A polarization index for overlapping groups *

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Abstract

The well-known index of income bi-polarization proposed by Wolfson (1994) requires two groups to be split according to the median income and, therefore, to be non-overlapping. Aim of this paper is to propose a new polarization index in the spirit of the Wolfson index. It allows for any possible partition of the population in two or more (also overlapping) groups. The new index maintains the simplicity and immediate comprehension of the Wolfson index, though being much more flexible. An application is then provided to German and Italian income data.

Keywords: Polarization index; Gini index decomposition; Overlapping groups; Increased bipolarity; Modified increased spread; Income distribution

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1 Introduction

In the last two decades the measurement of polarization has gained some attention in the literature on social indicators. Based on the seminal papers by Esteban and Ray (1994) and Wolfson (1994), two different strands of the literature have emerged, underlining different aspects of the phenomenon of income polarization. The first string, originating from Esteban and Ray (1994), focuses on the rise of separated income groups: polarization increases if the groups become more homogeneous internally, more separated from each other and more equal in size. This approach is followed, among others, by Gradín (2000), D´Ambrosio (2001), Zhang and Kanbur (2001), Duclos et al. (2004), Anderson (2004), Esteban et al. (2007), Lasso de la Vega and Urrutia (2006) and Anderson et al. (2009).

The second strand, going back to Wolfson (1994, 1997), describes the decline of the middle class, measuring how the center of the income distribution is emptied. This approach is often referred to as "bi-polarization" and assumes the presence of only two groups which are divided by the median income. Authors like Wolfson (1997), Wang and Tsui (2000), Chakravarty and Majumder (2001), Rodríguez and Salas (2003), Chakravarty et al. (2007), Deutsch et al. (2007), Bossert and Schworm (2008), Chakravarty and D'Ambrosio (2010) as well as Lasso de la Vega et al. (2010) develop and extend this alternative approach. See Chakravarty (2009) for a review of the main proposals in income polarization measurement.

Interestingly, for example Rodríguez and Salas (2003) show that there exist similarities between the two approaches, since the Wolfson index can be rewritten as a difference of the Gini index between groups and the Gini index within groups; thus revealing that, also according to Wolfson's measure, polarization increases with inequality between groups and decreases with inequality within groups. However, since the commonly used Gini decompositions are two-term decompositions only in case of non-overlapping groups, Wolfson's measure and its subsequent extensions have the shortfall of assuming only two non-overlapping groups, thus reducing considerably its applicability. In case of pure income polarization this might not be a serious drawback but if one is interested in what Gradín (2000) calls "socioeconomic" polarization, meaning that groups are formed by other characteristics than income, e.g. race or religion, groups are generally overlapping and this type of measures cannot be applied any more.

In general most of the existing contributions to polarization measurement focus on income as the group forming characteristic. In contrast, Zhang and Kanbur (2001) and Gradín (2000) derive polarization measures that allow for other characteristics than income to form groups and measure income polarization among those exogenously formed groups.

In this paper we want to combine the original Wolfson approach on income polarization with the possibility of exogenously formed groups. We propose a new polarization index based on a recent decomposition of the Gini index into two components, proposed by Okamoto (2009), that holds also in case of overlapping groups. Analogously to the Wolfson index, also the new index is based on the Gini index and is expressed as a difference of inequality between and within groups. However, our polarization index is more general than the Wolfson index, since it does not require the groups to be split according to the median income nor to be non-overlapping. It rather allows for any possible partition of the population, also with more than two groups. Therefore, our index still maintains the simplicity and immediate comprehension of the Wolfson index, though being much more flexible.

The paper is organized as follows. We first provide a short review of the existing literature on polarization measurement in Section 2. In Section 3 we introduce a new polarization measure that is based on the decomposition of the Gini coefficient proposed by Okamoto (2009) and that can be seen as a generalization of several approaches of polarization measurement, namely the approaches of Wolfson (1994), Wolfson (1997), D´Ambrosio (2001), Rodríguez and Salas (2003), Zhang and Kanbur (2001) and Silber et al. (2007). As a consequence of the extension to exogenously formed groups we have to adapt the classical polarization postulates (increased spread and increased bipolarity) and extend them to the new setting, which is done in Section 4. Next, in Section 5 the case of two groups, overlapping or not, is investigated. Section 6 covers the general case of $k \geq 2$ possibly overlapping groups. Afterwards, we illustrate the new measure through an empirical application based on Italian and German data in Section 7. Section 8 concludes and addresses the possible extension of our approach to a measure of polarization in many attributes.

2 A brief review of income polarization measurement

As already stressed in the introduction, the literature on univariate income polarization has its origins in the two pioneering papers of Wolfson (1994) and Esteban and Ray (1994). Independent of each other they claimed that polarization cannot be appropriately measured by indices of income inequality. They obtained special polarization measures, Esteban and Ray (1994) from an axiomatic point of view and Wolfson (1994) based on the phenomenon of the declining middle class. In this section we want to give a short review on the most common univariate polarization measures and their connection to well-known measures of inequality.

Let us first introduce some notation. Let $\boldsymbol{x}_j = \left(x_{j1}, \ldots, x_{jn_j}\right)$ be the income vector of individuals belonging to group j, and $\bar{x}_j$ be the corresponding mean income, $j = 1, \ldots, k$. The whole population has size $n = \sum_j n_j$. Denote by $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$ the income vector of the whole population, ordered according to the k groups, and let $\boldsymbol{N} = (n_1, \ldots, n_k)$ signify the vector of groups' sizes. The indices considered in this paper depend on $\boldsymbol{x}$ and $\boldsymbol{N}$. For simplicity, we suppress these arguments whenever possible.

Esteban and Ray (1994) set up a system of four axioms and derive the following polarization measure

$$P^{ER} = K \sum_{j=1}^{k} \sum_{l=1}^{k} \left(\frac{n_j}{n}\right)^{1+\alpha} \cdot \frac{n_l}{n} \cdot |\bar{x}_j - \bar{x}_l| \,, \tag{1}$$

where $K > 0$ is a normalizing constant and $\alpha \in [1; \alpha^*], \alpha^* \approx 1.6$ is the so called polarization sensitivity. Note that the inclusion of α makes the difference between polarization and inequality in this approach, since for $\alpha = 0$ and $K = \frac{1}{2 \cdot \bar{x} \cdot n^2}$ expression (1) gets back to the Gini index.[1] According to Esteban and Ray (1994), polarization depends on individuals' sense of identification and alienation. Individuals are assumed to feel identified with the individuals belonging to their same group, and alienated versus the individuals belonging to different groups. Hence, polarization increases with alienation, measured in terms of distance between groups' income means ($|\bar{x}_j - \bar{x}_l|$), and it decreases with identification, which depends on the group size to which the individual belongs ($\left(\frac{n_j}{n}\right)^{1+\alpha}$).

Since P^{ER} does not incorporate any within-group heterogeneity, Esteban et al. (2007) proposed an extension of the original index that is given by

$$P^{EGR} = K \left[\sum_{j=1}^{k} \sum_{l=1}^{k} \left(\frac{n_j}{n}\right)^{1+\alpha} \left(\frac{n_l}{n}\right) |\bar{x}_j - \bar{x}_l| \right] - \Big[\beta \cdot (G - G_B)\Big], \tag{2}$$

where G and G_B are the total and between-group Gini indices and $\beta \geq 0$ is an additional parameter which reflects the weight of the correction term.[2] Esteban et al. (2007) recommend that their measure should be applied after the original vector of incomes has been grouped by a statistical approach that minimizes within-group dispersion. Gradín (2000) proposed the same correction term to the original index, though in his approach groups can be defined according to variables different from income.

However, Lasso de la Vega and Urrutia (2006) point out that these extended measures violate some basic properties of polarization. In particular, greater inequality between groups and at the same time constant inequality within groups may go

[1] Being precisely, for $\alpha = 0$ we would obtain the Gini index for classified data. This is because Esteban and Ray (1994) argue that an individual feels perfect identification with each member of his or her own subgroup, regardless of possible income differences. Therefore, each income of a group can be replaced by the respective group mean.

[2] Esteban et al. (2007) claim that incomes should be normalized to unity, so they remove the scaling parameter K in their measure.

along with a decline in the polarization according to (2). Therefore they propose an alternative extension of the ER index of polarization which is given by

$$P^{LU} = K \sum_{j=1}^{k} \sum_{l=1}^{k} \left(\frac{n_j}{n}\right)^{1+\alpha} \left(\frac{n_l}{n}\right) (1 - G_j)^{\beta} |\bar{x}_j - \bar{x}_l| \tag{3}$$

with G_j being the Gini coefficient of group j, $\alpha \in [1, 1.6]$, $\beta \geq 0$ and $K > 0$.

The second strand of the literature on income polarization is based on the polarization measure of Wolfson (1994). Additional to the existing notation, let m denote the median of the incomes and $L(z)$ be the value of the Lorenz curve at the z-quantile of $\boldsymbol{x}$. Wolfson (1994) proposes a polarization curve analogous to the Lorenz curve for inequality measurement and defines his polarization measure to be

$$P^W = \frac{2\bar{x}}{m} \cdot (1 - 2 \cdot L(0.5) - G) \; . \tag{4}$$

In this formulation the link between polarization and inequality measurement is obvious since the Gini index is used explicitly. Using a subgroup decomposition of the Gini index we can make this link even more explicit. Let us assume that the population is divided into two groups, where the first group, L, consists of all individuals with incomes below the median income and the second group, H, consists of all individuals with higher incomes. Let us denote the two resulting income vectors as $\boldsymbol{x}_L$ and $\boldsymbol{x}_H$ with corresponding means $\bar{x}_L$ and $\bar{x}_H$ and group shares $\frac{n_L}{n} = \frac{n_H}{n} = \frac{1}{2}$, assuming n even. Also, let x_{Li} and x_{Hi} be income of the i-th individual in group L and H, respectively.

Then the overall Gini index G (or $G(\boldsymbol{x})$) can be decomposed into a within group G^W and a between-group G^B component as follows

$$G(\boldsymbol{x}) = \tfrac{1}{2\bar{x}n^2} \textstyle\sum_{i=1}^{n} \sum_{m=1}^{n} |x_i - x_m| \tag{5}$$

$$= \underbrace{\frac{\bar{x}_L}{4 \cdot \bar{x}} \cdot G(\boldsymbol{x}_L) + \frac{\bar{x}_H}{4 \cdot \bar{x}} \cdot G(\boldsymbol{x}_H)}_{G^W} + \underbrace{0.5 - L(0.5)}_{G^B} \; . \tag{6}$$

Using this decomposition one can rewrite the Wolfson measure according to

$$P^W = \frac{2\bar{x}}{m} \cdot (G^B - G^W) \, . \tag{7}$$

Therefore, the Wolfson polarization measure is a normalized function of the difference between the Gini index between groups G^B and the Gini index within groups G^W.[3] As Rodríguez and Salas (2003) pointed out, this formulation can be used

[3]This is not a new result; see e.g. Rodríguez and Salas (2003). We discuss it here, since we will compare this Gini decomposition with the one used for our new polarization index. Note that Rodríguez and Salas (2003) obtained the Wolfson expression as in (7) from a "geometric" reasoning based on the Lorenz curve.

to distinguish between polarization and inequality and to establish a link between the two strands of polarization measurement. Clearly there is a difference between adding up the two components, as it is done in inequality measurement, and taking the differences as it is in polarization measurement. Moreover, the between-group component can be seen as a measure of the alienation between groups and the within-group component as a measure of the identification within groups, two basic ideas in the Esteban and Ray (1994) approach.

Several other polarization measures are explicitly constructed as functions of inequality measures. In particular, Rodríguez and Salas (2003) propose an extension of Wolfson's polarization measure that includes an additional sensitivity parameter v. Their measure is based on a subgroup decomposition of the extended Gini coefficient introduced by Donaldson and Weymark (1980) in the case of population divided by the median, and is defined as

$$P^{RS}(\boldsymbol{x}, v) = G^{B}(\boldsymbol{x}, v) - G^{W}(\boldsymbol{x}, v)\,, \quad v \in [2,3]\,.$$

The idea of measuring polarization in terms of the difference of a between-group inequality component and a within-group inequality component (measured through the Gini index) is also present in the polarization index of Silber et al. (2007), which is defined as

$$P^{SDH} = \frac{G^{B} - G^{W}}{G}\,,$$

where it is assumed that the population is divided by the median income.

Rather than using the difference between the two inequality components, Zhang and Kanbur (2001) use the ratio of these two components and define their polarization measure to be

$$P^{ZK} = \frac{I^{B}}{I^{W}}\,,$$

where I^B and I^W are the between- and within-group component of Theil's measure of inquality I.[4] Since this measure is not defined if the within-group inequality is zero, Silber et al. (2007) propose the following modification

$$P^{ZK^*} = \frac{P^{ZK} - 1}{P^{ZK} + 1} = \frac{I^{B} - I^{W}}{I}\,.$$

[4]Zhang and Kanbur (2001) use Theil's measure instead of the Gini index. In principle, the formulation of their polarization measure allows for any decomposable inequality index that satisfies the Pigou-Dalton transfer axiom.

3 A new polarization index based on Gini decomposition

In this section we will construct a polarization measure that is in line both with the intuition of Wolfson bi-polarization and with the ideas of identification and alienation of Esteban and Ray (1994). Therefore, in the next paragraphs we will introduce some basic requirements, which our polarization index has to meet.

Firstly, polarization is regarded to increase with respect to the identification within the groups. Generally, it is assumed that the identification within a group is reflected by its inequality, in a sense that higher inequality within groups means lower identification and vice versa.

Secondly, polarization is seen to decrease with respect to the alienation between groups. Alienation is measured in an obvious way via between-group inequality (Esteban and Ray, 1994); the higher the inequality between the groups, the higher the alienation.

Note that this approach is in line with the intuition of Wolfson bi-polarization too. If we split the population at the median, then the inequality within the two new groups is smaller than the inequality within the original population, and the inequality between the two new groups becomes larger than zero, both increasing polarization.

Thirdly, a polarization measure has to account for the differences in group sizes, in such a way that for a given number of groups k and given within- and between-group inequality, the more equally sized the groups are, the higher polarization is. See, among others, Esteban and Ray (1994), D´Ambrosio (2001) or Gigliarano and Mosler (2009). In Wolfson's approach this issue is implicit since the groups are divided by the median, and hence are equally-sized.

In summary, a measure of polarization should increase with inequality between groups, decrease with inequality within groups. It may also increase with the degree of homogeneity among group sizes.[5]

With respect to the inequality measure we will focus on the well-known Gini index of inequality, mainly out of two reasons. First, out of many existing measures, the Gini index remains the most popular and commonly used measure of inequality. Second, over the decades several alternative decompositions of the Gini index have been proposed in the literature; see Bhattacharya and Mahalanobis (1967), Pyatt (1976), Yitzhaki and Lerman (1991), Dagum (1997). A recent proposal has been

[5]Observe that not only the polarization measures of the Wolfson string fit into this concept, but also the measures based on the Esteban and Ray's approach are in line with this idea. For example, in case of two groups the measure by Esteban et al. (2007) can be written as $P^{EGR} = G^B(\boldsymbol{x}) \cdot S(\boldsymbol{N}) - G^W(\boldsymbol{x})$. Here G^B denotes the inequality between, G^W the inequality within, and $S(\boldsymbol{N})$ the homogeneity of the group sizes. It can be seen immediately that P^{EGR} meets the three requirements mentioned above.

formulated by Okamoto (2009), which is the decomposition that we consider for measuring polarization and that we now briefly describe.

3.1 Okamoto's decomposition

Consider a vector $\mathbf{x}$ of incomes that are, w.l.o.g., increasingly ordered, i.e. $x_1 \leq x_2 \leq \ldots \leq x_n$. Let us recall the Gini mean difference

$$\Delta = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{m=1}^{n} |x_i - x_m| \,, \tag{8}$$

which corresponds to the numerator of the Gini index G.

The Gini mean difference and the Gini index for group j are, respectively:

$$\Delta_j(\boldsymbol{x}_j) = \frac{1}{n_j^2} \sum_{i=1}^{n_j} \sum_{m=1}^{n_j} |x_{ji} - x_{jm}| \,, \quad j = 1, \ldots, k, \tag{9}$$

and

$$G_j(\boldsymbol{x}_j) = \frac{\Delta_j(\boldsymbol{x}_j)}{2\bar{x}_j} = \frac{1}{2\bar{x}_j n_j^2} \sum_{i=1}^{n_j} \sum_{m=1}^{n_j} |x_{ji} - x_{jm}| \,, \quad j = 1, \ldots, k \,. \tag{10}$$

Finally, the Gini mean difference between group j and group l is defined as

$$\Delta_{jl}(\boldsymbol{x}_j, \boldsymbol{x}_l) = \frac{1}{n_j n_l} \sum_{i=1}^{n_j} \sum_{m=1}^{n_l} |x_{ji} - x_{lm}| \,, \quad j, l = 1, \ldots, k \,. \tag{11}$$

Obviously, $\Delta_{jj}(\boldsymbol{x}_j, \boldsymbol{x}_j) = \Delta_j(\boldsymbol{x}_j)$. Okamoto (2009) provides a new two-term-decomposition of the Gini index, given by

$$G = G^W_{Oka} + G^B_{Oka} \,, \tag{12}$$

where the within-group component is defined as

$$G^W_{Oka} = \sum_{j=1}^{k} \frac{n_j}{n} \frac{\bar{x}_j}{\bar{x}} G_j(\boldsymbol{x}_j) \tag{13}$$

and the between-group component is

$$G^B_{Oka} = \frac{1}{2\bar{x}} \sum_{j=1}^{k} \sum_{l=1}^{k} \frac{n_j n_l}{n^2} cv(\boldsymbol{x}_j, \boldsymbol{x}_l) \,, \tag{14}$$

where the term $cv(\boldsymbol{x}_j, \boldsymbol{x}_l)$ is named by Okamoto (2009) as the "Cramèr coefficient of variation" between groups j and l, since it recalls the two-sample Cramer-von Mises test statistic[6], and it is defined as

$$\begin{aligned} cv(\boldsymbol{x}_j, \boldsymbol{x}_l) &= \frac{1}{n_j n_l} \sum_{i=1}^{n_j} \sum_{m=1}^{n_l} |x_{ji} - x_{lm}| \\ &- \frac{1}{2n_j^2} \sum_{i=1}^{n_j} \sum_{m=1}^{n_j} |x_{ji} - x_{jm}| - \frac{1}{2n_l^2} \sum_{i=1}^{n_l} \sum_{m=1}^{n_l} |x_{li} - x_{lm}| \\ &= \Delta_{jl}(\boldsymbol{x}_j, \boldsymbol{x}_l) - \frac{\Delta_j(\boldsymbol{x}_j)}{2} - \frac{\Delta_l(\boldsymbol{x}_l)}{2} \,. \end{aligned} \tag{15}$$

Note that $cv(\boldsymbol{x}_j, \boldsymbol{x}_l) = 0$ if $j = l$. Alternatively, we may write (16) as

$$G_{Oka}^B = \frac{1}{2\bar{x}n^2} \sum_{j,l=1, j \neq l}^{k} \sum_{i=1}^{n_j} \sum_{m=1}^{n_l} |x_{ji} - x_{lm}| \,. \tag{16}$$

In most of the known decompositions of the Gini index, the between-group inequality component is based on differences of groups' mean incomes. In particular, the inequality between groups gets null if and only if the group means are the same, regardless of how different the subgroup distributions might be. This can be considered as a drawback for our class of polarization measures, as it would take into account only one parameter of the distribution. Differently, in the Okamoto's decomposition the inequality between groups takes into account the whole subgroup distribution rather than only difference in means. Therefore, when using Okamoto's decomposition, the between-group inequality is zero if and only if all group distributions are identical and concentrated at a single point.[7] This is a rather strong condition for having zero between-group inequality, but – at least in our opinion – it fits better to the concept of groups' alienation.

Looking at the within-group component G_{Oka}^W, Okamoto's decomposition aggregates the inequality within each group using as weight the share of the group and its relative mean income. As a consequence, the weights sum up to one, and the within-group component can be considered properly as a weighted average of the subgroups inequalities. This appears to be more natural than using the square of the group shares and their relative means as weights, as in most of the other Gini index decompositions.

[6]See, among others, Anderson (1962).

[7]Note that the equality of group means is, therefore, not a sufficient condition for $G_{Oka}^B = 0$. Hence, Okamoto's decomposition does not satisfy the property of additive decomposability defined in Shorrocks (1980), which is, however, considered quite restrictive, as it allows the between-group component to rely only on the first moment of each subgroup distribution; see, among others, Ebert (2010).

Particularly interesting for polarization measurement is that Okamoto (2009) allows for a two-term decomposition of the Gini index, while almost all the other proposals are three-term decompositions. One important exception is the Dagum's decomposition, which can be rewritten as the sum of the within-group Gini index and the gross between-group Gini component. However, the latter is not minimum when the subgroup distributions or means are all identical. Moreover, differently from the three-term decompositions, Okamoto's decomposition is consistent with multilevel grouping.

Our last reason for employing the Okamoto decomposition is that it is available not only for the unidimensional Gini index but for some of its multidimensional extensions as well, such as for the Distance Gini index and for the Volume Gini index proposed in Koshevoy and Mosler (1997). Since we also want to outline the possibility to construct multidimensional polarization indices, this is a nice property since it allows us to handle uni- and multidimensional polarization measures in the same framework.

Note that using the Okamoto decomposition for polarization measurement we do not obtain the Wolfson measure as a special case. Indeed, it can be shown that the expressions (4) and (7) of the Wolfson polarization index are equivalent only if the Gini index is decomposed using the Dagum (1997) or the Bhattacharya and Mahalanobis (1967) decompositions, but not the Okamoto method. However, this is not a serious drawback, since this approach will allow to analyze situations where the population is divided into subgroups according to characteristics other than income and where the groups income distribution overlap. For the sake of simplicity, henceforth, G^B and G^W will denote the Okamoto decomposition's components G^B_{Oka} and G^W_{Oka}, respectively.

3.2 A new polarization index

As already mentioned, our new polarization index should decrease with respect to the within-group inequality, increase with respect to the between-group inequality and depend on the homogeneity of group sizes.

To meet these requirements in a simple and transparent way, we propose a new polarization index which is defined according to

$$\tilde{P} = \tilde{P}(\boldsymbol{x}, \boldsymbol{N}) = S(\boldsymbol{N}) \cdot \left[G^B(\boldsymbol{x}, \boldsymbol{N}) - G^W(\boldsymbol{x}, \boldsymbol{N})\right] . \quad (17)$$

For G^B and G^W we use the components of Okamoto's decomposition (equations (14) and (13), respectively) and for $S(\boldsymbol{N})$ we employ the inverse of Herfindahl's measure of concentration,

$$S(\boldsymbol{N}) = \left[\sum_{j=1}^{k} \left(\frac{n_j}{n}\right)^2\right]^{-1} , \quad (18)$$

which equals k if all groups have the same size, and approaches 1 if all but one group tend to become empty.[8] Obviously, polarization index (17) is not normalized; it can actually take a negative value if the inequality within groups exceeds the inequality between groups. In order to normalize $\tilde{P}(\boldsymbol{x}, \boldsymbol{N})$, we normalize each of its components. Starting with the group sizes' homogeneity component, the inverse Herfindahl index is normalized by

$$\frac{1}{k-1}\left[\left[\sum_{j=1}^{k}\left(\frac{n_j}{n}\right)^2\right]^{-1} - 1\right]. \tag{19}$$

To normalize the inequality difference $D = G^B - G^W$ we have to determine its minimum and maximum. Then

$$\frac{D - \min D}{\max D - \min D} \tag{20}$$

ranges between 0 and 1. Note that the minimum value of D depends on the vector $\boldsymbol{N}$ of groups' sizes. Hence, determining the minimum value in case of overlapping groups is not a trivial exercise. A detailed derivation of the minimum and maximum values of D for two generic, also overlapping, groups is included in Appendix A. Finally, as our normalized index of polarization we introduce

$$P = P(\boldsymbol{x}, \boldsymbol{N}) = \frac{1}{k-1}\left[\left[\sum_{j=1}^{k}\left(\frac{n_j}{n}\right)^2\right]^{-1} - 1\right] \cdot \frac{D - \min D}{\max D - \min D}. \tag{21}$$

Note that in this analysis we do not intend to directly compare scenarios characterized by a different number of groups; e.g., ceteris paribus our measure does not assign a higher level of polarization in case of two equally sized groups than in case of three equally sized groups.[9]

In this section we have introduced a new polarization index which is based on the difference between inequality components obtained from the Okamoto's decomposition of the Gini index. In contrast to the Wolfson's approach, the new polarization index can be computed for any number of groups, which may also be overlapping. In the next section we will focus on the properties of the new polarization index.

[8] We use the factor $S(\boldsymbol{N})$ since the difference $G^B(\boldsymbol{x}, \boldsymbol{N}) - G^W(\boldsymbol{x}, \boldsymbol{N})$ does not always increase with increasing homogeneity.

[9] If different numbers of groups shall be compared, the index may be augmented by a factor that properly decreases with k. For example, one could consider as an alternative version of expression (19) the following $S(\boldsymbol{N}) = \frac{\left(\sum_{j=1}^{k}\left(\frac{n_j}{n}\right)^2\right)^{-1} - 1}{k-1}\frac{\sqrt{2}}{\sqrt{k}}$. But then normalization property (see the following postulate P5) would be no more satisfied.

4 Properties of the new index

An index of polarization is a function $P : \mathbb{R}_+^n \times \mathbb{N}^k \to \mathbb{R}, (\boldsymbol{x}, \boldsymbol{N}) \mapsto P(\boldsymbol{x}, \boldsymbol{N})$. It is supposed to satisfy a number of postulates, which we divide into two groups: the first group regards general aspects of measurement, while the second group contains those postulates which are specific to polarization and distinguish a polarization measure e.g. from a measure of inequality or concentration.

The group of general postulates of measurement includes the following postulates:

P1 **Continuity** $\boldsymbol{x} \mapsto P(\boldsymbol{x}, \boldsymbol{N})$ is continuous on $\mathbb{R}_+^n$.

P2 **Within-group Anonymity** $P(\boldsymbol{x}, \boldsymbol{N})$ is invariant to permutations of individual labels within groups. For every group j, given a permutation π_j of $\{1, 2, \ldots, n_j\}$, it holds

$$P(\boldsymbol{x}_1, , \ldots, \mathbf{y}_j, \ldots, \boldsymbol{x}_k, \boldsymbol{N}) = P(\boldsymbol{x}_1, , \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_k, \boldsymbol{N}) \,,$$

where $\boldsymbol{x}_j = (x_{j1}, x_{j2}, , \ldots, x_{jn_j})$ and $\mathbf{y}_j = (x_{j\pi_j(1)}, x_{j\pi_j(2)}, \ldots, x_{j\pi_j(n_j)})$.

P3 **Replication Invariance** For any $\ell \in \mathbb{N}$: $P(\boldsymbol{x}, \boldsymbol{N}) = P(\underbrace{(\boldsymbol{x}, \ldots, \boldsymbol{x})}_{\ell\, times}, \ell\boldsymbol{N})$.

P4 **Scale Invariance** $P(\boldsymbol{x}, \boldsymbol{N})$ is invariant to any change in the unit of measurement of income: for any scalar $\lambda > 0$, $P(\boldsymbol{x}, \boldsymbol{N}) = P(\lambda\boldsymbol{x}, \boldsymbol{N})$.

P5 **Normalization** $P(\boldsymbol{x}, \boldsymbol{N})$ should be normalized between 0 and 1.

For the new polarization index we can summarize its measurement properties with the following proposition:

Proposition 1. *The polarization index (21) satisfies the postulates P1 to P5 for any number of groups k.*

Proof: Obvious.

Additional to these general postulates of measurement there are two postulates which are specific to polarization:[10]

P6 **Increased Spread [IS]** Given two groups divided by the median, a regressive Pigou-Dalton transfer between two individuals belonging to different groups should increase polarization.

[10]For a detailed formulation of these propositions see, e.g., Wang and Tsui (2000).

P7 **Increased Bipolarity [IB]** Given two groups divided by the median, if the inequality within at least one group reduces, by means of a progressive Pigou-Dalton transfer, polarization should increase.

However, the two polarization postulates $P6$ (IS) and $P7$ (IB) are not appropriate if the groups do overlap, since they are defined referring to two groups divided by the median. Therefore, we now introduce a modified version of IS (Postulate $P6^*$ below) and of IB (Postulate $P7^*$ below), which are adapted to the case of overlapping groups and are defined for the general case of $k \geq 2$ groups.

P6* **Modified Increased Spread [MIS]**
Let us order the k groups increasingly according to their means, so that $\bar{x}_1 \leq \bar{x}_2 \leq \ldots \leq \bar{x}_k$. This property states that if (i) incomes x_{1i}, for $i = 1, \ldots, n_1$, of all individuals belonging to group 1 are decreased by the same positive value $\beta > 0$ (thus becoming $y_{1i} = x_{1i} - \beta$) and, at the same time, (ii) incomes x_{km}, for $m = 1, \ldots, n_k$, of all the individuals belonging to group k are increased by the same positive value $\alpha = \frac{n_1}{n_k}\beta$, such that the overall mean is not affected (thus becoming $y_{km} = x_{km} + \alpha$), then polarization should increase.

In other words, while the Increased Spread property considers a Pigou-Dalton transfer that involves only two individuals belonging to two non-overlapping groups, the Modified Increased Spread assumes that all individuals in the group with the lowest income mean become poorer and all individuals of the group with highest income mean become richer. This change should reasonably increase polarization.

P7* **Modified Increased Bipolarity [MIB]**
Let us order the k groups increasingly according to their means, so that $\bar{x}_1 \leq \bar{x}_2 \leq \ldots \leq \bar{x}_k$. If the inequality within at least one group reduces, then P should increase. In particular, if a progressive transfer occurs between two individuals g^* and h^* belonging to the same group (without loss of generality, we assume group 1), such that income x_{1h^*} is reduced by $\epsilon > 0$ and income x_{1g^*} is augmented by $\epsilon > 0$, then polarization should increase.

The Modified Increased Bipolarity property is an intuitive extension of the Increased Bipolarity to the case of more than two groups. It states that any progressive transfer occurring within any sub-group distribution should increase polarization.

Together with the original versions, these two modified postulates will be investigated in detail in Sections 5 and 6.

5 Polarization of two groups

We now discuss the polarization properties of the new index proposed in the case of $k = 2$ groups, both non-overlapping and overlapping. We investigate the postulates of Increased Spread (P6) and Increased Bipolarity (P7) as well as their extended versions of Modified Increased Spread (P6*) and Modified Increased Bipolarity (P7*) that cope with the situation of overlapping groups. We also discuss the two extreme scenarios of minimum and maximum polarization.

5.1 Two non-overlapping groups

Proposition 2 below discusses the scenarios of minimum and maximum polarization for the case of two non-overlapping groups.

Proposition 2. ***(Minimum and maximum polarization of two non-overlapping groups)***

a) The polarization index P defined in (21) is maximal (equal to 1) if the groups are of equal size, at maximal distance to each other and with no within-group inequality, that means, for a given overall mean $\bar{x}$, if $\boldsymbol{x}_1 = (0, \ldots, 0)$ and $\boldsymbol{x}_2 = (2\bar{x}, \ldots, 2\bar{x})$.

b) The case of minimal polarization depends on the groups' sizes. For two groups of equal size the polarization index P defined in (21) is minimal and equal to 0 if all individuals have the same income, that means, for a given overall mean $\bar{x}$, if $\boldsymbol{x}_1 = \boldsymbol{x}_2 = (\bar{x}, \ldots, \bar{x})$.

Regarding the polarization-specific postulates of Increased Spread (IS) and Increased Bipolarity (IB) it holds the following proposition:

Proposition 3. ***(IS and IB for two non-overlapping groups)***
Let $k = 2$ and assume that the two groups are divided by the median, so that they do not overlap. Then the polarization index P defined in (21) always satisfies IB postulate, while it satisfies IS postulate if the number of individuals, whose income is between the incomes of the two individuals involved in the regressive transfer, is smaller than $\frac{2}{3}(n-1)$.

For a proof of Proposition 3 see Appendices B.1 and B.3, respectively.

According to Proposition 3, in case of two non-overlapping groups, a sufficient condition for the Increased Spread (IS) property to hold is that at most two thirds of the population has income between the incomes of the two individuals involved in the IS transfer. Roughly speaking, consider an individual h with income below the median and an individual g with income above the median, such that the proportion of individuals between them does not exceed two thirds: if the incomes of individuals h and g get further, then the polarization index proposed will increase.

5.2 Two overlapping groups

Proposition 4 below discusses the scenarios of minimum and maximum polarization for the case of two overlapping groups.

Proposition 4. ***(Minimum and maximum polarization of two overlapping groups)***

a) The polarization index P defined in (21) is maximal (equal to 1) if the groups are of equal size, at maximal distance to each other and with no within-group inequality, that means, for a given overall mean $\bar{x}$, if $\boldsymbol{x}_1 = (0, \ldots, 0)$ and $\boldsymbol{x}_2 = (2\bar{x}, \ldots, 2\bar{x})$.

b) The case of minimal polarization depends on the groups' sizes. For two groups of equal size the polarization index P in (21) is minimal and equal to 0 if the two groups' distributions perfectly overlap and the within-group inequality is maximal, that means, for a given overall mean $\bar{x}$, if $\boldsymbol{x}_1 = \boldsymbol{x}_2 = (0, \ldots, 0, \frac{n\bar{x}}{2})$.

Regarding the polarization-specific postulate of Modified Increased Spread (MIS), it holds:

Proposition 5. ***(MIS for two overlapping groups)***
In the case of $k = 2$ groups, the postulate of Modified Increased Spread ($P6^*$) is satisfied by the polarization index P defined in (21) if

$$\sum_{i=1}^{n_1} \left[(\alpha + \beta) \sum_{m=1}^{n_2} [\mathbf{1}(y_{2m} > y_{1i}) - \mathbf{1}(y_{2m} < y_{1i})] + \sum_{\substack{m:(x_{2m} < x_{1i}) \\ \wedge (y_{2m} > y_{1i})}} (\alpha + \beta - 2\,(x_{1i} - x_{2m})) \right] \geq 0,$$

where $\mathbf{1}(\cdot)$ is the indicator function, and the symbol $\wedge$ represents the logical conjunction "and". The proof is shown in Appendix B.2.

The condition expressed in Proposition 5 means that the Modified Increased Spread postulate is satisfied if, for any individual i belonging to group 1, the number of individuals of group 2 who are richer than him is higher than the number of individuals of group 2 who are poorer than him, both after (see expression $\sum_m \mathbf{1}(y_{2m} > y_{1i}) - \mathbf{1}(y_{2m} < y_{1i})$) and before (see expression $\sum_m \alpha + \beta - 2 \cdot (x_{1i} - x_{2m})$) the transformation required by the $P6^*$ postulate; shortly, polarization increases if there is few overlap between the two groups.

An easy to verify sufficient condition for property $P6^*$ is given by $m(\boldsymbol{x}_1) \leq m(\boldsymbol{x}_2)$, where $m(\boldsymbol{x}_1)$ and $m(\boldsymbol{x}_2)$ are the median incomes of the first and second group. Roughly said, the polarization index P satisfies the Modified Increased Spread postulate if the medians of the two groups are ordered in the same way as their means (i.e. $\bar{x}_1 \leq \bar{x}_2$ and $m(\boldsymbol{x}_1) \leq m(\boldsymbol{x}_2)$).

Next we investigate the postulate of Modified Increased Bipolarity (MIB) in the case of two overlapping groups.

Proposition 6. ***(MIB for two overlapping groups)***
In the case of $k = 2$ groups, the postulate of Modified Increased Bipolarity ($P7^*$) is met by the polarization index P defined in (21) if

$$\left(F_1(h^*) - F_1(g^*) + \frac{1}{n_1}\right)\left(\frac{n_2}{2n} + 1\right) > \frac{n_2}{n}\left(F_2(h^*) - F_2(g^*)\right) ,$$

where F_l is the cumulative distribution function of group l, with $l = 1, 2$. See the proof in Appendix B.3.

If the inequality within one group reduces, then P should increase. Since this change affects both G^W and G^B, the overall effect on the polarization index P proposed in (21) depends on the degree of overlap between the groups' distributions. Intuitively, Proposition 6 requires that the majority of the individuals with income between x_{1g^*} and x_{1h^*} should belong to group 1 rather than group 2.

6 The general case of $k \geq 2$ overlapping groups

Now let $k \geq 2$ groups be given that possibly overlap. We consider the same modified polarization postulates $P6^*$ (MIS) and $P7^*$ (MIB) as before. However, now the conditions under which P satisfies them are more intricate.

Proposition 7. ***(MIS for many overlapping groups)***
Let $k \geq 2$. Then the postulate of Modified Increased Spread ($P6^*$) holds if

$$A + B + C \geq 0,$$

with

$$A = \sum_{j \neq \{1,k\}} \sum_{i=1}^{n_1} \left(\beta \sum_{m=1}^{n_j} \left(\mathbf{1}(x_{jm} > y_{1i}) - \mathbf{1}(x_{jm} < y_{1i})\right) + \sum_{m:(y_{1i} < x_{jm} < x_{1i})} \beta - 2\left(x_{1i} - x_{jm}\right) \right)$$

$$B = \sum_{j \neq \{1,k\}} \sum_{i=1}^{n_k} \left(\alpha \sum_{m=1}^{n_j} \left(\mathbf{1}(x_{jm} < y_{ki}) - \mathbf{1}(x_{jm} > y_{ki})\right) + \sum_{m:(x_{ki} < x_{jm} < y_{ki})} \alpha - 2\left(x_{jm} - x_{ki}\right) \right)$$

$$C = \sum_{i=1}^{n_1} \left((\alpha + \beta) \sum_{m=1}^{n_k} \left(\mathbf{1}(y_{km} > y_{1i}) - \mathbf{1}(y_{km} < y_{1i})\right) + \sum_{\substack{m:(x_{km} < x_{1i}) \\ \wedge (y_{km} > y_{1i})}} \alpha + \beta - 2\left(x_{1i} - x_{km}\right) \right).$$

The proof is given in Appendix B.4.

As for $k = 2$, the condition given in Proposition 7 means that the overlap must not be 'too large'. In particular, expressions A and B in Proposition 7 monitor the degree of overlap, respectively, between group 1 and the other groups and between group k and the other groups. Expression A is positive if the number of individuals whose incomes' ordering is in accordance with the ordering of their group means (that is, $x_{jm} > y_{1i}$ and $\bar{x}_j > \bar{x}_1$) is higher than the number of individuals whose incomes' ordering is reversed if compared to the their means ($x_{jm} < y_{1i}$ and $\bar{x}_j > \bar{x}_1$). An analogous interpretation holds for expression B. Expression C refers, instead, to the overlap between the two groups involved in the transformation, that are group 1 and group k, and it coincides with the condition required in Proposition 5 for the case of 2 groups.

Regarding the postulate of Modified Increasing Bipolarity we obtain the following result:

Proposition 8. ***(MIB for many overlapping groups)***
Let $k \geq 2$. Then the postulate of Modified Increased Bipolarity ($P7^*$) is satisfied if

$$\left(F_1(h^*) - F_1(g^*) + \frac{1}{n_1}\right)\left(1 + \frac{n - n_1}{2n}\right) > \sum_{l=2}^{k} \frac{n_l}{n}\left(F_l(h^*) - F_l(g^*)\right),$$

where F_l is the cumulative distribution function of group l, with $l = 1, \ldots, k$. See proof in the Appendix B.5.

The MIB postulate states that any progressive transfer occurring within any subgroup distribution should increase polarization. In particular, if the inequality within at least one group (here, w.l.o.g., we consider group 1) is reduced, this change affects both G^W and G^B. Proposition 8 says that the overall effect on the polarization index P defined in (21) depends on the degree of overlap between the groups' distributions. In particular, the polarization measure P will increase if the overlap among groups is not too high, that is, if most of the individuals whose income lies between x_{1g^*} and x_{ih^*} belong to same group 1 rather than to other groups.

7 An empirical application

The empirical application is aimed at comparing the new polarization index with other polarization measures proposed in the literature and is focused on measuring polarization of incomes in Germany and Italy. We consider two different scenarios. First, we measure (income) bi-polarization in the sense of Wolfson, meaning that two groups are formed by separating the population at the median income. In the second part of the application we consider a situation where groups are identified not by income differences but rather by differences in the place of residence. This is a special case of what Gradín (2000) calls "socioeconomic" polarization. We will refer to this second scenario as "regional polarization" throughout this section.

7.1 Data

For our application we use data from two different sources. The first source is the EU Statistics on Income and Living Conditions (henceforth EU-SILC) provided by Eurostat. Since this survey is designed for collecting timely and comparable cross-sectional and longitudinal multidimensional micro data on income, poverty, social exclusion and living conditions, it is perfectly suited for measuring and comparing polarization for Germany and Italy; see, e.g., Eurostat (2007). Unfortunately, for Germany we do not have regional codes, so we are not able to measure regional polarization for Germany based on this survey. Therefore we incorporate a second source, the German Socio-Economic Panel (henceforth SOEP); see, among others, Wagner et al. (2007).[11]

For both sources we have income data referring to all years from 2005 to 2011. We use household data and equivalized household disposable income as the variable of interest.[12] For computing regional polarization we use the place of residence, splitting Germany into West and East and Italy into North, Central and South; hence, we handle situations with two as well as three groups.

7.2 Bi-polarization

We first consider the more familiar scenario of (income) bi-polarization as in Wolfson's approach. We apply both the Wolfson and the new polarization measure to German and Italian data, splitting each population in two groups by the country-specific income median. This allows us to compare the two countries and the two measures as well. To get a closer insight into the two measures we also report the corresponding Gini index decompositions, as well as the Gini index of the low and the high income groups; see Table 1 and 2. We also report 95% bootstrap confidence intervals, which have been obtained from 2000 bootstrap replicates of the original samples. Since we are in the special situation of two equally sized groups, we can

[11] Both sources lead to comparable results when we compute the bi-polarization measures for Germany.

[12] For computing the equivalized income we use the modified OECD scale, which assigns a value of 1 to the household head, a value of 0.5 to each additional adult and a value of 0.3 to each child below 14 years old.

observe the relation between the two within-group components, viz. that the within-group component of the Wolfson measure is half the within-group component of the new measure.

With respect to measuring (income) bi-polarization, both polarization measures (the Wolfson's and the new index proposed) show similar results. In particular, according to both measures, income bi-polarization in Italy has significantly decreased from 2006 to 2012 (see Table 1). On the contrary, in Germany polarization has not changed significantly over the period under consideration; this result does not depend on the polarization measure employed.

Looking at the Wolfson measure, levels of polarization are much higher in Italy than in Germany for all years. The new polarization measure, instead, registers much smaller differences between the two countries, although the direction of change is the same. Looking, moreover, at the within- and between-group Gini index, we note that also the overall inequality is higher in Italy than in Germany.

[Tables 1 and 2 about here]

7.3 Regional polarization

We now consider a situation where identification and alienation are not based solely on income disparities but also on differences in the place of residence. In this context the Wolfson and any other bi-polarization measure are of no use, so we will compare the new measure with the index P^{LU} proposed by Lasso de la Vega and Urrutia (2006), defined as in expression (3). For the latter, two different settings for the parameters α and β will be considered.[13]

Before performing this second analysis, one should verify that the conditions required by the Modified Increased Spread and the Modified Increased Polarity properties (see Propositions 5 to 8) are satisfied in this context. The Modified Increased Spread property holds both for Germany and for Italy in all years, while the Modified Increased Polarity property may depend on the income levels and on the groups of the two individuals involved in the transfer. We have simulated all possible transfers and in all cases the condition holds for Italy as well as for Germany. (The data are not shown here, but available from the authors upon request.)

Estimates of both polarization measures (P^{LU} and the new index proposed) are shown in Tables 3 and 4, again accompanied by associated bootstrap confidence intervals and by the within- and between-group components of the Okamoto's Gini

[13]For K we have chosen $K = k^{2+\alpha}/\left[2(k-1)\log(k \cdot \mu - k + 1)\right]$ with k being the number of groups. Assuming that the minimal mean income of a group is one – keep in mind that the measure is not defined for mean incomes of zero – this choice of K normalizes the measure between 0 and 1. Therefore K depends on the number of groups and differs between Germany and Italy.

decomposition. We first notice that for both countries the values of the new measure are now lower than in the previous analysis of income bi-polarization. This is due to the fact that the inequality within regional groups is much higher than in the groups divided by the median income, while the income disparities among regional groups are much lower than the disparities across the income groups. This seems plausible since in this second analysis we allow the groups to overlap so the difference between groups can potentially be much lower.[14]

Tables 3 and 4 clearly show that regional polarization is significantly higher in Italy than in Germany for all years, according both to Lasso de la Vega and Urrutia (2006) and to the new measure. This is mostly due to the fact that the inequality within the Italian macro-regions (North, Center and South) is higher. The between-group component is quite low in both countries, meaning that there is a high degree of overlap across the regions' income distributions. Over time both polarization measures reveal no significant changes in regional polarization in Italy and in Germany.

[Tables 3 and 4 about here]

8 Concluding remarks

In this paper we have proposed a new polarization measure that resembles the structure of the polarization index proposed by Wolfson (1994), while simultaneously reflecting the ideas of identification and alienation introduced by Esteban and Ray (1994). The new polarization index is based on a recent decomposition of the Gini index by Okamoto (2009), which divides the overall inequality into a within and a between group component. By making use of this decomposition, the new index maintains the simplicity and immediate comprehension of the Wolfson index, while being much more flexible.

The increased flexibility refers to two aspects of polarization measurement: (i) the new index allows for the presence of more than two groups, and (ii) the groups may also overlap. The new approach, therefore, accommodates both researchers who are interested in measuring the traditional income bi-polarization and those who assume that the population is made up by more than two groups, even if the groups are formed by other characteristics than income – a situation that generally introduces some overlap between the groups. The empirical application has shown how the new approach enriches the analysis based on standard measures of income polarization and how it contributes to disentangle the different faces of polarization analysis.

Finally, similar to socio-economic inequality, polarization may be seen as a multivariate phenomenon that regards not only income but also attributes like wealth,

[14] If the overlap is very large the between group component can even tend to zero, independent of the overall or within-group inequality.

education and health variables. Okamoto (2009) has provided analogous decompositions for two multidimensional extensions of the Gini index, such as the Distance Gini index and the Volume Gini index (Koshevoy and Mosler, 1997). Multivariate polarization indices can be based on these decompositions and developed along the lines of Gigliarano and Mosler (2009). But this is up to future research.

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Table 1: Income polarization in Italy

year	Wolfson	G^W	G^B	New	G^W_{Oka}	G^B_{Oka}	G_{Low}	G_{High}
2005	0.2673	0.1040	0.2193	0.6048	0.2080	0.1153	0.1966	0.2125
	[0.2613 ; 0.2734]	[0.1017 ; 0.1064]	[0.2162 ; 0.2224]	[0.6014 ; 0.6079]	[0.2037 ; 0.2125]	[0.1132 ; 0.1174]	[0.1924 ; 0.2010]	[0.2066 ; 0.2184]
2006	0.2680	0.1051	0.2204	0.6034	0.2102	0.1153	0.1961	0.2157
	[0.2612 ; 0.2743]	[0.1028 ; 0.1075]	[0.2172 ; 0.2237]	[0.5998 ; 0.6066]	[0.2057 ; 0.2148]	[0.1131 ; 0.1174]	[0.1918 ; 0.2006]	[0.2098 ; 0.2220]
2007	0.2556	0.1017	0.2136	0.6057	0.2033	0.1119	0.1932	0.2074
	[0.2501 ; 0.2612]	[0.0994 ; 0.1039]	[0.2105 ; 0.2166]	[0.6025 ; 0.6090]	[0.1992 ; 0.2078]	[0.1099 ; 0.1139]	[0.1890 ; 0.1974]	[0.2015 ; 0.2135]
2008	0.2584	0.1057	0.2170	0.6000	0.2114	0.1113	0.1947	0.2180
	[0.2528 ; 0.2636]	[0.1034 ; 0.1079]	[0.2143 ; 0.2198]	[0.5964 ; 0.6036]	[0.2070 ; 0.2163]	[0.1094 ; 0.1134]	[0.1907 ; 0.1987]	[0.2118 ; 0.2247]
2009	0.2552	0.1030	0.2141	0.6032	0.2061	0.1110	0.1944	0.2108
	[0.2497 ; 0.2610]	[0.1007 ; 0.1056]	[0.2111 ; 0.2172]	[0.5994 ; 0.6068]	[0.2011 ; 0.2115]	[0.1089 ; 0.1131]	[0.1900 ; 0.1987]	[0.2042 ; 0.2179]
2010	0.2477	0.1094	0.2177	0.5930	0.2188	0.1083	0.2085	0.2229
	[0.2405 ; 0.2544]	[0.1064 ; 0.1128]	[0.2146 ; 0.2211]	[0.5876 ; 0.5979]	[0.2127 ; 0.2253]	[0.1055 ; 0.1110]	[0.2036 ; 0.2135]	[0.2147 ; 0.2317]
2011	0.2470	0.1105	0.2181	0.5911	0.2210	0.1076	0.2075	0.2263
	[0.2402 ; 0.2531]	[0.1078 ; 0.1133]	[0.2151 ; 0.2213]	[0.5861 ; 0.5957]	[0.2157 ; 0.2270]	[0.1050 ; 0.1103]	[0.2030 ; 0.2122]	[0.2190 ; 0.2345]

Note: $Wolfson$ is to the Wolfson index. G^W and G^B refer to the Gini index decomposition as in expression (5). New corresponds to the new polarization measure proposed, while G^W_{Oka} and G^B_{Oka} refers to the Okamoto's decomposition as in expression (12). G_{Low} and G_{High} correspond to the Gini index of the group below the median and above the median, respectively. Values within the brackets refer to the 95% bootstrap confidence intervals.

Table 2: Income polarization in Germany

year	Wolfson	G^W	G^B	New	G^W_{Oka}	G^B_{Oka}	G_{Low}	G_{High}
2005	0.2089	0.0985	0.1902	0.5961	0.1972	0.0915	0.1525	0.2173
	[0.2013 ; 0.2190]	[0.0953 ; 0.1014]	[0.1863 ; 0.1947]	[0.5916 ; 0.6002]	[0.1913 ; 0.2031]	[0.0884 ; 0.0947]	[0.1486 ; 0.1568]	[0.2093 ; 0.2257]
2006	0.2095	0.0977	0.1896	0.5976	0.1954	0.0919	0.1491	0.2163
	[0.2034 ; 0.2211]	[0.0934 ; 0.1012]	[0.1855 ; 0.1949]	[0.5918 ; 0.6025]	[0.1883 ; 0.2026]	[0.0892 ; 0.095]	[0.1453 ; 0.1536]	[0.2067 ; 0.2265]
2007	0.2107	0.0976	0.1894	0.5971	0.1956	0.0915	0.1507	0.2159
	[0.2019 ; 0.2236]	[0.0926 ; 0.1016]	[0.1851 ; 0.1949]	[0.5898 ; 0.6029]	[0.1878 ; 0.2043]	[0.0885 ; 0.0951]	[0.1465 ; 0.1561]	[0.2052 ; 0.2281]
2008	0.2173	0.0938	0.1878	0.6032	0.1881	0.0935	0.1486	0.2063
	[0.2059 ; 0.2283]	[0.0907 ; 0.0966]	[0.1834 ; 0.1930]	[0.5987 ; 0.6077]	[0.1823 ; 0.1941]	[0.0902 ; 0.0970]	[0.1450 ; 0.1526]	[0.1981 ; 0.2146]
2009	0.2282	0.0924	0.1929	0.6043	0.1878	0.0975	0.1523	0.2049
	[0.2138 ; 0.2374]	[0.0894 ; 0.097]	[0.1872 ; 0.1977]	[0.5996 ; 0.6089]	[0.1813 ; 0.1949]	[0.0934 ; 0.1008]	[0.1475 ; 0.1565]	[0.1956 ; 0.2145]
2010	0.2318	0.0929	0.1941	0.6056	0.1881	0.0989	0.1541	0.2040
	[0.2215 ; 0.2401]	[0.090 ; 0.0965]	[0.1897 ; 0.1984]	[0.6009 ; 0.6098]	[0.1827 ; 0.1939]	[0.0959 ; 0.1017]	[0.1503 ; 0.1581]	[0.1962 ; 0.2121]
2011	0.2192	0.0939	0.1905	0.6042	0.1886	0.0958	0.1512	0.2057
	[0.2093 ; 0.2271]	[0.0904 ; 0.0996]	[0.1864 ; 0.1941]	[0.5953 ; 0.6111]	[0.1812 ; 0.1989]	[0.0917 ; 0.0990]	[0.1471 ; 0.1549]	[0.1953 ; 0.2203]

Note: *Wolfson* is to the Wolfson index. G^W and G^B refer to the Gini index decomposition as in expression (5). *New* corresponds to the new polarization measure proposed, while G^W_{Oka} and G^B_{Oka} refers to the Okamoto's decomposition as in expression (12). G_{Low} and G_{High} correspond to the Gini index of the group below the median and above the median, respectively. Values within the brackets refer to the 95% bootstrap confidence intervals.

Table 3: Regional polarization in Italy

year	P^{LU}		New	G^{W}_{Oka}	G^{B}_{Oka}	G_{North}	G_{Center}	G_{South}
	$\alpha=\beta=1$	$\alpha=1.6\,,\beta=1$						
2005	0.0244	0.0267	0.3841	0.3129	0.0113	0.3019	0.3140	0.3357
	[0.0224 ; 0.0264]	[0.0245 ; 0.0289]	[0.3788 ; 0.3892]	[0.3074 ; 0.3183]	[0.010 ; 0.0128]	[0.2942 ; 0.3092]	[0.3022 ; 0.3266]	[0.3262 ; 0.3450]
2006	0.0235	0.0257	0.3839	0.3140	0.0115	0.3024	0.3218	0.3321
	[0.0214 ; 0.0256]	[0.0233 ; 0.0280]	[0.3224 ; 0.3893]	[0.3086 ; 0.3198]	[0.0101 ; 0.013]	[0.2952 ; 0.3096]	[0.3083 ; 0.3370]	[0.3217 ; 0.3440]
2007	0.0225	0.0245	0.3891	0.3042	0.0104	0.2948	0.3090	0.3200
	[0.0207 ; 0.0244]	[0.0226 ; 0.0267]	[0.3269 ; 0.3939]	[0.2987 ; 0.3095]	[0.0092 ; 0.0117]	[0.2875 ; 0.3021]	[0.2961 ; 0.3233]	[0.3113 ; 0.3282]
2008	0.0238	0.0260	0.3871	0.3089	0.0112	0.2987	0.3158	0.3251
	[0.0220 ; 0.0256]	[0.0240 ; 0.0280]	[0.3821 ; 0.3917]	[0.3040 ; 0.3136]	[0.010 ; 0.0127]	[0.2913 ; 0.3058]	[0.306 ; 0.3260]	[0.3171 ; 0.3333]
2009	0.0228	0.0250	0.3880	0.3058	0.0097	0.2991	0.3052	0.3202
	[0.0210 ; 0.0246]	[0.0229 ; 0.0270]	[0.3825 ; 0.3934]	[0.3001 ; 0.3112]	[0.0084 ; 0.011]	[0.2909 ; 0.3076]	[0.2945 ; 0.3162]	[0.3122 ; 0.3282]
2010	0.0256	0.0281	0.3872	0.3102	0.0124	0.2978	0.3147	0.3338
	[0.0236 ; 0.0276]	[0.0258 ; 0.0304]	[0.3816 ; 0.3926]	[0.3044 ; 0.3154]	[0.011 ; 0.0139]	[0.2898 ; 0.3056]	[0.3042 ; 0.3258]	[0.3226 ; 0.3453]
2011	0.0247	0.0270	0.3857	0.3118	0.0115	0.3004	0.3182	0.3312
	[0.0229 ; 0.0266]	[0.0250 ; 0.0291]	[0.3808 ; 0.3907]	[0.3067 ; 0.3169]	[0.0102 ; 0.013]	[0.2937 ; 0.3077]	[0.3082 ; 0.3284]	[0.3220 ; 0.3400]

Note: P^{LU} is to the Lasso de la Vega and Urrutia (2006) index. *New* corresponds to the new polarization measure proposed, while G^{W}_{Oka} and G^{B}_{Oka} refers to the Okamoto's decomposition as in expression (12). G_{North}, G_{Center} and G_{South} correspond to the Gini index of the North, Center and South of Italy, respectively. Values within the brackets refer to the 95% bootstrap confidence intervals.

Table 4: Regional polarization in Germany

year	P^{LU}		New	G^W_{Oka}	G^B_{Oka}	G_{West}	G_{East}
	$\alpha=\beta=1$	$\alpha=1.6\,,\beta=1$					
2005	0.0143	0.0165	0.2962	0.2783	0.0041	0.2861	0.2403
	[0.0125 ; 0.0161]	[0.0145 ; 0.0185]	[0.2835 ; 0.3088]	[0.2715 ; 0.2854]	[0.0032 ; 0.0051]	[0.2783 ; 0.2942]	[0.2300 ; 0.2517]
2006	0.0148	0.0171	0.2965	0.2769	0.0045	0.2850	0.2371
	[0.0129 ; 0.0167]	[0.0150 ; 0.0193]	[0.2831 ; 0.3104]	[0.2691 ; 0.2861]	[0.0035 ; 0.0056]	[0.2760 ; 0.2957]	[0.2267 ; 0.2482]
2007	0.0149	0.0173	0.2931	0.2763	0.0045	0.2837	0.2390
	[0.0131 ; 0.0168]	[0.0152 ; 0.0194]	[0.2791 ; 0.3067]	[0.2680 ; 0.2856]	[0.0036 ; 0.0057]	[0.2739 ; 0.2945]	[0.2293 ; 0.2488]
2008	0.0140	0.0163	0.2944	0.2718	0.0043	0.2783	0.2400
	[0.0121 ; 0.0159]	[0.0141 ; 0.0185]	[0.2805 ; 0.3095]	[0.2642 ; 0.2798]	[0.0033 ; 0.0055]	[0.2693 ; 0.2875]	[0.2281 ; 0.2528]
2009	0.0136	0.0159	0.2819	0.2775	0.0040	0.2833	0.2477
	[0.0115 ; 0.0157]	[0.0135 ; 0.0184]	[0.2677 ; 0.2974]	[0.2689 ; 0.2870]	[0.0031 ; 0.0051]	[0.2734 ; 0.2943]	[0.2342 ; 0.2626]
2010	0.0139	0.0161	0.2933	0.2797	0.0039	0.2876	0.2415
	[0.0122 ; 0.0155]	[0.0142 ; 0.0180]	[0.2809 ; 0.3064]	[0.2730 ; 0.2869]	[0.0031 ; 0.0049]	[0.2797 ; 0.2959]	[0.2323 ; 0.2508]
2011	0.0129	0.0149	0.2761	0.0036	0.2815	0.2509	
	[0.0111 ; 0.0146]	[0.0129 ; 0.0168]	[0.2865 ; 0.3127]	[0.2695 ; 0.2826]	[0.0028 ; 0.0046]	[0.2740 ; 0.2889]	[0.2398 ; 0.2614]

Note: P^{LU} is to the Lasso de la Vega and Urrutia (2006) index. *New* corresponds to the new polarization measure proposed, while G^W_{Oka} and G^B_{Oka} refers to the Okamoto's decomposition as in expression (12). G_{West} and G_{East} correspond to the Gini index of the West and East of Germany, respectively. Values within the brackets refer to the 95% bootstrap confidence intervals.

Appendices

A Proof of minimum and maximum value of $D = G^B - G^W$ for two generic groups

If we have two generic groups, which can be formed by any characteristic and consequently may also overlap, the formulas for the two Gini index components of the Okamoto decomposition are

$$
\begin{aligned}
G^W_{Oka} &= \sum_{j=1}^{2} \frac{n_j}{n} \frac{\bar{x}_j}{\bar{x}} G_j(\boldsymbol{x}_j) = \sum_{j=1}^{2} \frac{n_j}{n} \frac{\bar{x}_j}{\bar{x}} \frac{\Delta_j}{2 \cdot \bar{x}_j} \\
&= \frac{1}{2n\bar{x}} \left(n_1\Delta_1 + n_2\Delta_2\right)
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
G^B_{Oka} &= \frac{1}{2 \cdot \bar{x}} \sum_{j=1}^{k} \sum_{l=1}^{k} \frac{n_j \cdot n_l}{n^2} cv(\boldsymbol{x}_j, \boldsymbol{x}_l) = \frac{1}{\bar{x}} \sum_{j=1}^{2} \sum_{l>j} \frac{n_j \cdot n_l}{n^2} cv(\boldsymbol{x}_j, \boldsymbol{x}_l) \\
&= \frac{n_1 n_2}{n^2 \bar{x}} \left(\Delta_{12} - \frac{\Delta_1}{2} - \frac{\Delta_2}{2}\right) .
\end{aligned} \tag{23}
$$

Consequently, the difference $D = G^B_{Oka} - G^W_{Oka}$ for the case of two generic groups is

$$
\begin{aligned}
D &= \frac{n_1 n_2}{n^2 \bar{x}} \left(\Delta_{12} - \frac{\Delta_1}{2} - \frac{\Delta_2}{2}\right) - \frac{1}{2n\bar{x}} \left(n_1\Delta_1 + n_2\Delta_2\right) \\
&= \frac{1}{2n\bar{x}} \left(\frac{2n_1 n_2}{n}\Delta_{12} - n_1\Delta_1 \left(\frac{n_2}{n} + 1\right) - n_2\Delta_2 \left(\frac{n_1}{n} + 1\right)\right) \\
&= \frac{1}{2n^2\bar{x}} \left(2n_1 n_2 \Delta_{12} - n_1\Delta_1 \left(n_2 + n\right) - n_2\Delta_2 \left(n_1 + n\right)\right) .
\end{aligned} \tag{24}
$$

To compute the minimal and maximal value of the difference D expressed in (24) we compare four different scenarios:

1. $\boldsymbol{x}_1 = (\bar{x}_1, \ldots, \bar{x}_1)$ and $\boldsymbol{x}_2 = (\bar{x}_2, \ldots, \bar{x}_2)$

2. $\boldsymbol{x}_1 = (\bar{x}_1, \ldots, \bar{x}_1)$ and $\boldsymbol{x}_2 = (0, \ldots, 0, n_2 \cdot \bar{x}_2)$
3. $\boldsymbol{x}_1 = (0, \ldots, 0, n_1 \cdot \bar{x}_1)$ and $\boldsymbol{x}_2 = (\bar{x}_2, \ldots, \bar{x}_2)$
4. $\boldsymbol{x}_1 = (0, \ldots, 0, n_1 \cdot \bar{x}_1)$ and $\boldsymbol{x}_2 = (0, \ldots, 0, n_2 \cdot \bar{x}_2)$

In particular, scenario 1. simulates the situation of no inequality within both groups; scenarios 2. and 3. refer to the case of perfect inequality within one group and no inequality in the other group; situation 4. refers to perfect inequality within both groups. The following table gives an overview about the values of the three Gini Mean Differences that are part of expression (24), for the four different scenarios.

Scenario	Δ_1	Δ_2	Δ_{12}
1	0	0	$\bar{x}_2 - \bar{x}_1$
2	0	$2 \cdot \bar{x}_2 \cdot (1 - \frac{1}{n_2})$	$\frac{n_2-2}{n_2} \cdot \bar{x}_1 + \bar{x}_2$
3	$2 \cdot \bar{x}_1 \cdot (1 - \frac{1}{n_1})$	0	$\frac{(n_1-1)}{n_1} \cdot \bar{x}_2 + \frac{\lvert \bar{x}_2 - n_1 \cdot \bar{x}_1 \rvert}{n_1}$
4	$2 \cdot \bar{x}_1 \cdot (1 - \frac{1}{n_1})$	$2 \cdot \bar{x}_2 \cdot (1 - \frac{1}{n_2})$	$\frac{(n_1-1)}{n_1} \cdot \bar{x}_2 + \frac{(n_2-1)}{n_2} \cdot \bar{x}_1 + \frac{\lvert n_2 \cdot \bar{x}_2 - n_1 \cdot \bar{x}_1 \rvert}{n_1 \cdot n_2}$

1. For the first scenario, in case of zero inequality within the two groups it holds that

$$\begin{aligned} D &= \frac{n_1 \cdot n_2}{n^2 \cdot \bar{x}} (\bar{x}_2 - \bar{x}_1) \\ &= \frac{n_1}{n \cdot \bar{x}} (\bar{x} - \bar{x}_1) \end{aligned}$$

   It is obvious that in this situation the measure has its maximum at $\bar{x}_1 = 0$ with a value of $D = \frac{n_1}{n}$ and its minimum at $\bar{x}_1 = \bar{x}$ with $D = 0$.

2. For the second scenario, if the first group has no inequality and the second group perfect inequality it holds that

$$\begin{aligned} D &= \frac{n_1 \cdot n_2}{n^2 \cdot \bar{x}} \left(n_1 \cdot n_2 \left(\frac{n_2 - 2}{n_2} \cdot \bar{x}_1 + \bar{x}_2 \right) - (n_1 + n) \cdot (n_2 - 1) \cdot \bar{x}_2 \right) \\ &= \frac{1}{n^2 \cdot \bar{x}} \Bigg((n_1 \cdot n_2 - (n_1 + n) \cdot (n_2 - 1)) \cdot \frac{n}{n_2} \cdot \bar{x} \\ &+ \left(n_1 \cdot (n_2 - 2) - \frac{n_1}{n_2} (n_1 \cdot n_2 - (n_1 + n) \cdot (n_2 - 1)) \right) \cdot \bar{x}_1 \Bigg) \\ &= \frac{1}{n^2 \cdot \bar{x}} \left((n + n_1 - n \cdot n_2) \cdot \frac{n}{n_2} \cdot \bar{x} + n_1 \cdot \left(n_2 - 2 - \frac{n}{n_2} - \frac{n_1}{n_2} + n \right) \cdot \bar{x}_1 \right) \end{aligned}$$

Since $n_2 - 2 - \frac{n}{n_2} - \frac{n_1}{n_2} + n > 0$ for $n_2 > 1$ the measure reaches its maximum if $\bar{x}_1 = \bar{x}$ with a value of $D = \frac{1}{n^2}\left(n^2 \cdot (\frac{1}{n_2} - 1) + n_1 \cdot n_2 - 2 \cdot n_1 - \frac{n_1^2}{n_2} + n \cdot n_1\right)$ and reaches its minimum if $\bar{x}_1 = 0$ with a value of $D = \frac{n + n_1 - n \cdot n_2}{n \cdot n_2}$.

3. The third scenario considers the case of perfect inequality within the first group and no inequality within the second group. Here it holds that

$$
\begin{aligned}
D &= \frac{1}{n^2 \cdot \bar{x}} \cdot \left(n_1 \cdot n_2 \left(\frac{n_1 - 1}{n_1} \cdot \bar{x}_2 + \frac{|\bar{x}_2 - n_1 \cdot \bar{x}_1|}{n_1}\right) - n_1 \cdot (n_2 + n) \cdot \bar{x}_1 \cdot \left(1 - \frac{1}{n_1}\right)\right) \\
&= \frac{1}{n^2 \cdot \bar{x}} \left((n_1 - 1) \cdot n \cdot \bar{x} - \bar{x}_1 \cdot (n_1 \cdot (n_1 - 1) + (n_2 + n) \cdot (n_1 - 1)) + |n \cdot \bar{x} - n_1 \cdot \bar{x}_1 \cdot (n_2 + 1)|\right) \\
&= \begin{cases} \frac{1}{n^2 \cdot \bar{x}} \left(n_1 \cdot n \cdot \bar{x} - \bar{x}_1 \cdot (n_1 \cdot (n_1 - 1) + (n_2 + n) \cdot (n_1 - 1) + n_1 \cdot (n_2 + 1))\right), & \text{if } || > 0 \\ \frac{1}{n^2 \cdot \bar{x}} \left((n_1 - 2) \cdot n \cdot \bar{x} - \bar{x}_1 \cdot (n_1 \cdot (n_1 - 1) + (n_2 + n) \cdot (n_1 - 1) - n_1 \cdot (n_2 + 1))\right), & \text{if } || < 0 \end{cases}
\end{aligned}
$$

Since $n_1 \cdot (n_1 - 1) + (n_2 + n) \cdot (n_1 - 1) - n_1 \cdot (n_2 + 1) > 0$ the measure reaches its maximum if $\bar{x}_1 = 0$ with a value of $D = \frac{n_1}{n}$ (in fact it degenerates to a situation where there is no inequality within the first group) and its minimum at $\bar{x}_1 = \bar{x}$ with a value of $D = \frac{n_1 \cdot (1 - n_1)}{n^2}$.

4. The fourth scenario assumes perfect inequality within both groups. Then new polarization measure is

$$
\begin{aligned}
D &= \frac{1}{n^2 \bar{x}} \left(n_1 n_2 \left(\frac{n_1 - 1}{n_1} \bar{x}_2 + \frac{n_2 - 1}{n_2} \bar{x}_1 + \frac{|n_2 \bar{x}_2 - n_1 \bar{x}_1|}{n_1 n_2}\right)\right. \\
&\quad \left. - n_1 (n_2 + n) \bar{x}_1 \left(1 - \frac{1}{n_1}\right) - n_2 (n_1 + n) \bar{x}_2 \left(1 - \frac{1}{n_2}\right)\right) \\
&= \frac{1}{n^2 x} \left((2n - n_1) x_1 (1 - n_1) + n_1 (n - n_1 - 1) x_1 + (nx - n_1 x_1)(n_1 - 1) + |nx - 2 n_1 x_1|\right. \\
&\quad \left. + (nx - n_1 x_1)(n + n_1) \left(\frac{1}{n - n_1} - 1\right)\right)
\end{aligned}
$$

Here we have to distinguish between the case with $\bar{x}_1 < \frac{n\bar{x}}{2n_1}$ and with $\bar{x}_1 \geq \frac{n\bar{x}}{2n_1}$. If we look at the derivative of the polarization measure with respect to $\bar{x}_1$, we obtain

$$
\begin{aligned}
\frac{\partial D}{\partial \bar{x}_1} &= \begin{cases} \frac{n_1}{n^2 x} \left((n - n_1 - 1) - (n_1 - 1) - 2 - (n + n_1) \left(\frac{1}{n - n_1} - 1\right) + (2n - n_1) \left(\frac{1}{n_1} - 1\right)\right), & \text{if } || > 0 \\ \frac{n_1}{n^2 x} \left((n - n_1 - 1) - (n_1 - 1) + 2 - (n + n_1) \left(\frac{1}{n - n_1} - 1\right) + (2n - n_1) \left(\frac{1}{n_1} - 1\right)\right), & \text{if } || < 0 \end{cases} \\
&= \begin{cases} \frac{1}{n^2 \bar{x}} \left(2n - 3n_1 - \frac{n_1 (n + n_1)}{n - n_1}\right), & \text{if } \bar{x}_1 < \frac{n\bar{x}}{2n_1} \\ \frac{1}{n^2 \bar{x}} \left(2n + n_1 - \frac{n_1 (n + n_1)}{n - n_1}\right), & \text{if } \bar{x}_1 \geq \frac{n\bar{x}}{2n_1} \end{cases}
\end{aligned}
$$

If we look at the first part of the derivative, it is positive for $1 < n_1 < \frac{n}{2}(3-\sqrt{5})$ and negative for $\frac{n}{2}(3-\sqrt{5}) < n_1 < n$. The second part is positive for $1 < n_1 < \frac{n}{2}(\sqrt{5}-1)$ and negative for $\frac{n}{2}(\sqrt{5}-1) < n_1 < n$.

Combining these information, the minimum and maximal values of the polarization measure in this situation depend on the size of the groups. They are given in the following table

	$\boldsymbol{x}_{min}$	D_{min}	$\boldsymbol{x}_{max}$	D_{max}	
$\boldsymbol{x}_1$ $\boldsymbol{x}_2$	$(0,\ldots,0)$ $(0,\ldots,0,n\cdot\bar{x})$	$\frac{n+n_1-n(n-n_1)}{n(n-n_1)}$	$(0,\ldots,n_1\bar{x})$ $(0,\ldots,(n-n_1)\bar{x})$	$\frac{3n-2n_1-n^2}{n^2}$	if $1 < n_1 < \frac{n}{2}(3-\sqrt{5})$
$\boldsymbol{x}_1$ $\boldsymbol{x}_2$	$(0,\ldots,n_1\bar{x})$ $(0,\ldots,0,(n-n_1)\cdot\bar{x})$	$\frac{3n-2n_1-n^2}{n^2}$	$(0,\ldots,0)$ $(0,\ldots,n\bar{x})$	$\frac{n+n_1-n(n-n_1)}{n(n-n_1)}$	if $\frac{n}{2}(3-\sqrt{5}) < n_1 \leq \frac{n}{2}$
$\boldsymbol{x}_1$ $\boldsymbol{x}_2$	$(0,\ldots,\frac{n\bar{x}}{2})$ $(0,\ldots,\frac{n\bar{x}}{2})$	$\frac{\frac{2n}{n_1}-2n-3+\frac{n+n_1}{n-n_1}}{2n}$	$(0,\ldots,0)$ $(0,\ldots,n\bar{x})$	$\frac{n+n_1-n(n-n_1)}{n(n-n_1)}$	if $\frac{n}{2} < n_1 < \frac{n}{2}(\sqrt{5}-1)$
$\boldsymbol{x}_1$ $\boldsymbol{x}_2$	$(0,\ldots,n_1\bar{x})$ $(0,\ldots,0,(n-n_1)\cdot\bar{x})$	$\frac{n+2n_1-n^2}{n^2}$	$(0,\ldots,0)$ $(0,\ldots,n\bar{x})$	$\frac{n+n_1-n(n-n_1)}{n(n-n_1)}$	if $\frac{n}{2}(\sqrt{5}-1) < n_1 < n$

Table 5: Minimum and maximum in scenario 4

If we compare the minimum and maximum values of the different situations, then the overall minimum and maximum values of D are the ones shown in Table 6.

Table 6: Minimum and maximum values of D, in case of two generic groups.

	$\boldsymbol{x}_{min}$	D_{min}	$\boldsymbol{x}_{max}$	D_{max}	
$\boldsymbol{x}_1$ $\boldsymbol{x}_2$	$(0,\ldots,0)$ $(0,\ldots,0,n\cdot\bar{x})$	$\frac{n+n_1-n(n-n_1)}{n(n-n_1)}$	$(0,\ldots,0)$ $(\bar{x}_2,\ldots,\bar{x}_2)$ s.t. $\bar{x}_2=\frac{n\cdot\bar{x}}{n-n_1}$	$\frac{n_1}{n}$	if $1 < n_1 < \frac{n}{2}(3-\sqrt{5})$
$\boldsymbol{x}_1$ $\boldsymbol{x}_2$	$(0,\ldots,n_1\bar{x})$ $(0,\ldots,0,(n-n_1)\cdot\bar{x})$	$\frac{3n-2n_1-n^2}{n^2}$	$(0,\ldots,0)$ $(\bar{x}_2,\ldots,\bar{x}_2)$ s.t. $\bar{x}_2=\frac{n\cdot\bar{x}}{n-n_1}$	$\frac{n_1}{n}$	if $\frac{n}{2}(3-\sqrt{5}) < n_1 \leq \frac{n}{2}$
$\boldsymbol{x}_1$ $\boldsymbol{x}_2$	$(0,\ldots,\frac{n\bar{x}}{2})$ $(0,\ldots,\frac{n\bar{x}}{2})$	$\frac{\frac{2n}{n_1}-2n-3+\frac{n+n_1}{n-n_1}}{2n}$	$(0,\ldots,0)$ $(\bar{x}_2,\ldots,\bar{x}_2)$ s.t. $\bar{x}_2=\frac{n\cdot\bar{x}}{n-n_1}$	$\frac{n_1}{n}$	if $\frac{n}{2} < n_1 < \frac{n}{2}(\sqrt{5}-1)$
$\boldsymbol{x}_1$ $\boldsymbol{x}_2$	$(0,\ldots,n_1\bar{x})$ $(0,\ldots,0,(n-n_1)\cdot\bar{x})$	$\frac{n+2n_1-n^2}{n^2}$	$(0,\ldots,0)$ $(\bar{x}_2,\ldots,\bar{x}_2)$ s.t. $\bar{x}_2=\frac{n\cdot\bar{x}}{n-n_1}$	$\frac{n_1}{n}$	if $\frac{n}{2}(\sqrt{5}-1) < n_1 < n$

B Proofs of the Properties

B.1 Increased Spread for two non-overlapping groups

Consider two groups divided by the median. W.l.o.g. we assume the population size n to be even, so that $n_1 = n_2 = \frac{n}{2}$. The vector of ordered incomes of group 1 (below the median) is denoted with $\boldsymbol{x}_1 = (x_{11}, x_{12}, \ldots, x_{1\frac{n}{2}})$, while the vector of ordered incomes of group 2 (above the median) is given by $\boldsymbol{x}_2 = (x_{2(\frac{n}{2}+1)}, \ldots, x_{2n})$. Consider the new ordered income vectors $\boldsymbol{y}_1 = (y_{11}, y_{12}, \ldots, y_{1\frac{n}{2}})$ and $\boldsymbol{y}_2 = (y_{2(\frac{n}{2}+1)}, \ldots, y_{2n})$ obtained from $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ by a regressive Pigou-Dalton (PD) transfer between the individual belonging to group 1 with income x_{1g^*} and the individual belonging to group 2 with income x_{2h^*} , such that $y_{1g^*} = x_{1g^*} - \epsilon$ and $y_{2h^*} = x_{2h^*} + \epsilon$. The PD transfer does not allow any change in the order of the individuals; therefore, no individual has income between $x_{1g^*} - \epsilon$ and x_{1g^*} nor between x_{2h^*} and $x_{2h^*} + \epsilon$. For all the other elements of $\boldsymbol{y}_1$ it holds that $y_{1i} = x_{1i}$ for $i \neq \{g^*\}$, and for the other elements of $\boldsymbol{y}_2$ it holds that $y_{2i} = x_{2i}$ for $i \neq \{h^*\}$. Note also that $\bar{y} = \bar{x}$.
The effect of this transfer on the difference between the Gini index between groups and the Gini index within (according to the Okamoto decomposition) is given by

$$
\begin{aligned}
\Delta G^B - \Delta G^W &= \frac{n_1 n_2}{n^2 \bar{x}} \left[cv(\boldsymbol{y}_1, \boldsymbol{y}_2) - cv(\boldsymbol{x}_1, \boldsymbol{x}_2)\right] - \sum_{j=1}^{2} \frac{n_j}{n} \left[\frac{1}{2\bar{x}} \Delta_j(\boldsymbol{y}_j) - \frac{1}{2\bar{x}} \Delta_j(\boldsymbol{x}_j)\right] \\
&= \frac{1}{4\bar{x}} \left[\Delta_{12}(\boldsymbol{y}_1, \boldsymbol{y}_2) - \Delta_{12}(\boldsymbol{x}_1, \boldsymbol{x}_2)\right] - \frac{3}{8\bar{x}} \left[\Delta_1(\boldsymbol{y}_1) - \Delta_1(\boldsymbol{x}_1)\right. \\
&+ \left.\Delta_2(\boldsymbol{y}_2) - \Delta_2(\boldsymbol{x}_2)\right] \\
&= \frac{1}{4\bar{x}} \left[\frac{4}{n^2} \sum_{i=1}^{n/2} \sum_{m=\frac{n}{2}+1}^{n} (|y_{1i} - y_{2m}| - |x_{1i} - x_{2m}|)\right. \\
&- \frac{6}{n^2} \sum_{i=1}^{n/2} \sum_{m=1}^{n/2} (|y_{1i} - y_{1m}| - |x_{1i} - x_{1m}|) \\
&- \left.\frac{6}{n^2} \sum_{i=\frac{n}{2}+1}^{n} \sum_{m=\frac{n}{2}+1}^{n} (|y_{2i} - y_{2m}| - |x_{2i} - x_{2m}|)\right]
\end{aligned}
$$

The first term within the brackets amounts to $\frac{4}{n^2}\epsilon(n+2)$ as the two changing incomes increase their respective distance from each other by 2ϵ, and all others by ϵ. The second term is $2[\epsilon(\frac{n}{2} - g^* + 1) - \epsilon(g^* - 1)]$, the third term is $2[\epsilon(h^* - \frac{n}{2}) - \epsilon(n - h^*)]$, as each income of the lower class is related to each of the upper class and viceversa. Adding up yields:

$$
\Delta G^B - \Delta G^W = \frac{6\epsilon}{n^2 \bar{x}} \left[\frac{2}{3}(n-1) - (h^* - g^*)\right], \tag{25}
$$

where g^* refers to the rank of income x_{1g^*} in the increasingly ordered vector $\boldsymbol{x}_1 = (x_{11}, x_{12}, \ldots, x_{1\frac{n}{2}})$, while $h^* - \frac{n}{2}$ refers to the rank of income x_{2h^*} in the increasingly ordered vector $\boldsymbol{x}_2 = (x_{2(\frac{n}{2}+1)}, \ldots, x_{2n})$.
We note that the difference $\Delta G^B - \Delta G^W$ is positive if and only if $h^* - g^* < \frac{2}{3}(n-1)$ holds, that is if the number of individuals between individual g^* and individual h^* (given by $(h^* - g^*)$) is smaller than $\frac{2}{3}(n-1)$.
Therefore, the new polarization index will increase in the case of a regressive Pigou-Dalton transfer between two individuals belonging to two different groups divided by the median, if the number of individuals between them is smaller than $\frac{2}{3}(n-1)$.

B.2 Modified Increased Spread for two overlapping groups

Consider two groups with group-specific income vectors $\boldsymbol{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n_1})$ and $\boldsymbol{x}_2 = (x_{21}, x_{22}, \ldots, x_{2n_2})$. Assume that all incomes belonging to group 2 are increased by a value $\alpha > 0$ (i.e. $y_{2m} = x_{2m} + \alpha$) while all incomes belonging to group 1 are reduced by a common factor $\beta = \frac{n_2}{n_1}\alpha > 0$ (i.e. $y_{1i} = x_{1i} - \beta$).

The variation in the within-group component is given by

$$\Delta G^W = \sum_{j=1}^{2} \frac{n_j}{n} \left(\frac{\bar{y}_j}{\bar{y}} G_j(\boldsymbol{y}_j) - \frac{\bar{x}_j}{\bar{x}} G_j(\boldsymbol{x}_j) \right) = \sum_{j=1}^{2} \frac{n_j}{n} \left(\frac{1}{2\bar{y}} \Delta_j(\boldsymbol{y}_j) - \frac{1}{2\bar{x}} \Delta_j(\boldsymbol{x}_j) \right) = 0,$$

since $\bar{y} = \bar{x}$ and thanks to the translation invariance property of the Gini mean difference, according to which $\Delta_j(\boldsymbol{y}_j) = \Delta_j(\boldsymbol{x}_j)$ for each group.

The change in the between-group component G^B corresponds to

$$\begin{aligned}
\Delta G^B &= \frac{n_1 n_2}{n^2 \bar{x}} \Delta cv(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
&= \frac{n_1 n_2}{n^2 \bar{x}} \sum_{i=1}^{n_1} \sum_{m=1}^{n_2} \frac{1}{n_1 n_2} \left(|x_{1i} - \beta - (x_{2m} + \alpha)| - |x_{1i} - x_{2m}| \right) \\
&= \frac{1}{n^2 \bar{x}} \sum_{i=1}^{n_1} \left[(\alpha + \beta) \cdot \left(\sum_{m=1}^{n_2} \mathbf{1}(x_{2m} > x_{1i}) - \sum_{m=1}^{n_2} \mathbf{1}(x_{2m} + \alpha < x_{1i} - \beta) \right) \right. \\
&+ \left. \sum_{m:(x_{2m} < x_{1i}) \wedge (x_{2m} + \alpha > x_{1i} - \beta)} \left(\alpha + \beta - 2\left(x_{1i} - x_{2m}\right)\right) \right].
\end{aligned}$$

Since there is no change in G^W the sign of ΔG^B determines if polarization increases or decreases with such a transfer.

B.3 (Modified) Increased Bipolarity for two groups (overlapping and not)

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be an increasingly ordered income vector. Consider another ordered income vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ obtained from $\boldsymbol{x}$ by a Pigou-Dalton (PD) transfer between two individuals g^* and h^* belonging to group 1, such that $y_{1g^*} = x_{1g^*} + \epsilon$ and $y_{1h^*} = x_{1h^*} - \epsilon$, with $x_{1g^*} < x_{1g^*} + \epsilon < \ldots < \ldots \leq x_{1h^*} - \epsilon < x_{1h^*}$, for any $\epsilon > 0$. The PD transfer does not allow any change in the order of the individuals; therefore, no individual has income between $x_{1h^*} - \epsilon$ and x_{1h^*} nor between x_{1g^*} and $x_{1g^*} + \epsilon$. For all the other elements of $\boldsymbol{y}$ it holds that $y_i = x_i$ for $i \neq \{g^*, h^*\}$. Note also that $\bar{y} = \bar{x}$, $\bar{y}_1 = \bar{x}_1$ and $\bar{y}_2 = \bar{x}_2$.

The obvious effect of this kind of PD transfer is that the within-group inequality decreases:

$$\begin{aligned}
\Delta G^W(\boldsymbol{x}) &= G^W(\boldsymbol{y}) - G^W(\boldsymbol{x}) \\
&= \frac{n_1}{n}\frac{\bar{x}_1}{\bar{x}}\frac{1}{2\bar{x}_1 n_1^2}\sum_{i=1}^{n_1}\sum_{m=1}^{n_1}\left(|y_{1i} - y_{1m}| - |x_{1i} - x_{1m}|\right) \\
&= \frac{1}{nn_1\bar{x}}\Big[\sum_{m=1,\ldots,n_1: m\neq\{g^*,h^*\}}\Big(|y_{1g^*} - y_{1m}| - |x_{1g^*} - x_{1m}| \\
&+ |y_{1h^*} - y_{1m}| - |x_{1h^*} - x_{1m}|\Big) + |y_{1h^*} - y_{1g^*}| - |x_{1h^*} - x_{1g^*}|\Big] \\
&= -\frac{2\epsilon}{nn_1\bar{x}}\left(\sum_{m=1,\ldots,n_1: m\neq\{g^*,h^*\}}\mathbf{1}(x_{1g^*} < x_{1m} < x_{1h^*}) + 1\right) < 0.
\end{aligned}$$

We now focus on the effect of this kind of PD transfer on the between-inequality component:

$$\Delta G^B(\boldsymbol{x}) = \frac{n_1 n_2}{n^2\bar{x}}\Delta cv(\boldsymbol{x}_1, \boldsymbol{x}_2),$$

where

$$cv(\boldsymbol{x}_1, \boldsymbol{x}_2) = \underbrace{\frac{1}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{m=1}^{n_2}|x_{1i} - x_{2m}|}_{I_{1,2}} - \underbrace{\frac{1}{2n_1^2}\sum_{i=1}^{n_1}\sum_{m=1}^{n_1}|x_{1i} - x_{1m}|}_{II_{1,1}} - \underbrace{\frac{1}{2n_2^2}\sum_{i=1}^{n_2}\sum_{m=1}^{n_2}|x_{2i} - x_{2m}|}_{II_{2,2}}$$

Since the variation of $II_{2,2}$ is null, then $\Delta G^B(\boldsymbol{x}) = \frac{n_1 n_2}{n^2 \bar{x}} \left(\Delta I_{1,2} - \Delta II_{1,1}\right)$, where

$$\begin{aligned}
\Delta I_{1,2} &= \frac{1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{m=1}^{n_2} \left(|y_{1i} - y_{2m}| - |x_{1i} - x_{2m}|\right) \\
&= \frac{1}{n_1 n_2} \sum_{m=1}^{n_2} \left(|y_{1g^*} - y_{2m}| - |x_{1g^*} - x_{2m}| + |y_{1h^*} - y_{2m}| - |x_{1h^*} - x_{2m}|\right) \\
&= \frac{1}{n_1 n_2} \sum_{m=1}^{n_2} \underbrace{\left(|x_{1g^*} + \epsilon - x_{2m}| - |x_{1g^*} - x_{2m}| + |x_{1h^*} - \epsilon - x_{2m}| - |x_{1h^*} - x_{2m}|\right)}_{A_{2m}} \\
&= \frac{1}{n_1 n_2} \left(\sum_{m: x_{2m} < x_{1g^*}} A_{2m} + \sum_{m: x_{1g^*} + \epsilon < x_{2m} < x_{1h*} - \epsilon} A_{2m} + \sum_{m: x_{2m} > x_{1h^*}} A_{2m} \right) \\
&= \frac{1}{n_1 n_2} \left(0 - 2\epsilon \cdot \sum_{m=1}^{n_2} \mathbf{1}(x_{1g^*} + \epsilon < x_{2m} < x_{1h*} - \epsilon) + 0 \right),
\end{aligned}$$

and

$$\begin{aligned}
\Delta II_{1,1} &= \frac{1}{2n_1^2} \sum_{i=1}^{n_1} \sum_{m=1}^{n_1} \left(|y_{1i} - y_{1m}| - |x_{1i} - x_{1m}|\right) \\
&= -\frac{\epsilon}{n_1^2} \left(\sum_{m=1}^{n_1} \mathbf{1}(x_{1g^*} < x_{1m} < x_{1h^*}) + 1 \right) < 0.
\end{aligned}$$

Summing up,

$$\Delta G^B(\boldsymbol{x}) = \frac{2\epsilon n_2}{n^2 \bar{x}} \left(-\frac{\sum_{m=1}^{n_2} \mathbf{1}(x_{1g^*} < x_{2m} < x_{1h*})}{n_2} + \frac{\sum_{m=1}^{n_1} \mathbf{1}(x_{1g^*} < x_{1m} < x_{1h*}) + 1}{2n_1} \right).$$

Hence, a PD transfer within any of the two generic group increases $G^B(\boldsymbol{x})$ if

$$\frac{\sum_{m=1}^{n_1} \mathbf{1}(x_{1g^*} < x_{1m} < x_{1h*}) + 1}{2n_1} > \frac{\sum_{m=1}^{n_2} \mathbf{1}(x_{1g^*} < x_{2m} < x_{1h*})}{n_2}.$$

If we rewrite this expression in terms of the within group distribution functions, $G^B(\boldsymbol{x})$ increases if

$$\frac{1}{2n_1} + \frac{F_1(h^*) - F_1(g^*)}{2} > F_2(h^*) - F_2(g^*)\,.$$

Note that in the case of non-overlapping groups, the increase of $G^B(\boldsymbol{x})$ will always be positive since in this case it holds that $F_2(h^*) - F_2(g^*) = 0$.

If we combine the changes in $G^B(\boldsymbol{x})$ and $G^W(\boldsymbol{x})$, the overall condition for Increased Bipolarity is

$$\frac{1}{n_1}\left(\sum_{m=1}^{n_1}\mathbf{1}(x_{1g^*} < x_{1m} < x_{1h*}) + 1\right)\left(\frac{n_2}{2n}+1\right) > \frac{1}{n}\sum_{m=1}^{n_2}\mathbf{1}(x_{1g^*} < x_{2m} < x_{1h*})$$

or, in terms of the group distribution functions,

$$\left(F_1(h^*) - F_1(g^*) + \frac{1}{n_1}\right)\left(\frac{n_2}{2n}+1\right) > \frac{n_2}{n}\left(F_2(h^*) - F_2(g^*)\right).$$

B.4 Modified Increased Spread for $k \geq 2$ generic groups

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be an increasingly ordered income vector of a population which can be split into k groups with group income vectors $\boldsymbol{x}_j = (x_{j1}, x_{j2}, \ldots, x_{jn_1}), j = 1, \ldots, k$. Consider another ordered income vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ obtained from $\boldsymbol{x}$ by means of the following group-specific transfers: all incomes belonging to group k are increased by a value $\alpha > 0$ (i.e. $y_{ki} = x_{ki} + \alpha$) while all incomes belonging to group 1 are reduced by a common factor $\beta = \frac{n_k}{n_1}\alpha > 0$ (i.e. $y_{1i} = x_{1i} - \beta$). This transformation does affect only the between-group inequality G^B, while the within group inequality G^W remains constant.

For the change of G^W it holds that

$$\Delta G^W = \sum_{j=1}^{k}\frac{n_j}{n}\left(\frac{\bar{y}_j}{\bar{y}}G_j(\boldsymbol{y}_j) - \frac{\bar{x}_j}{\bar{x}}G_j(\boldsymbol{x}_j)\right) = \sum_{j=1}^{k}\frac{n_j}{n}\left(\frac{1}{2\bar{y}}\Delta_j(\boldsymbol{y}_j) - \frac{1}{2\bar{x}}\Delta_j(\boldsymbol{x}_j)\right) = 0\,.$$

The change in the G^B component corresponds to

$$
\begin{aligned}
\Delta G^B &= \frac{1}{2\bar{x}} \sum_{j=1}^{k} \sum_{l=1}^{k} \frac{n_j n_l}{n^2} \Delta cv(\boldsymbol{x}_j, \boldsymbol{x}_l) \\
&= \frac{1}{2\bar{x}} \left(\underbrace{\sum_{j \neq \{1,k\}} \sum_{l \neq \{1,k\}} \frac{n_j n_l}{n^2} \Delta cv(\boldsymbol{x}_j, \boldsymbol{x}_l)}_{A} + \underbrace{2 \sum_{l \neq \{1,k\}} \frac{n_1 n_l}{n^2} \Delta cv(\boldsymbol{x}_1, \boldsymbol{x}_l)}_{B} \right. \\
&+ \left. \underbrace{2 \sum_{l \neq \{1,k\}} \frac{n_k n_l}{n^2} \Delta cv(\boldsymbol{x}_k, \boldsymbol{x}_l)}_{C} + \underbrace{2 \frac{n_1 n_k}{n^2} \Delta cv(\boldsymbol{x}_1, \boldsymbol{x}_k)}_{D} \right),
\end{aligned}
$$

with

$$
\begin{aligned}
A &= 0; \\
B &= \frac{2}{n^2} \sum_{l \neq \{1,k\}} \sum_{i=1}^{n_1} \sum_{m=1}^{n_l} \left(|x_{1i} - \beta - x_{lm}| - |x_{1i} - x_{lm}| \right) \\
&= \frac{2}{n^2} \sum_{l \neq \{1,k\}} \sum_{i=1}^{n_1} \sum_{m=1}^{n_l} \left[\beta \cdot \left(\mathbf{1}(x_{1i} < x_{lm}) - \mathbf{1}(x_{1i} - \beta > x_{lm}) \right) \right. \\
&+ \left. \left(\beta - 2\left(x_{1i} - x_{lm} \right) \right) \mathbf{1}(x_{1i} - \beta < x_{lm} < x_{1i}) \right];
\end{aligned}
$$

$$
\begin{aligned}
C &= \frac{2}{n^2} \sum_{l \neq \{1,k\}} \sum_{i=1}^{n_k} \sum_{m=1}^{n_l} \left(|x_{ki} + \alpha - x_{lm}| - |x_{ki} - x_{lm}| \right) \\
&= \frac{2}{n^2} \sum_{l \neq \{1,k\}} \sum_{i=1}^{n_l} \sum_{m=1}^{n_k} \left[\alpha \cdot \left(1(x_{li} < x_{km}) - \mathbf{1}(x_{li} > x_{km} + \alpha) \right) \right. \\
&+ \left. \left(\alpha - 2\left(x_{li} - x_{km} \right) \right) \mathbf{1}(x_{km} < x_{li} < x_{km} + \alpha) \right];
\end{aligned}
$$

$$
\begin{aligned}
D &= \frac{2}{n^2} \sum_{i=1}^{n_1} \sum_{m=1}^{n_k} \left(|x_{1i} - \beta - x_{km} - \alpha| - |x_{1i} - x_{km}| \right) \\
&= \frac{2}{n^2} \sum_{i=1}^{n_1} \sum_{m=1}^{n_k} \left((\alpha + \beta) \cdot \left(\mathbf{1}(x_{1i} < x_{km}) - \mathbf{1}(x_{1i} - \beta > x_{km} + \alpha) \right) \right. \\
&+ \left. \left(\alpha + \beta - 2\left(x_{1i} - x_{km} \right) \right) \mathbf{1}(x_{1i} > x_{km}) \& (x_{i1} - \beta < x_{km} + \alpha) \right).
\end{aligned}
$$

If α and β are small this change is positive if

$$\left\{ \sum_{l\neq\{1,k\}} \sum_{i=1}^{n_1} \sum_{m=1}^{n_l} \beta \cdot (\mathbf{1}(x_{1i} \leq x_{lm}) - \mathbf{1}(x_{1i} > x_{lm})) \right.$$
$$+ \sum_{l\neq\{1,k\}} \sum_{i=1}^{n_l} \sum_{m=1}^{n_k} \alpha \cdot (\mathbf{1}(x_{li} \leq x_{km}) - \mathbf{1}(x_{li} > x_{km}))$$
$$\left. + \sum_{i=1}^{n_1} \sum_{m=1}^{n_k} (\alpha+\beta) \cdot (\mathbf{1}(x_{1i} \leq x_{km}) - \mathbf{1}(x_{1i} > x_{km})) \right\} > 0\,.$$

An easy to verify sufficient condition for this inequality is $m(\boldsymbol{x}_1) \leq m(\boldsymbol{x}_2) \leq \ldots \leq m(\boldsymbol{x}_k)$, where $m(\boldsymbol{x}_j)$ is the median income of the j^{th} group.

B.5 Modified Increased Polarity for $k \geq 2$ generic groups

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ be an increasingly ordered income vector. Consider another ordered income vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ obtained from $\boldsymbol{x}$ by a Pigou-Dalton (PD) transfer between two individuals g^* and h^* belonging to the same group (w.l.o.g., we consider group 1), such that $y_{1g^*} = x_{1g^*} + \epsilon$ and $y_{1h^*} = x_{1h^*} - \epsilon$, with $x_{1g^*} < x_{1g^*} + \epsilon < \ldots < \ldots \leq x_{1h^*} - \epsilon < x_{1h^*}$, for any $\epsilon > 0$. The PD transfer does not allow any change in the order of the individuals; therefore, nobody can stay between $x_{1h^*} - \epsilon$ and x_{1h^*} nor between x_{1g^*} and $x_{1g^*} + \epsilon$. Note also that $\bar{y} = \bar{x}$ and that $\bar{y}_j = \bar{x}_j$, for each $j = \{1, 2, \ldots, k\}$.

The obvious effect of this kind of PD transfer is that the within-group inequality decreases. For ΔG^W it holds that (see B.3 for details)

$$\Delta G^W(\boldsymbol{x}) = -\frac{2\epsilon}{nn_1\bar{x}} \left(\sum_{m=1}^{n_1} 1(x_{1g^*} < x_{1m} < x_{1h^*}) + 1 \right) < 0.$$

For the change of G^B it holds that (see B.3 for details)

$$\Delta G^B(\boldsymbol{x}) = \frac{2\epsilon}{n^2\bar{x}} \sum_{l=2}^{k} \left(\frac{n_l}{2n_1} \left(\sum_{m=1}^{n_1} \mathbf{1}(x_{1g^*} < x_{1m} < x_{1h*}) + 1 \right) - \sum_{m=1}^{n_l} \mathbf{1}(x_{1g^*} < x_{lm} < x_{1h*}) \right).$$

Hence, a PD transfer within group 1 between individuals g^* and h^* increases polarization if

$$\sum_{l=2}^{k}\left(\frac{n_l}{2nn_1}\left(\sum_{m=1}^{n_1}\mathbf{1}(x_{1g^*} < x_{1m} < x_{1h*}) + 1\right) - \frac{1}{n}\sum_{m=1}^{n_l}\mathbf{1}(x_{1g^*} < x_{lm} < x_{1h*})\right)$$
$$+ \quad \frac{1}{n_1}\left(\sum_{m=1}^{n_1}\mathbf{1}(x_{1g^*} < x_{1m} < x_{1h^*}) + 1\right) > 0,$$

that is

$$\frac{1}{n_1}\left(\sum_{m=1}^{n_1}\mathbf{1}(x_{1g^*} < x_{1m} < x_{1h*}) + 1\right)\cdot\left(1 + \frac{n - n_1}{2n}\right) - \frac{1}{n}\sum_{l=2}^{k}\sum_{m=1}^{n_l}\mathbf{1}(x_{1g^*} < x_{lm} < x_{1h*}) > 0,$$

or, in terms of the group distribution functions,

$$\left(F_1(h^*) - F_1(g^*) + \frac{1}{n_1}\right)\left(1 + \frac{n - n_1}{2n}\right) > \sum_{l=2}^{k}\frac{n_l}{n}\left(F_l(h^*) - F_l(g^*)\right) .$$