm to approximate the asymptotic distribution is presented in Section 3.3 and a procedure for detecting multiple breaks is described in Section 3.4.

3.1. SMM Estimation

We are interested in estimating the parameter vector $\theta_{uT:vT}$ for the subsample ranging from $\lfloor uT \rfloor$ to $\lfloor vT \rfloor$, where $u < v$ and $u, v \in [\varepsilon, 1]$, with $\varepsilon > 0$. This is achieved by using the simulated method of moments (SMM), where the estimator is defined as

$$\hat{\theta}_{uT:vT,S} := \underset{\theta \in \Theta}{\arg\min}\ Q_{uT:vT,S}(\theta),$$

where $Q_{uT:vT,S}(\theta) := g_{uT:vT,S}(\theta)' \hat{W}_{(uT:vT)} g_{uT:vT,S}(\theta)$ is the objective function, $g_{uT:vT,S}(\theta) := \hat{m}_{uT:vT} - \tilde{m}_S(\theta)$ and $\hat{W}_{(uT:vT)}$ a positive definite weight matrix with probability limit W. The moment conditions $\hat{m}_{uT:vT}$ are $k \times 1$ vectors of appropriately chosen pairwise dependence measures $\hat{m}^{ij}_{uT:vT}$ (possibility averaged over equidependent pairs), computed from the residuals $\{\hat{\eta}_t\}_{t=\lfloor uT \rfloor}^{\lfloor vT \rfloor}$, whereas $\tilde{m}_S(\theta)$ is the corresponding vector of true dependence measures. Note that the dependence measures implied by the factor copula model are typically not available in closed form and they have to be obtained by simulation. Therefore, the classical method of moments (MM) or generalized method of moments (GMM) cannot be used here. The true dependence measures are approximated using S simulations $\{\tilde{\eta}_t\}_{t=1}^{S}$ from $\boldsymbol{F}_{\boldsymbol{X}_t}$, and hence the objective function, the estimator, and consequently our detector defined in equation (2.3) depend on the number of simulations S.

For the dependence measures of the pair (η_i, η_j), we use Spearman's rank correlation ρ^{ij} and the quantile dependence λ^{ij}_q. These are defined as

$$\begin{aligned}
\rho^{ij} :=& 12 \int_0^1 \int_0^1 C_{ij}(u_i, v_j) du_i dv_j - 3 \\
\lambda^{ij}_q :=& \begin{cases} P[F_i(\eta_i) \le q | F_j(\eta_j) \le q] = \frac{C_{ij}(q,q)}{q}, & q \in (0, 0.5] \\ P[F_i(\eta_i) > q | F_j(\eta_j) > q] = \frac{1-2q+C_{ij}(q,q)}{1-q}, & q \in (0.5, 1) \end{cases}
\end{aligned}.$$

The sample counterparts for the observations between $\lfloor uT \rfloor$ and $\lfloor vT \rfloor$ are defined as

$$\hat{\rho}^{ij} := \frac{12}{\lfloor vT - uT \rfloor} \sum_{t=\lfloor uT \rfloor}^{\lfloor vT \rfloor} \hat{F}_i^{uT:vT}(\hat{\eta}_{it}) \hat{F}_j^{uT:vT}(\hat{\eta}_{jt}) - 3$$

$$\hat{\lambda}_q^{ij} := \begin{cases} \frac{\hat{C}_{ij}^{uT:vT}(q,q)}{q}, & q \in (0, 0.5] \\ \frac{1-2q+\hat{C}_{ij}^{uT:vT}(q,q)}{1-q}, & q \in (0.5, 1) \end{cases},$$

where $\hat{F}_i^{uT:vT}(y) := \frac{1}{\lfloor vT-uT \rfloor} \sum_{t=\lfloor uT \rfloor}^{\lfloor vT \rfloor} \mathbb{1}\{\hat{\eta}_{it} \leq y\}$ and $\hat{C}_{ij}^{uT:vT}(u,v) := \frac{1}{\lfloor vT-uT \rfloor} \sum_{t=\lfloor uT \rfloor}^{\lfloor vT \rfloor} \mathbb{1}\{\hat{F}_i^{uT:vT}(\hat{\eta}_{it}) \leq u, \hat{F}_j^{uT:vT}(\hat{\eta}_{jt}) \leq v\}$. The simulated counterparts of these dependence measures based on the simulations $\{\tilde{\eta}_t\}_{t=1}^S$ are defined analogically and are denoted by $\tilde{\rho}^{ij}$ and $\tilde{\lambda}_q^{ij}$.

In summary, the SMM estimator minimizes the weighted difference between suitable sample dependence measures and their model counterparts obtained by simulation. Depending on the precise model specification, the pairwise dependence measures are averaged for groups, which have the same factor loadings. For more information on SMM estimation and a suitable way to average the pairwise dependence measures for equidependence or block equidependence models see Oh and Patton (2013) and Oh and Patton (2017).

3.2. Asymptotics

To derive the asymptotic distribution of our detector (2.3), we consider Assumption 1 and Assumptions 2-5 (given in the appendix) and follow similar steps as in Manner et al. (2017). The difference is that we replace the scale factor $s\sqrt{T}$ by $m\sqrt{T}$ and that we derive the following distributional limit for the process $m\sqrt{T} g_{1+(s-m)T:sT,S}(\theta)$, for $\frac{S}{T} \to k \in (0, \infty]$ and

$T, S \to \infty$,

$$
\begin{aligned}
& m\sqrt{T} g_{1+(s-m)T:sT,S}(\theta) = m\sqrt{T}\left(\hat{m}_{1+(s-m)T:sT} - \tilde{m}_S(\theta)\right) \\
=& m\sqrt{T}\left(\hat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) - m\sqrt{T}\left(\tilde{m}_S - m_0(\theta)\right) \\
=& m\sqrt{T}\left(\hat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) - \sqrt{\frac{T}{S}} m\sqrt{S}\left(\tilde{m}_S - m_0(\theta)\right) \\
\xrightarrow{d}& A(s) - \frac{m}{\sqrt{k}} A(1).
\end{aligned}
$$

Here $A(s)$ is a Gaussian process defined in the proof of Theorem 1 in the appendix. The limit result follows by using a steady transformation of the moment process defined in the proof of Theorem 1 (see appendix) and choosing $s = m = 1$ for the simulation term.

Theorem 1. Under the null hypothesis $H_0 : \theta_1 = \cdots = \theta_{mT} = \theta_{mT+1} = \ldots$ and under Assumptions 1 in Section 2.2 and Assumption 2-5 in the appendix, we obtain for $m \geq \varepsilon > 0$

$$
m\sqrt{T}\left(\hat{\theta}_{1+(s-m)T:sT,S} - \theta_0\right) \stackrel{d}{\Longrightarrow} A^*(s)
$$

as $T, S \to \infty$ in the space of Càdlàg functions on the interval $[m, 1]$ and $\frac{S}{T} \to k \in (0, \infty]$. Here, $A^*(s) = (G'WG)^{-1} G'W(A(s) - \frac{m}{\sqrt{k}} A(1))$ and θ_0 is the (constant) value of θ_t under the null. Note that G is the derivative matrix of g, which is the probability limit of $g_{.,S}$.

With Theorem 1 we obtain for $T, S \to \infty$

$$
\begin{aligned}
& m\sqrt{T}(\hat{\theta}_{1+(s-m)T:sT,S} - \hat{\theta}_{1:mT,S}) \\
=& m\sqrt{T}(\hat{\theta}_{1+(s-m)T:sT,S} - \theta_0) - m\sqrt{T}(\hat{\theta}_{1:mT,S} - \theta_0) \\
\xrightarrow{d}& A^*(s) - A^*(m).
\end{aligned}
$$

From this we can conclude the asymptotic behaviour of our detector under H_0, which we state in Corollary 1.

Corollary 1. Under the null hypothesis $H_0 : \theta_1 = \cdots = \theta_{mT} = \theta_{mT+1} = \ldots$ and if all mentioned Assumptions hold, we obtain for our detector

$$\begin{aligned}D_{T,S}(s) =& m^2 T(\hat{\theta}_{1+(s-m)T:sT,S} - \hat{\theta}_{1:mT,S})'(\hat{\theta}_{1+(s-m)T:sT,S} - \hat{\theta}_{1:mT,S}) \\ \xrightarrow{d}& (A^*(s) - A^*(m))'(A^*(s) - A^*(m)) =: Q(s)\end{aligned}$$

as $T, S \to \infty$ and $\frac{S}{T} \to k \in (0, \infty]$.

Similarly, we may define a monitoring detector that is based directly on the moment conditions. This allows for monitoring of the corresponding dependence measures in a model-free way. Under the assumed factor copula model it can be used to monitor the stability of the model parameters. Furthermore, it has the added advantage of being computationally much less demanding since no model parameters have to be estimated and it does not depend on any simulated quantities. The following corollary defines such a detector and describes its asymptotic behaviour.

Corollary 2. Under the null hypothesis $H_0 : \theta_1 = \cdots = \theta_{mT} = \theta_{mT+1} = \ldots$ and if all mentioned Assumptions hold, we obtain

$$\begin{aligned}M_T(s) :=& m^2 T(\hat{m}_{1+(s-m)T:sT} - \hat{m}_{1:mT})'(\hat{m}_{1+(s-m)T:sT} - \hat{m}_{1:mT}) \\ \xrightarrow{d}& (A(s) - A(m))'(A(s) - A(m)) =: R(s)\end{aligned}$$

as $T \to \infty$.

With the limit distribution of our detector $Q(s)$, we define the boundary value c in our monitoring procedure as the upper α-quantile of

$$\sup_{s \in [m,1]} Q(s) = \sup_{s \in [m,1]} (A^*(s) - A^*(m))'(A^*(s) - A^*(m)), \quad m \geq \varepsilon > 0. \tag{3.1}$$

Thus, $\lim_{T,S \to \infty} P(\tau_T < \infty | H_0) = \lim_{T,S \to \infty} P(\inf_k \{k \leq T : D_{T,S}(k) > c\} < \infty | H_0) = \alpha.$

In the same way the critical value of the moment monitoring procedure is determined as the upper α-quantile of $\sup_{s \in [m,1]} R(s)$.

For the estimation of the break point k^*, once H_0 is rejected, we propose

$$\hat{k} := \underset{\gamma \leq i \leq \tau_T - mT - 1}{\operatorname{argmax}} i^2(\tau_T - mT)(\hat{\theta}_{1+mT:mT+i,S} - \hat{\theta}_{1+mT:\tau_T-1,S})'(\hat{\theta}_{1+mT:mT+i,S} - \hat{\theta}_{1+mT:\tau_T-1,S}), \tag{3.2}$$

where we only consider the information from $mT + 1$ to $\tau_m - 1$. Note that we need to trim a sufficient fraction $\lfloor \gamma T \rfloor$ of the beginning, where $\gamma > 0$ to receive reasonable SMM estimates. In a similar way, the size of the rolling window mT should not be choosen to small. Note that the stopping time and the break point estimator for the moment monitoring procedure are defined analogically to the parameter monitoring procedure. As mentioned above, the moment based monitoring procedure is easy to implement and can be calculated fast, but in general it has lower power than the parametric procedure. Furthermore, as outlined in Manner et al. (2017), another disadvantage is that it does not allow testing the constancy of a subset of the parameters, but only can detect breaks in the whole copula. It may, however, be used to test for breaks in the dependence in selected regions of the support such as the lower tail. We leave this possibility for future research.

The limit distributions of $D_{T,S}$ and M_T are not known in closed form. To overcome this issue we have to simulate the critical values using an i.i.d. bootstrap procedure, which is described in the next section.

3.3. Bootstrap Distribution

First note that the limit result mainly consists of the limit distribution of the moment vectors, which can be computed relatively fast, compared to the detector that requires solving a minimization problem. This fact is used for the construction of the bootstrap. In order to approximate the limiting distribution under the null we use an i.i.d. bootstrap consisting of the following steps:

i) Sample with replacement from $\{\tilde{\eta}_i\}_{i=1}^T$ to obtain B bootstrap samples $\{\tilde{\eta}_i^{(b)}\}_{i=1}^T$, for

$b = 1, \dots, B$, where $\{\tilde{\eta}_i\}_{i=1}^T$ stacks the initial residual data $\{\hat{\eta}_i\}_{i=1}^{mT}$ and simulated data $\{\tilde{\eta}_i^*\}_{i=mT+1}^T$ from the assumed model, using the parameter estimate $\hat{\theta}_{1:mT,S}$ from the initial sample period.

ii) Use $\{\tilde{\eta}_i^{(b)}\}_{i=1+t-mT}^t$ to compute $\hat{m}_{1+t-mT:t}^{(b)}$ and $t = mT, \dots, T$ and use $\{\tilde{\eta}_i^{(b)}\}_{i=1}^T$ to obtain $\hat{m}_{1:T}^{(b)}$, for $b = 1, \dots, B$.

iii) Calculate the bootstrap version of the limiting distribution of our detector

$$K^{(b)} := \max_{t \in \{mT, \dots, T\}} \left(A^{*(b)}\left(\frac{t}{T}\right) - A^{*(b)}(m) \right)' \left(A^{*(b)}\left(\frac{t}{T}\right) - A^{*(b)}(m) \right),$$

with $A^{*(b)}\left(\frac{t}{T}\right) := (\hat{G}'\hat{W}_T\hat{G})^{-1}\hat{G}'\hat{W}_T A^{(b)}(\frac{t}{T})$ and $A^{(b)}(\frac{t}{T}) = m\sqrt{T}\left(\hat{m}_{1+t-mT:t}^{(b)} - \hat{m}_{1:T}^{(b)}\right)$, where $\hat{G}$ is the two sided numerical derivative estimator of G, evaluated at point $\theta_{1:mT,S}$, computed with the historical sample $\{\hat{\eta}_i\}_{i=1}^{mT}$. We can compute the k-th column of $\hat{G}$ by

$$\hat{G}^k = \frac{g_{T,S}(\hat{\theta}_{1:mT,S} + e_k\varepsilon_{T,S}) - g_{T,S}(\hat{\theta}_{1:mT,S} - e_k\varepsilon_{T,S})}{2\varepsilon_{T,S}}, \quad k \in \{1, \dots, p\},$$

where e_k is the k-th unit vector, whose dimension is $p \times 1$ and $\varepsilon_{T,S}$ has to be chosen in a way that it fulfils $\varepsilon_{T,S} \to 0$ and $\min\{\sqrt{T}, \sqrt{S}\}\varepsilon_{T,S} \to \infty$.

iv) Compute B versions of $K^{(b)}$ and determine the boundary value c such that

$$\frac{1}{B}\sum_{b=1}^{B} \mathbb{1}\{K^{(b)} > c\} \stackrel{!}{=} 0.05.$$

This bootstrap method is similar to the bootstrap used in Manner et al. (2017), where iii) is adapted to the monitoring situation. The same intuitive argument holds for the validity of the bootstrap, which is only based on natural estimators of the respective terms. Furthermore, draws from the empirical distribution are close to draws from the population distribution and the structure of the limiting distribution allows for a direct computation without the need for centering. Under the alternative of a change point the bootstrap quantiles are bounded because the i.i.d. bootstrap destroys the temporal structure of the time series and thus mimicks a stationary distribution. Our Monte Carlo simulations below confirm that the bootstrap indeed results in reasonably sized tests.

3.4. Multiple Break Testing

In practice if one is interested in detecting multiple structural breaks in factor copula models in real time, we propose the following procedure that consists of steps applying the monitoring procedure proposed in this paper and the retroperspective change point test for factor copulas from Manner et al. (2017). In particular, the retroperspective test is used to test for the constant parameter assumption (2.2) in the initial sample period and to detect the break point location once the monitoring procedure stops.

1) Compute the restrospective change point statistic $\sup_{s\in[\varepsilon,m]} P_{sT,S}$ from Manner et al. (2017) for the initial mT observation. If a changepoint is detected go to step 2a). If no changepoint is detected go to step 2b).

2a) Estimate the breakpoint location and remove all pre-change observations. Restock the subsample to mT observations and return to step 1). If there are not enough observations left to restock the subsample to mT observations go to step 4).

2b) Take the sample as initial sample period. Apply the monitoring procedure to the residuals, i.e. compute $D_{T,S}(s)$ for $s \in (m,1]$. Compute the bootstrap critical value c as described in Section 3.3. If a changepoint is detected go to step 3). If no changepoint is detected go to step 4).

3) Estimate the location of the changepoint. Then, remove the pre-change observations, use the first mT observations of the resulting dataset as the new initial sample and return to step 1). If there are not enough observations left to restock the subsample to mT observations go to step 4).

4) Terminate the procedure.

In the same way this procedure can be adapted for the moment monitoring procedure. Simulation results for single and multiple break testing, using the moment or the parameter monitoring procedure can be found in the next section. An obvious issue with this procedure is its multiple testing nature, in particular given that a pre-test has to be applied to the initial sample period to ensure that Assumption 1 holds. One should adapt the confidence levels accordingly and be aware of this when interpreting testing results.

4. SIMULATIONS

We now want to investigate the size and power and the estimation of the break point location of our monitoring procedure. We consider the simple one factor copula model, i.e. the copula implied by

$$[X_{1t}, \dots, X_{Nt}]' =: X_t = \beta_t \boldsymbol{Z}_t + \boldsymbol{q}_t, \tag{4.1}$$

where $Z_t \overset{init}{\sim} \textsf{Skew t}\,(\sigma^2, \nu^{-1}, \lambda)$ and $q_t \overset{iid}{\sim} \textsf{t}\,(\nu^{-1})$ for $t = 1, \dots, T$. We fix $\sigma^2 = 1$, $\nu^{-1} = 0.25$ and $\lambda = -0.5$, so that our model is parametrized by the factor loading parameter $\theta_t = \beta_t$. The sequential parameter estimates $\hat{\theta}_t = \hat{\theta}_{1-mT+t:t}$ for $t = mT, \dots, T$ in the detector are computed using the SMM approach with $S = 25 \cdot mT$ simulations. For this we use five dependence measures, namely Spearman's rank correlation and the 0.05, 0.10, 0.90, 0.95 quantile dependence measures, averaged across all pairs. Critical values for the monitoring procedure are computed using $B = 1000$ bootstrap replications. The nominal size of the tests is chosen to be 5% and we use 301 Monte Carlo replications for all settings.[1]

[1]All computations for this paper were implemented in Matlab, parallelized and performed using CHEOPS, a scientific High Performance Computer at the Regional Computing Center of the University of Cologne (RRZK) funded by the DFG.

4.1. Size and Single Break Case

The size of the testing period is always fixed to be $T = 1500$.

		$N = 10$	$N = 20$	$N = 30$
i)	$mT = 250$	0.059	0.059	0.056
	$mT = 400$	0.066	0.066	0.066
	$mT = 500$	0.066	0.069	0.073
ii)	$mT = 250$	0.079	0.083	0.076
	$mT = 400$	0.069	0.079	0.079
	$mT = 500$	0.076	0.096	0.076

Table 1: Empirical size for $\theta_0 = 1.0$, $T = 1500$ and 301 simulations, using i) the whole sample up to time point T and using ii) the initial data set and simulated data from $mT + 1$ up to T.

We begin with the case of testing against a single break. The rejection rate under the null are presented in Table 1 for $\theta_0 = 1$, for various combinations of the length of the initial sample mT and dimension N, where the critical values are calculated using one of the following two possibilities:

i) Calculate the critical value c using the whole, in general not known, data up to time point T. This mimics the situation that the test is used in a retrospective fashion, i.e. once all T observations are available.

ii) Calculate the critical value c using the initial data set together with the data from $mT + 1$ up to T, based on the estimated parameter vector $\hat{\theta}_{1:mT,S}$.

The test is slightly oversized for both settings. The empirical size is slightly higher for the second procedure ii), due to the fluctuation in the parameter estimation in the SMM procedure, but overall between 0.05 and 0.1.

To study the power of the procedure, we generate data with a break point at $\frac{T}{2}$, where the data is simulated with $\theta_t = 1$ for $t \in \{1, \ldots, \frac{T}{2}\}$, denoted as θ_0 and with $\theta_t = \{1.2, 1.4, 1.6, 1.8, 3.0\}$

for $t \in \{\frac{T}{2}+1, \ldots, T\}$, denoted as θ_1. With power we mean the probability that our monitoring procedure stops with in the monitored testing period ($\tau_T < \infty$). The upper panel of Table 2 reveals that the power of the procedure increases with the size of the initial sample for the two possibilities i) and ii). The moment monitoring procedure based on M_T has similar size characteristics but lower power compared to the parameter-based procedure. This result is in line with the results for the retrospective test in Manner et al. (2017).

The second and third panels of the table present the (average) relative stopping times and break point estimates using (3.2). The table reveals, that the averaged stopping time, given that a break has been detected, occurs with a significant delay after the true break point. It is closer to the true location $\frac{1}{2}$ for a smaller monitoring window, due to the greater impact of new data and, of course, for an increase of the step size between θ_0 and θ_1. If the step size is large enough ($\theta_1 = 3.0$) the monitoring procedure consistently stops shortly after the true break point.

The averaged estimated break point locations based on equation (3.2) are closer to the true break point. It always detects the break before the stopping time. For small shifts in θ it estimates the break too late, whereas for large shifts in θ break are estimated a little too early. A larger initial sample always results in later estimates of the break point. Note that the moment monitoring tends to result in later stopping times and break point estimates in all cases.

4.2. Multiple Breaks

For the analysis of multiple breaks we allow for breaks at $\frac{T}{3}$ and $\frac{2T}{3}$ with sample size $T = 1500$, and dimensions $N = 10$ and $N = 20$. The parameter varies from $\theta_0 = 1.0$ for $t \in \{1, \ldots, \frac{T}{3}\}$ to $\theta_1 = 1.5$ for $t \in \{\frac{T}{3}+1, \ldots, \frac{2T}{3}\}$ and $\theta_2 = 0.8$ for $t \in \{\frac{2T}{3}+1, \ldots, T\}$. The results using the procedure proposed in Section 3.4 can be found in Table 3. The tables report the averaged stopping times, averaged break point estimates and rejection rates for the first, second, and

			$\theta_0 = 1.0$	$\theta_1 = 1.2$	$\theta_1 = 1.4$	$\theta_1 = 1.6$	$\theta_1 = 1.8$	$\theta_1 = 3.0$
rej	i)	$mT = 250$	0.059	0.375	0.787	0.973	1.000	1.000
		$mT = 400$	0.066	0.435	0.877	0.993	1.000	1.000
		$mT = 500$	0.066	0.465	0.910	1.000	1.000	1.000
	ii)	$mT = 250$	0.079	0.409	0.787	0.967	1.000	1.000
		$mT = 400$	0.069	0.468	0.860	0.990	1.000	1.000
		$mT = 500$	0.076	0.485	0.894	1.000	1.000	1.000
	m_T	$mT = 250$	0.053	0.193	0.482	0.780	0.944	1.000
		$mT = 400$	0.076	0.223	0.671	0.944	0.993	1.000
		$mT = 500$	0.063	0.306	0.738	0.960	1.000	1.000
$\frac{\tau_T}{T}$	i)	$mT = 250$		0.715	0.667	0.625	0.579	0.513
		$mT = 400$		0.751	0.689	0.629	0.588	0.523
		$mT = 500$		0.767	0.677	0.639	0.596	0.525
	ii)	$mT = 250$		0.698	0.660	0.619	0.581	0.518
		$mT = 400$		0.733	0.675	0.626	0.587	0.525
		$mT = 500$		0.759	0.699	0.638	0.595	0.527
	m_T	$mT = 250$		0.718	0.695	0.662	0.627	0.525
		$mT = 400$		0.738	0.725	0.672	0.625	0.528
		$mT = 500$		0.765	0.741	0.679	0.626	0.530
$\frac{\hat{k}}{T}$	i)	$mT = 250$		0.516	0.487	0.483	0.473	0.457
		$mT = 400$		0.544	0.508	0.493	0.487	0.479
		$mT = 500$		0.562	0.522	0.497	0.492	0.487
	ii)	$mT = 250$		0.511	0.484	0.479	0.471	0.464
		$mT = 400$		0.538	0.502	0.491	0.485	0.483
		$mT = 500$		0.562	0.518	0.497	0.491	0.489
	m_T	$mT = 250$		0.517	0.495	0.487	0.486	0.485
		$mT = 400$		0.541	0.518	0.500	0.495	0.492
		$mT = 500$		0.561	0.534	0.507	0.499	0.496

Table 2: Rejection frequency (rej), average stopping time $\frac{\tau_T}{T}$ and average breakpoint estimate $\frac{\hat{k}}{T}$ for $\theta_0 = 1$, $T = 1500$ $N = 10$ and 301 simulations for the parameter monitoring procedure, where critical values c computed with the two possibilities i) and ii) and for the moment monitoring procedure. Data was generated with a break in $\frac{T}{2}$ and post-break parameter θ_1.

the joint first and second break events.

The rejection rates increase with the size of the initial sample period mT. Power increases in the dimension N, although this effect is only moderate for both tests. As before, the tests based on $D_{T,S}$ has larger power than the one based on M_T. We also note that the second

		τ_T^1	$\hat{k}^1$	rej_1	τ_T^2	$\hat{k}^2$	rej_2	$(\tau_T^1\ \tau_T^2)$	$(\hat{k}^1\ \hat{k}^2)$	rej_{all}
Parameter based										
$N=10$	$mT=250$	0.458	0.336	0.800	0.805	0.667	0.851	(0.458 0.801)	(0.337 0.665)	0.777
	$mT=400$	0.479	0.365	0.861	0.836	0.711	0.954	(0.479 0.836)	(0.365 0.716)	0.854
$N=20$	$mT=250$	0.457	0.338	0.810	0.799	0.661	0.864	(0.456 0.794)	(0.339 0.659)	0.787
	$mT=400$	0.475	0.363	0.874	0.827	0.708	0.970	(0.475 0.827)	(0.362 0.713)	0.867
Moment based										
$N=10$	$mT=250$	0.493	0.343	0.570	0.806	0.663	0.618	(0.495 0.796)	(0.344 0.661)	0.558
	$mT=400$	0.517	0.369	0.691	0.843	0.721	0.834	(0.517 0.840)	(0.369 0.729)	0.691
$N=20$	$mT=250$	0.491	0.338	0.591	0.808	0.660	0.671	(0.493 0.797)	(0.338 0.658)	0.588
	$mT=400$	0.513	0.366	0.730	0.839	0.714	0.884	(0.513 0.834)	(0.366 0.721)	0.730

Table 3: Average detected break point location $\frac{\hat{k}^i}{T}$, stopping time $\frac{\tau_T^i}{T}$ and rejection frequency using 301 simulations for the parameter monitoring procedure. Data was generated with breaks at $\frac{T}{3}$ and $\frac{2T}{3}$, with $T = 1500$, $N = 10, 20$, $\theta_0 = 1.0$, $\theta_1 = 1.5$, $\theta_2 = 0.8$. Results are based on the parameter based detector $D_{T,S}$ (top panel) and the moment based detector (bottom panel).

break point is detected more frequently than the first one. Furthermore, if the monitoring procedure detects the first break point it is very likely that the second break point is detected as well, which can be seen by the almost identical rejection rates of rej_1 and rej_{all}. Again, the average stopping time is much later than the true break, but the estimated break point $\hat{k}$ is able to detect the breaks reasonably well. Thus, we can conclude that the procedure works fairly well for the case of multiple breaks and that both the power of detecting changes and estimating the break locations can be achieved in a fairly reliable manner.

5. EMPIRICAL APPLICATION

In this section we apply our test to a real data set. We use daily log returns of stock prices over a time span ranging from 29.01.2002 to 01.07.2013 of ten large firms, namely Citigroup, HSBC Holdings ($), UBS-R, Barclays, BNP Paribas, HSBC Holdings (ORD), Mitsubishi, Royal Bank, Credit Agricole and Bank of America. This implies a monitored period of size $T = 2980$ and $N = 10$. Figure 5.1 is a plot of the stock values in US-$ of the ten assets over the whole monitored period.

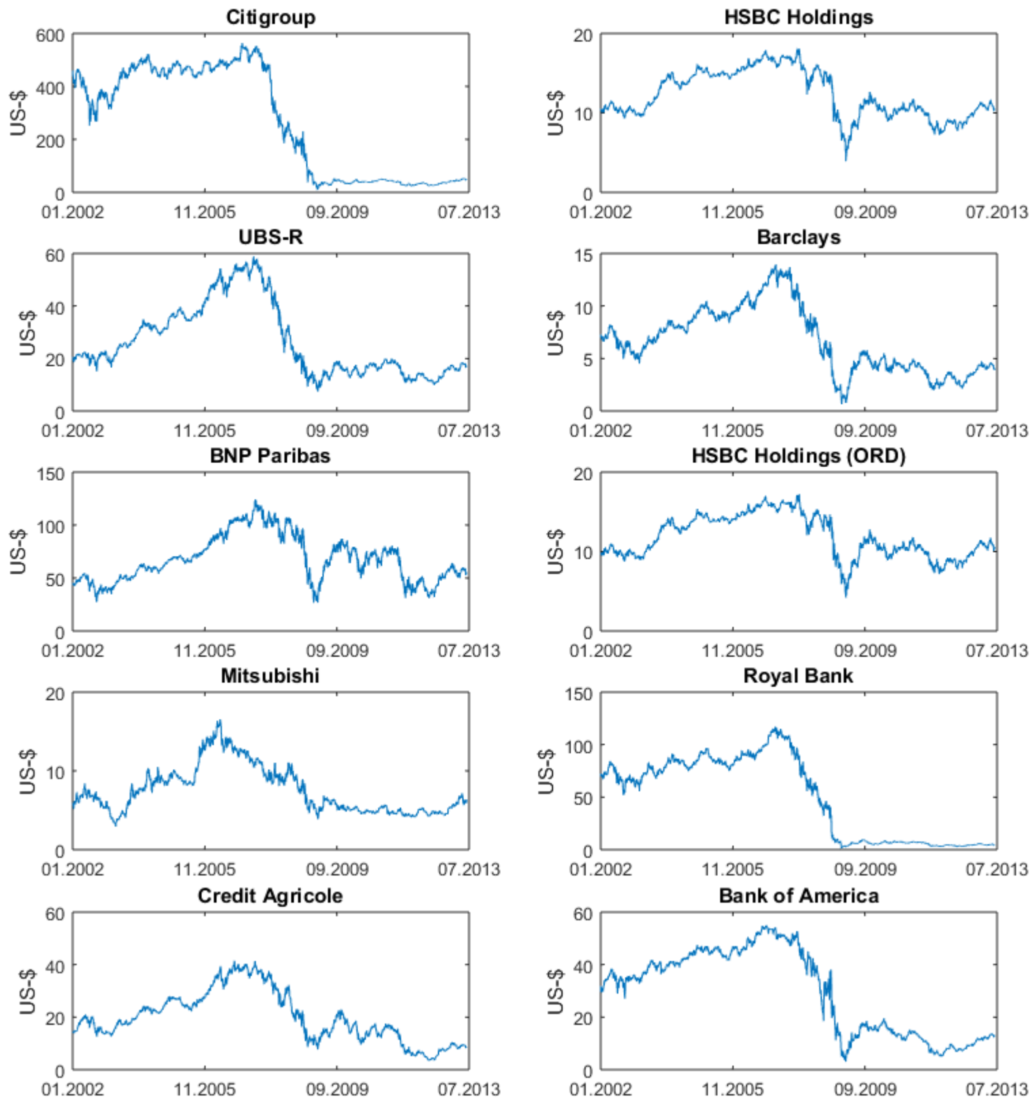

Figure 5.1: Asset values S_t^i in US-\$ in our considered portfolio for data between 29.01.2002 and 01.07.2013, $T = 2980$ and $N = 10$.

We use the same factor copula model as in (4.1) and we fix $\nu = 2.855$ and $\lambda = -0.0057$ for the monitoring procedure, i.e. we only monitor the factor loading parameter. These fixed values correspond the parameter estimates from the initial sample period of size $mT = 400$.

For the conditional mean and variance we specify the following AR(1)-GARCH(1,1).

$$r_{i,t} = \alpha + \beta r_{i,t-1} + \sigma_{i,t}\eta_{it},$$

$$\sigma_{it}^2 = \gamma_0 + \gamma_1\sigma_{i,t-1}^2 + \gamma_2\eta_{i,t-1}^2,$$

for $t = 2, \ldots, 2980$, and $i = 1, \ldots, 10$. Note that for the monitoring procedure the parameters of the conditional mean and variance models are always reestimated on the same rolling window sample of size mT.

5.1. Monitoring Procedure

A rolling window parameter analysis of the whole data set with window size 400 can be seen in Figure 5.2, indicating parameter changes between 2006 and 2009.

The results of the monitoring procedure of the whole considered period can be seen in Table 4. We choose the historical period $mT = 400$ from 29.01.2002 to 11.08.2003, where we first estimate the marginal AR(1)-GARCH(1,1) model to obtain the residuals. We use the retrospective constant parameter test from (Manner et al., 2017) to test the hypothesis of no parameter change with in the historical data set and the null hypothesis can not be rejected. Note that for the retrospective parameter test a burn in period of 20 % of the behold data is used. We then apply our constructed monitoring procedure. The monitoring procedure stops at the 21.11.2008 and the estimated break point location is found at the 19.07.2007, where we used the retrospective parameter break point estimate with data from the end of the historical data set 12.08.2003 to the stopping time 21.11.2008.

Figure 5.3 is a plot of $D_{T,S}$ for every time point between $mT+1$ (12.08.2003) and the stopping point, where $D_{T,S}$ exceeds the 0.95-quantile value of (3.1) equal to 4.4512.

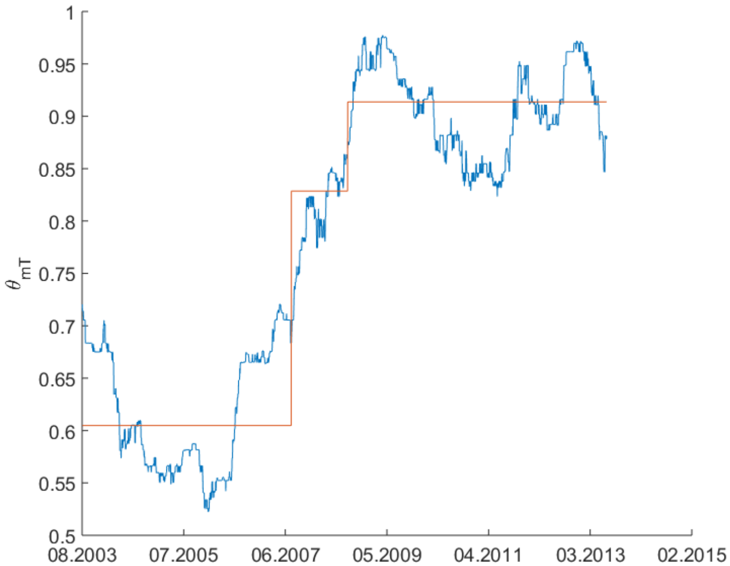

Figure 5.2: Rolling window estimate of θ_{mT} for $mT = 400$ and $N = 10$ between 11.08.2003 and 01.07.2013, with parameter values estimated from break to break. Each parameter value is associated to the end time point of the rolling window.

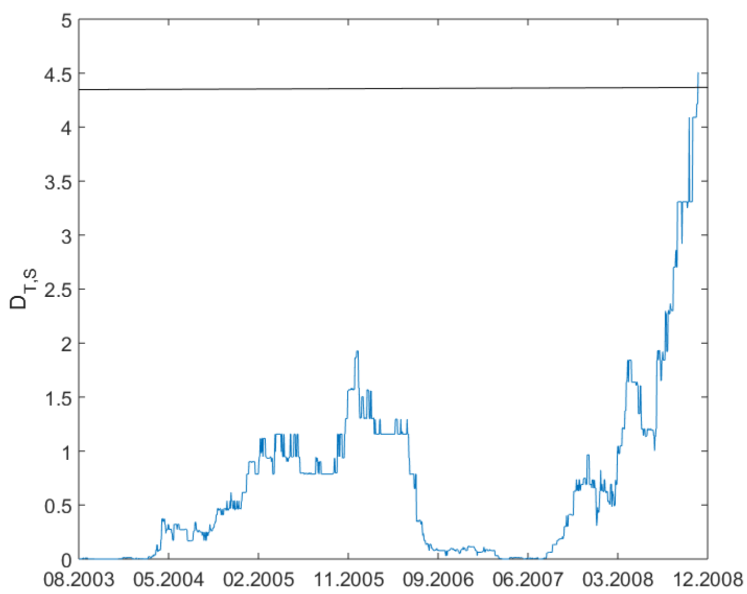

Figure 5.3: $D_{T,S}(s)$ for $T = 2980$, $mT = 400$ and $N = 10$. Stopping date at 21.11.2008 and $c = 4.4512$.

Monitored/Testing Period	τ_T	$\hat{k}$	T
29.01.2002-11.08.2003			400
12.08.2003-01.07.2013	21.11.2008	19.07.2007	2580
20.07.2007-29.01.2009		11.08.2008	400
12.08.2008-23.02.2010			400
24.02.2010-01.07.2013			874

Table 4: Stopping time τ_T, estimated break point location $\hat{k}$ and associated sample size T for monitored or tested periods using the monitoring procedure or the retroperspective parameter test.

We then cut of all the data in front of the estimated break point location (19.07.2007) and test for the null hypothesis of no parameter change in the period from 20.07.2007 to 29.01.2009 of size $mT = 400$, using again the retroperspective parameter test and the null is rejected. The estimated break point is found at the 11.08.2008.

For the next subsample we try the period from 12.08.2008 to 23.02.2010 and get a retroperspective test statistic value $S_{T,S}$ of 1.4442 and a quantile value of 2.5156, hence the null hypothesis can not be rejected and we choose this period as our new historical period and restart our monitoring procedure from 24.02.2010 to 01.07.2013. The detector $D_{T,S}$ does not cross the boundary value $c = 12.0020$ and the procedure stops at the end of the monitored period, without rejecting the null.

5.2. Value-at-Risk Predictions

Given the growing need for managing financial risk, risk prediction plays an increasing role in banking and finance. The value-at-risk (VaR), is the most prominent measure of financial market risk. Despite it having been criticized as being theoretically not efficient and numerically problematic (see Dowd and Blake, 2006), it is still the most widely used risk measure in practice. The number of methods for such calculations continues to increase. The theoretical and computational complexity of VaR models for calculating capital requirements is also increasing. Some examples include the use of extreme value theory see McNeil and Frey

(2000), quantile regression methods see Manganelli and Engle (2004), and Markov switching techniques see Gray (1996) and Klaassen (2002).

First, we want to define the Value at Risk (VaR). We define the log return of a single asset i to a time point t as $r_t^i = ln(S_t^i) - ln(S_{t-1}^i)$, where S_t^i is the stock value of asset i to a specific time point t. The change in the portfolio value over the time interval $[t-1, t]$ is then

$$\Delta V_t = \sum_{i=1}^{N} w_i r_t^i,$$

where w_i are portfolio weights. The (negative) α-quantile of the distribution of $\Delta V := \{\Delta V_t\}_{t=1}^T$ is the day t Value-at-risk at level α.

Here we want to show that our monitoring procedure can help improve the day-ahead predictions of the VaR based on a factor copula model.

The VaR predictions based on the monitoring procedure for the factor copula model are computed as follows. In general, based on $\mathcal{F}_t$, the information available at time t, we want to predict the VaR for period $t+1$. The prediction of the VaR is always based on the following four steps.

1. Simulate M draws from the copula model $\tilde{\boldsymbol{u}}_{t+1} \sim C(\cdot, \hat{\theta}_t)$, where $\tilde{\boldsymbol{u}}_{t+1} = [\tilde{\boldsymbol{u}}_{1,t+1}, \ldots, \tilde{\boldsymbol{u}}_{N,t+1}]$ is an $M \times N$ matrix of simulated observation and $\hat{\theta}_t$ is an appropriate parameter estimate based on information up to time t.

2. Use the inverse marginal distribution function of the standardized residuals η to transform every component of $\tilde{\boldsymbol{u}}_{t+1}$ to $\tilde{\boldsymbol{\eta}}_{t+1} = \left[F_1^{-1}(\tilde{\boldsymbol{u}}_{1,t+1}), \ldots, F_N^{-1}(\tilde{\boldsymbol{u}}_{N,t+1})\right]$, where $F_i^{-1}(\cdot)$ is estimated by the inverse integrated kernel density estimator of the residuals $\hat{\boldsymbol{\eta}}$ with a sufficiently large number of evaluation points.

3. Compute the simulated returns $\tilde{r}_{t+1} := [\tilde{r}_{t+1}^1, \ldots, \tilde{r}_{t+1}^N]' = \boldsymbol{\mu}(\hat{\phi}_t) + \boldsymbol{\sigma}(\hat{\phi}_t)\tilde{\boldsymbol{\eta}}_{t+1}$, where $\hat{\phi}_t$ are the estimated parameters from models for the conditional mean and variance using information up to time t.

4. Form the portfolio of interest from the simulated returns and compute the appropriate quantile from the distribution of the portfolio to obtain the VaR prediction for time $t + 1$.

This procedure for predicting the VaR is generic. The monitoring procedure for the copula parameter θ_t is used to determine the appropriate information set on which the parameter estimate in Step 1. is based. The basic idea is to use as much information as possible as long as no changepoint is detected. In case a changepoint is found only the most recent observations should be used to estimate θ_t. Recall that mT observations for which the dependence is assumed to be constant are available at the beginning of the sample. Further, denote $\hat{\theta}_{s:t}$ the estimator of the copula parameter based on the observations from time s to t. At each point in time t, compute $D_{T,S}(t)$.

i. Before a changepoint is detected, i.e. as long as $D_{T,S}(t) < c$ the draws from the copula in Step 1 above are based on $\hat{\theta}_{1:t}$

ii. Assume the monitoring procedure stops at time $t = \hat{\tau}$, i.e. when $D_{T,S}(t) > c$. Compute the breakpoint estimate $\hat{k}$ using (3.2). Use the estimate $\hat{\theta}_{\hat{k}:t}$ in Step 1 above. If $\hat{k} - t < 400$, i.e. if less than 400 observations are available use $\hat{\theta}_{t-400:t}$. In other words, after a breakpoint is identified use either all observations after the breakpoint estimate or the most recent 400 observations to estimate the copula parameter.[2]

iii. If $\hat{k} - t \leq mT$ proceed as in Step ii. Otherwise use the window $[\hat{k}, \hat{k} + mT]$ as the new initial sample and apply the monitoring procedure. As long as no further breakpoint is detected the parameter estimate $\hat{\theta}_{\hat{k}:t}$ is used. When the monitoring procedure stops again return to Step ii.

[2]The minimum number of observations required for model estimation depends on the complexity of the chosen model. However, for the type of model we are considering here we found that one needs at least 400 observations to obtain reliable and numerically stable parameter estimates.

The results for the online VaR evaluation based on $M = 1500$ simulations for each period and for $\alpha = 0.05$ can be seen in Figure 5.4. As an alternative, we consider the same model without the monitoring procedure. In that case the copula parameter is estimated using the full sample available at time t using an expanding window. The model for the margins is an AR(1)-GARCH(1,1) in both cases. Visually, the online procedure tracks the 5 % VaR well. The empirical VaR exceedance rate is, in fact, 6.05% (156 exceedances in 2580 days) and therefore reasonably close to 5 %. In the model without structural breaks, where the parameters are estimated from the beginning of the sample on, the exceedance rate is higher with 7.71% (199 exceedances). With a binomial test (compare Berens, Wied, Weiß, and Ziggel, 2014, we test the null hypothesis of unconditional coverage, i.e.,

$$\mathbb{E}\left(\frac{1}{T}\sum_{t=1}^{T} I_t(0.05)\right) = \alpha = 0.05,$$

where α is the VaR coverage probability and

$$I_t(0.05) = \begin{cases} 0, & \text{if } \Delta V_t \geq -VaR_{0.05} \\ 1, & \text{if } \Delta V_t < -VaR_{0.05}. \end{cases}$$

One expects 129 exceedances under H_0 and at the 1% significance level the critical value of the test is 158 exceedances. This implies that the null of unconditional coverage is rejected in the model without structural breaks, but not in the model with structural breaks.

6. CONCLUSION

We propose a new monitoring procedure for detecting structural breaks in factor copula models and analyse the behaviour under the null hypothesis of no change. Due to the discontinuity of the SMM objective function this requires additional effort to derive a functional limit

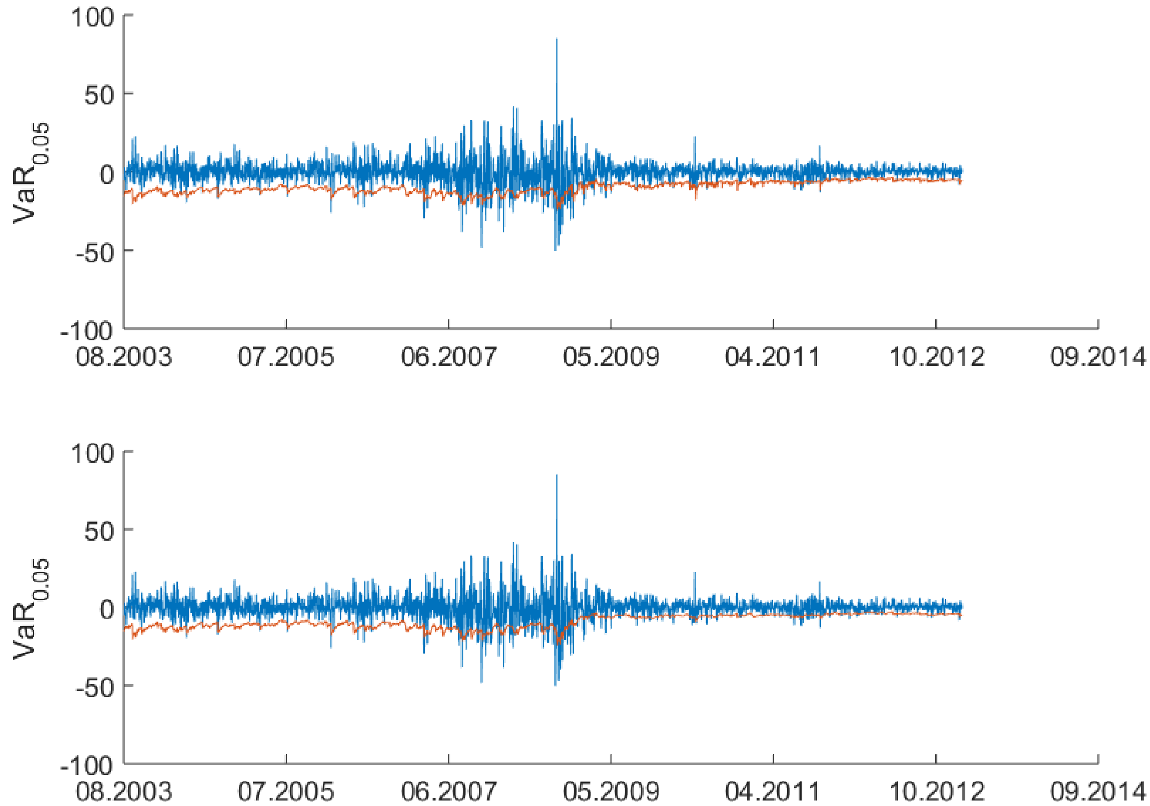

Figure 5.4: Portfolio returns ΔV_t and the $\alpha = 0.05$ predicted Value-at-Risk based on the monitoring procedure, allowing for structural breaks (upper panel) and without (lower panel) for the period between 29.01.2002 and 01.07.2013.

theorem for the model parameters. The presence of nuisance parameters in the asymptotic distribution of the two proposed detectors requires a bootstrap approximation for parts of the asymptotic distribution. The case of detecting multiple breaks is also treated. In simulations, the proposed procedures show good size and power properties in single and multiple break settings in finite samples. An empirical application to a set of 10 stock returns of large financial firms indicates the presence of break points around July 2007 and August 2008, time points of the heights of the last financial crisis. The proposed online Value-at-Risk procedure shows the usefulness of the monitoring procedure in portfolio management.

7. ASSUMPTIONS AND PROOF

7.1. Assumption

Assumption 2 and Assumption 3 ensure that the estimated rank correlation and quantile dependencies converge to their respective population counterparts.

Assumption 2. i) The distribution function of the innovations F_η and the joint distribution function of the factors $F_X(\theta)$ are continuous.

ii) Every bivariate marginal copula $C_{ij}(u_i, u_j; \theta)$ of $\mathbf{C}(u; \theta)$ has continuous partial derivatives with respect to $u_i \in (0,1)$ and $u_j \in (0,1)$.

The assumption is similar to Assumption 1 in (Oh and Patton, 2013), but the assumption on the copula is relaxed in the sense that the restriction of u_i and v_i is relaxed to the open interval $(0,1)$.

Assumption 3. Define $\gamma_{0t} := \sigma_t^{-1}(\hat{\phi})\dot{\mu}_t(\hat{\phi})$ and $\gamma_{1kt} := \sigma_t^{-1}(\hat{\phi})\dot{\sigma}_{kt}(\hat{\phi})$, where $\dot{\mu}_t(\phi) := \frac{\partial \mu_t(\phi)}{\partial \phi'}$ and $\dot{\sigma}_{kt}(\phi) := \frac{\partial [\sigma_t(\phi)]_{\text{k-th column}}}{\partial \phi'}$ for $k = 1, \ldots, N$. Define

$$d_t = \eta_t - \hat{\eta}_t - \left(\gamma_{0t} + \sum_{k=1}^{N} \eta_{kt}\gamma_{1kt}\right)(\hat{\phi} - \phi_0),$$

with η_{kt} is the k-th row of η_t and γ_{0t} and γ_{1kt} are $\mathcal{F}_{t-1}$-measurable, where $\mathcal{F}_{t-1}$ contains information from the past as well as possible information from exogenous variables.

i) $\frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor} \gamma_{0t} \xrightarrow{p} s\Gamma_0$ and $\frac{1}{T}\sum_{t=1}^{\lfloor sT \rfloor} \gamma_{1kt} \xrightarrow{p} s\Gamma_{1k}$, uniformly in $s \in [\varepsilon, 1]$, $\varepsilon > 0$, where Γ_0 and Γ_{1k} are deterministic for $k = 1, \ldots, N$.

ii) $\frac{1}{T}\sum_{t=1}^{T} E(\|\gamma_{0t}\|)$, $\frac{1}{T}\sum_{t=1}^{T} E(\|\gamma_{0t}\|^2)$, $\frac{1}{T}\sum_{t=1}^{T} E(\|\gamma_{1kt}\|)$ and $\frac{1}{T}\sum_{t=1}^{T} E(\|\gamma_{1kt}\|^2)$ are bounded for $k = 1, \ldots, N$.

iii) There exists a sequence of positive terms $r_t > 0$ with $\sum_{i=1}^{\infty} r_t < \infty$, such that the sequence $\underset{1 \le t \le T}{max} \frac{\|d_t\|}{r_t}$ is tight.

iv) $\underset{1\le t\le T}{max}\frac{\|\gamma_{0t}\|}{\sqrt{T}} = o_p(1)$ and $\underset{1\le t\le T}{max}\frac{|\eta_{kt}|\|\gamma_{1kt}\|}{\sqrt{T}} = o_p(1)$ for $k = 1, \ldots, N$.

v) $(\alpha_T(s), \sqrt{T}(\hat{\phi}-\phi_0))$ weakly converges to a continuous Gaussian process in $\mathcal{D}([0,1]^N)\times\mathbb{R}^r$, where $\mathcal{D}$ is the space of all Càdlàg-functions on $[0,1]^N$, with

$$\alpha_T(s) := \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor sT\rfloor}\left\{\prod_{k=1}^{N}\mathbb{1}\{U_{kt}\le u_k\} - \mathbf{C}(u;\theta)\right\}.$$

vi) $\frac{\partial F_\eta}{\partial \eta_k}$ and $\eta_k\frac{\partial F_\eta}{\partial \eta_k}$ are bounded and continuous on $\overline{\mathbb{R}}^N = [-\infty,\infty]^N$ for $k = 1, \ldots, N$.

vii) For $\mathbf{u} \in [0,1]^N$ and $\hat{\mathbf{F}}^{1+(s-m)T:st}(\hat{\eta}_t) = (\hat{F}_1^{1+(s-m)T:st}(\hat{\eta}_{1t}), \ldots, \hat{F}_N^{1+(s-m)T:st}(\hat{\eta}_{Nt}))$, the sequential empirical copula process

$$\frac{1}{\sqrt{T}}\left[\sum_{t=1+\lfloor (s-m)T\rfloor}^{\lfloor sT\rfloor}\mathbb{1}\{\hat{\mathbf{F}}^{1+(s-m)T:st}(\hat{\eta}_t)\le \mathbf{u}\} - C(\mathbf{u})\right]$$

converges in distribution to some limit process $A^*(s,\mathbf{u})$

Note that Assumption 3.7 is plausible and follows from a combination of the results in Bücher, Kojadinovic, Rohmer, and Segers (2014) and Remillard (2017).

The next assumption is needed for consistency of the successively estimated parameters. It is the same as Assumption 3 in (Oh and Patton, 2013) with the difference that part (iv) is adapted to our situation.

Assumption 4. i) $g_0(\cdot)$ is the limit function of $g_{\cdot,S}(\cdot)$ and it holds $g_0(\theta) = 0$ only for $\theta = \theta_0$ (the value of all θ_t under the null).

ii) The space Θ of all θ is compact.

iii) Every bivariate marginal copula $C_{ij}(u_i, u_j; \theta)$ of $\mathbf{C}(u;\theta)$ is Lipschitz-continuous for $(u_i, u_j) \in (0,1)\times(0,1)$ on Θ.

iv) The sequential weighting matrix $\hat{W}_{(s-m)T:sT}$ is $O_p(1)$ and $\underset{s\in[m,1]}{sup}\|\hat{W}_{(s-m)T:sT} - W\| \xrightarrow{p} 0$ for $m \ge \varepsilon > 0$.

Finally, we need an assumption for distributional results, which is the same as Assumption 4 in (Oh and Patton, 2013) with a difference in part iii).

Assumption 5. i) θ_0 is an interior point of Θ.

ii) $g_0(\theta)$ is differentiable at θ_0 with derivative G such that $G'WG$ is non singular.

iii) $\forall s \in [m,1], \varepsilon > 0$: $g_{\cdot,S}(\theta_{(s-m)T:sT,S})'\hat{W}g_{\cdot,S}(\theta_{(s-m)T:sT,S}) \leq \inf_{\theta\in\Theta} g_{\cdot,S}(\theta)'\hat{W}g_{\cdot,S}(\theta) + o_p^*((m^2T)^{-1})$, where $o_p^*((m^2T)^{-1})$ converges on the right hand side to zero and is therefore strictly positive.

7.2. Proof

Proof of Theorem 1

We consider the dependence measures Spearman's rho and quantile dependence measures, which are functions only depending on bivariate copulas.

Under the null and all mentioned Assumptions, we first want to show

$$m\sqrt{T}(\hat{m}_{(s-m)T:sT} - m_0(\theta_0)) \xrightarrow{d} A(s), \qquad T \to \infty, \quad \forall s \in [m,1], m \geq \varepsilon > 0$$

where $A(s)$ is a Gaussian process and θ_0 the value of all θ_t under the null.

By Assumption 3.7 (1) the sequential empirical copula of the N-dimensional random vectors fulfills

$$\begin{aligned}
\mathbb{C}_T :=& m\sqrt{T}\left[\hat{C}_{1+(s-m)T:sT}(\mathbf{u}) - C(\mathbf{u})\right] \\
=& \frac{1}{\sqrt{T}}\left[\sum_{t=1+\lfloor (s-m)T\rfloor}^{\lfloor sT\rfloor} \mathbb{1}\{\hat{\mathbf{F}}^{1+(s-m)T:sT}(\hat{\eta}_t) \leq \mathbf{u}\} - C(\mathbf{u})\right] \\
\xrightarrow[(1)]{d}& = A^*(s,\mathbf{u}), \quad T \to \infty, \quad \forall s \in [m,1], m \geq \varepsilon > 0,
\end{aligned}$$

where $\mathbf{u} \in [0,1]^N$, $\mathbf{u}^{(j)} \in [0,1]^N$ defined by $\mathbf{u}_i^{(j)} = \mathbf{u}_j$, if $i = j$ and 1 otherwise and $\hat{\mathbf{F}}^{1+(s-m)T:sT}(\hat{\eta}_t) = (\hat{F}_1^{1+(s-m)T:sT}(\hat{\eta}_{1t}), \ldots, \hat{F}_N^{1+(s-m)T:sT}(\hat{\eta}_{Nt}))$. Here, $\hat{F}_j^{1+(s-m)T:sT}$ denotes the marginal empirical distribution function of the j-th component calculated from the data time point $1+\lfloor(s-m)T\rfloor$ up to the time point $[sT]$. Moreover $\mathbb{B}(s-m,s,\mathbf{u}) := \mathbb{Z}(s,\mathbf{u})-\mathbb{Z}(s-m,\mathbf{u})$, where $\mathbb{Z}(s,\mathbf{u})$ is a tight centered continuous Gaussian process with

$$\mathsf{Cov}(\mathbb{Z}(s,\mathbf{u}),\mathbb{Z}(t,\mathbf{v})) = \mathsf{min}(s,t)\mathsf{Cov}(\mathbb{1}(\mathbf{F}(\eta)\leq \mathbf{u}),\mathbb{1}(\mathbf{F}(\eta)\leq \mathbf{v})).$$

Note that Spearman's rho between the i-th and j-th component is given by

$$12\int_0^1\int_0^1 C(1,\ldots,1,u_i,1,\ldots,1,u_j,1,\ldots,1)du_idu_j - 3$$

and that the quantile dependencies are projections of the N-dimensional copula onto one specific point divided by some prespecified constant. Define the function $m^{ij}(C)$ as the function which generates a vector of all considered dependence measures (Spearman's rho and/or quantile dependencies for different levels) between the i-th and j-th component out of the copula C. Without loss of generality consider the equicontinuity case, then the function

$$\begin{aligned} m(C): D[0,1]^N &\rightarrow \mathbb{R}^k \\ C &\rightarrow m(C) = \frac{2}{N(N-1)}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} m^{ij*}(C) \end{aligned}$$

is continuous and we directly obtain

$$\begin{aligned} m\sqrt{T}(\hat{m}_{1+(s-m)T:sT} - m_0(\theta)) =& m\sqrt{T}\left[m(C^s) - m(C)\right] \\ & \xrightarrow{d} \frac{2}{N(N-1)}\left(\sum_{i,j} m^{ij}(A^*(s,\mathbf{u}))\right) =: A(s) \end{aligned}$$

as $T\to\infty$ with $s\in[m,1], m\geq\varepsilon>0$. Here, $m^{ij}(\cdot)$ is the same function as $m^{ij*}(\cdot)$ with the only difference that the formula for Spearman's rho between the i-th and j-th component is replaced by

$$12\int_0^1\int_0^1 C(1,\ldots,1,u_i,1,\ldots,1,u_j,1,\ldots,1)du_idu_j.$$

Then we receive for $\frac{S}{T}\to k\in(0,\infty]$ and $T,S\to\infty$

$$
\begin{aligned}
&m\sqrt{T}g_{1+(s-m)T:sT,S}(\theta) = m\sqrt{T}\left(\hat{m}_{1+(s-m)T:sT} - \tilde{m}_S(\theta)\right)\\
=&m\sqrt{T}\left(\hat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) - m\sqrt{T}\left(\tilde{m}_S - m_0(\theta)\right)\\
=&m\sqrt{T}\left(\hat{m}_{1+(s-m)T:sT} - m_0(\theta)\right) - \sqrt{\frac{T}{S}}m\sqrt{S}\left(\tilde{m}_S - m_0(\theta)\right)\\
\xrightarrow{d}&A(s) - \frac{m}{\sqrt{k}}A(1).
\end{aligned}
$$

The limit result then follows with the same proof steps as in Manner et al. (2017), using the given limit result for $m\sqrt{T}g_{1+(s-m)T:sT,S}(\theta)$ and replacing the scale factor $s\sqrt{T}$ by $m\sqrt{T}$. This completes the proof. $\square$

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